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Comparative Study of Explicit Solutions to Wave Dispersion Equation

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The dispersion equation in the Airy wave theory is a transcendental equation, which usually requires an iterative technique to obtain a numerically exact solution of wave length for a given wave period and water depth. As an alternative, various kinds of approximate and explicit solutions (AESs) have been proposed. This paper presents the results of a comparative study on errors of 30 AESs, including 10 new ones. The conclusions are that in case of the AESs valid for a whole range of water depth conditions, one of the AESs proposed in this study has an maximum error of around ± 0.0001 %, which is more proper than the others in its accuracy and compactness of solution, and that neither of the AESs applicable only to a restricted water depth condition may be recommended for use due to inconvenience associated with a restrictive condition.

Key Words : wave dispersion equation, exact solution by the Newton method, approximate and explicit solutions

1. Introduction

The relationship between wave length L and wave period T associated with water depth h in the Airy wave theory is described by the well-known dispersion relation. Since this is a transcendental equation, an iterative technique like the Newton method has been used to obtain a highly accurate estimate of the wave length, that is a numerically exact solution for any given wave period and water depth. However, this method is inconvenient to use not only from a practical but also from an educational point of view. This is probably the reason why many AESs have been proposed to estimate the wave length thus far. In this study, the errors of 30 AESs, including 10 new ones are investigated. As a result, an AES to compute the wave length with high level or middle level accuracy respectively is recommended, taking account of magnitude of error and compactness of expression.

2. Equation for computation of wave length and numerical method

The relationship between wave length L and wave period T associated with water depth h is given as follows.

$$L=\left(\frac{gT^2}{2\pi}\right)\tanh\frac{2\pi h}{L}=L_0\tanh\frac{2\pi h}{L},\quad L_0=\frac{gT^2}{2\pi} \tag{1}$$

where subscript '$_0$' means deep water and g the acceleration of gravity. Eq.(1) is rewritten using the wave numbers $k=2\pi/L$ and $k_0=2\pi/L_0$ as

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$$k_0 h = kh \cdot \tanh kh \tag{2}$$

To arrive at a high-accuracy numerical solution to the dispersion relation of Eq.(1), we use a Newton method. Taking

$$\alpha = k_0 h = 2\pi (h/L_0) = \left(2\pi/T\sqrt{g/h}\right)^2 \tag{3}$$

we can replace Eq.(2) with

$$\alpha = kh \cdot \tanh kh = \beta \cdot \tanh \beta, \quad \beta = kh \tag{4}$$

The equation to be used in the computation then becomes:

$$\begin{aligned} f(\beta_n) &= \alpha - \beta_n \tanh\beta_n, \quad f'(\beta_n) = -\tanh\beta_n - \beta_n \mathrm{sech}^2 \beta_n \\ \beta_{n+1} &= \beta_n - f(\beta_n)/f'(\beta_n), \quad \left|(\beta_{n+1} - \beta_n)/\beta_n\right| < \varepsilon, \quad \varepsilon = 10^{-10} \end{aligned} \tag{5}$$

where superscript ' $'$ ' means differentiation and ' n ' the n-th iteration. As an initial value,

$$\begin{aligned} \beta_0 &= \alpha = 2\pi(h/L_0) \; ; \qquad \alpha \ge 1 \quad \text{or} \quad h/L_0 \ge 1/2\pi \approx 0.1592 \\ \beta_0 &= \alpha^{1/2} = \{2\pi(h/L_0)\}^{1/2} \; ; \quad \alpha < 1 \quad \text{or} \quad h/L_0 < 1/2\pi \approx 0.1592 \end{aligned} \tag{6}$$

can be employed. Goda[1], [2] recommends using the following equation to avoid the appearance of an inflection point in the first equation of Eq.(5) (with $f''(\beta) = 0, \quad \beta \cdot \tanh\beta = 1, \quad \beta \approx 1.200, \quad h/L_0 = 0.191$).

$$f(\beta_n) = \beta_n - \alpha \cdot \coth\beta_n \tag{7}$$

Alternatively, Eq.(5) can be rewritten to avoid the appearance of an inflection point as:

$$f(\beta_n) = \alpha/\beta_n - \tanh\beta_n \tag{8}$$

Numerical computation based on the first equation of Eq.(5) or Eq.(7) or Eq.(8) within the range of h/L_0 =10^{-6} ~1 (with increment of $\Delta(h/L_0)$=10^{-6}) yielded the same answer within 7 or 8 digits, and gave no numerical problems.

3. Explicit equations for approximate computation of wave length and their accuracy

AESs that have been published so far can be divided into 2 groups : I AESs applicable for a full range of water depth conditions h/L_0 and II AESs valid for a limited range of h/L_0 . Each group can be sub-classified into ① AESs with simple form but lower accuracy and ②AESs with complicated or lengthy form but higher accuracy. Also, AESs of group II① can be separated into i) shallower water use and ii) deeper water use (see **Table 1**). No AESs of group II② for deeper water use have been published.

Table 1 Grouping of approximate and explicit solutions

	Simple, low accuracy ①		Complicated or lengthy, high accuracy ②
Full range of depth I			Newton method
			Padé approximation
Limited range of depth II	Shallower water (i)	Deeper water (ii)	Shallower water (i)

For the present study, the numerical computations are conducted for the range of h/L_0 =10^{-4}~1 with an increment of $\Delta(h/L_0)$=10^{-4}. The error of each AES relative to the exact solution $\tilde{\varepsilon}$ is defined as:

$$\tilde{\varepsilon} = (L_a/L_{exa} - 1)\times 100 \tag{9}$$

where the subscript 'a' means an approximated wave length, calculated with an AES and the subscript 'exa' means the exact wave length, computed numerically with the dispersion relationship of Eq.(1), using the Newton method.

Each of the AESs classified into group I① (valid for all water depths, simple expression, low accuracy) and its range of relative error are given below in sequence of increasing accuracy. Any of the cited AESs reduces to the exact solution in the limits of deep water ($\alpha \to \infty$) and very shallow water($\alpha \to 0$) respectively.

1) The Eckart[3] solution (Eckart)

$$L_a = L_0 (\tanh k_0 h)^{1/2} = L_0 \{\tanh (2\pi h/L_0)\}^{1/2} = L_0 (\tanh \alpha)^{1/2} \quad (10)$$

or

$$k_a h = \beta_a = \alpha(\coth\alpha)^{1/2}, \quad k_a = 2\pi/L_a \quad (11)$$

$$\tilde{\varepsilon} = 0 \sim 5.24\ (h/L_0 = 0.111)\ \% \quad (12)$$

The term in parenthesis in Eq.(12) indicates a value of h/L_0 where positive or negative maximum error is produced.

2) The Iwagaki[4] solution (Iwagaki)

$$\beta_a = \alpha \cdot \coth\left\{\alpha^{1/2}\left(1+\alpha^{1/2}/2\pi\right)\right\} \quad (13)$$

$$\tilde{\varepsilon} = -3.05\ (h/L_0 = 0.287) \sim 3.14\ (h/L_0 = 0.023)\ \% \quad (14)$$

3) The Carvlho[5] 14th solution (Carv14)

$$\beta_a = \alpha\left(1+\alpha^{-2}\right)^{1/4} \quad (15)$$

$$\tilde{\varepsilon} = -2.45\ (h/L_0 = 0.366) \sim 3.28\ (h/L_0 = 0.068)\ \% \quad (16)$$

The above AES is named here after the solution number given by Carvlho[5]. The same type of numbering is used below. Carvlho[5] collects 17 AESs for which the absolute value of the maximum relative error ranges from 0.012 % to 5.24 %. As his AESs with higher accuracy are based on successive substitutions of lower AES into the dispersion equation, the argument of hyperbolic tangent function becomes more and more complicated with increasing accuracy of the AES. In spite of his efforts, the accuracy cannot be labeled high due to the minimum of the maximum relative error of 0.012 % in his AESs. For this reason, his AESs with higher accuracy are not attractive but Eq.(15) and the 3 AESs mentioned below draw attention because of their simple forms.

4) The Fenton and MacKee[6] solution(FM)

$$\beta_a = \alpha\left(\coth\alpha^{m/2}\right)^{1/m}\ ;\ m = 1.5 \quad (17)$$

$$\tilde{\varepsilon} = -1.39\ (h/L_0 = 0.321) \sim 1.66\ (h/L_0 = 0.054)\ \% \quad (18)$$

5) The Yamaguchi and Nonaka 1st solution(YN1), which is a modified Fenton and MacKee solution

$$\beta_a = \alpha\left(\coth\alpha^{m/2}\right)^{1/m}\ ;\ m = 1.485 \quad (19)$$

$$\tilde{\varepsilon} = -1.52\ (h/L_0 = 0.315) \sim 1.55\ (h/L_0 = 0.052)\ \% \quad (20)$$

This is basically the same as Eq.(17), but the power m is adjusted so that positive and negative maximum errors take almost the same absolute value.

6) The Carvlho[5] 9th solution(Carv9)

$$\beta_a = \alpha \cdot \coth\left(\sinh \alpha^{1/2}\right) \quad (21)$$

$$\tilde{\varepsilon} = -1.12\ (h/L_0 = 0.237) \sim 0\ \% \quad (22)$$

7) The Guo[7] solution (Guo)

$$\beta_a = \alpha\Big/\left\{1-\exp\left(-\alpha^{m/2}\right)\right\}^{1/m}\ ;\ m = 2.4901 \quad (23)$$

$$\tilde{\varepsilon} = -0.75\ (h/L_0 = 0.284) \sim 0.75\ (h/L_0 = 0.043)\ \% \quad (24)$$

Guo[7] gives both m=2.4908 and m=2.4901. The latter value is used here because the positive and negative maximum errors are then nearly equal (in an absolute sense).

8) The Yamaguchi and Nonaka 2nd solution(YN2), which is a combination of the modified Fenton and MacKee solution Eq.(19) and the dispersion equation Eq.(4).

$$\beta_a = \alpha \cdot \coth\beta_F, \quad \beta_F = \alpha\left(\coth\alpha^{m/2}\right)^{1/m}\ ;\ m = 1.378 \quad (25)$$

$$\tilde{\varepsilon} = -0.73\ (h/L_0 = 0.029) \sim 0.73\ (h/L_0 = 0.187)\ \% \quad (26)$$

This aims at improving accuracy in the wave length computation by substitution of the YN1-based estimate into the dispersion relationship. The same method is used by Carvlho[5]. The maximum error reduces to nearly half the maximum error produced by the FM formula or the YN1 formula. The power m is adjusted here to obtain almost the same positive and negative maximum errors (in an absolute sense). Repetition of the same procedure complicates the computation and is inefficient as it does not significantly improve the result.

9) The Carvlho[5] 5th solution(Carv5)

$$\beta_a = \alpha \cdot \coth\left(1.2^{\alpha} \cdot \alpha^{1/2}\right) \tag{27}$$

$$\tilde{\varepsilon} = -0.21\ (h/L_0 = 0.278) \sim 0.27\ (h/L_0 = 0.063)\ \% \tag{28}$$

10) The Carvlho[5] 4th solution(Carv4)

$$\beta_a = \alpha \Big/ \left\{ (\tanh\alpha)^{1/4} \cdot \left[\tanh\left\{(\sinh\alpha)^{1/2}\right\}\right]^{1/2} \right\} \tag{29}$$

$$\tilde{\varepsilon} = -0.12\ (h/L_0 = 0.198) \sim 0.20\ (h/L_0 = 0.423)\ \% \tag{30}$$

The Carv5 formula has a simpler form than the Carv4 formula but gives a slightly larger error.

Each of the above-mentioned 10 AESs reduces to the exact solution in the limit of deep water($\alpha \to \infty$) or very shallow water ($\alpha \to 0$) respectively. **Figs. 1 and 2** show the relation between the relative error $\tilde{\varepsilon}$ and h/L_0 for all of the AESs. The global behavior of the error with h/L_0 in each solution becomes clear from the figures.

The Carv5 formula with its error below 0.3 % in the above-mentioned AESs is recommended for a full range use of h/L_0 , taking into account the magnitude of error and simplicity of the solution form.

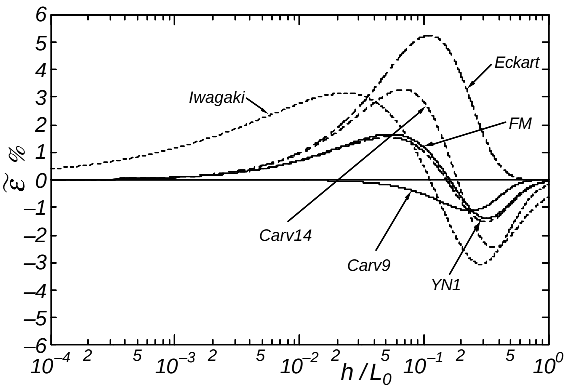

Fig. 1 Relation between relative error and h/L_0 (1).

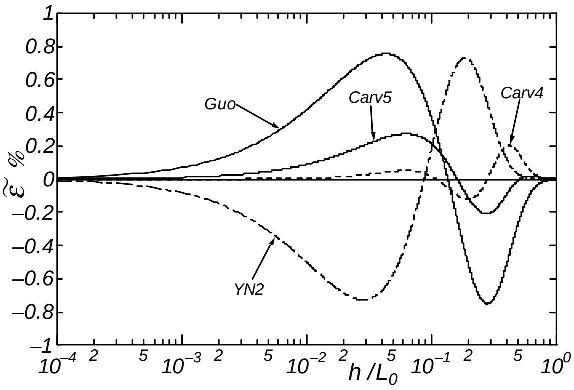

Fig. 2 Relation between relative error and h/L_0 (2).

Next, the more highly accurate AESs belonging to group I② (valid for all water depths, complicated or lengthy expressions, high accuracy) are classified into 2 families. The first family is based on the first iterative solution of the dispersion equation obtained with the Newton method, with one of the AESs 1) to 10) applied as an initial value(as proposed by Fenton[8]). The second family is based on using a Padé approximation. Each of the AESs of the first family and their range of relative error is written in order of accuracy as follows. Any of the cited AESs reduces to the exact solution in the limits of deep water ($\alpha \to \infty$) and very shallow water ($\alpha \to 0$) respectively. A solution can be written as

$$\left.\begin{aligned} & f(\beta_0) = \alpha - \beta_0 \tanh\beta_0, \quad f'(\beta_0) = -\tanh\beta_0 - \beta_0 \mathrm{sech}^2\beta_0 \\ & \beta_1 = \beta_0 - \frac{f(\beta_0)}{f'(\beta_0)} = \beta_0 + \frac{\alpha - \beta_0\tanh\beta_0}{\tanh\beta_0 + \beta_0\mathrm{sech}^2\beta_0} \\ & \quad = \frac{\alpha + \beta_0^2 \mathrm{sech}^2\beta_0}{\tanh\beta_0 + \beta_0\mathrm{sech}^2\beta_0} = \frac{\alpha + \beta_0^2\left(1 - \tanh^2\beta_a\right)}{\tanh\beta_0 + \beta_0\left(1 - \tanh^2\beta_0\right)} \end{aligned}\right\} \tag{31}$$

11-1) The Fenton[8] solution(Fenton), which is an iterative solution to the Eckart equation Eq.(11)

$$\begin{aligned} k_a h &= \left(\alpha + \beta_a^2 \mathrm{sech}^2\beta_a\right) \big/ \left(\tanh\beta_a + \beta_a \mathrm{sech}^2\beta_a\right) \\ &= \left\{\alpha + \beta_a^2\left(1 - \tanh^2\beta_a\right)\right\} \big/ \left\{\tanh\beta_a + \beta_a\left(1 - \tanh^2\beta_a\right)\right\} \\ \beta_a &= \alpha(\coth\alpha)^{1/m}, \quad m = 2 \end{aligned} \tag{32}$$

$$\tilde{\varepsilon} = -5.1\times10^{-2}\ (h/L_0 = 0.070) \sim 8.4\times10^{-3}\ (h/L_0 = 0.218)\ \% \tag{33}$$

Since the Newton method is the same for all other AESs in this family, the following description is limited to the solution for β_a and the error range. The former takes the same form as the solution mentioned previously, but in some cases the power m is different from the previous result.

11-2) The Yamaguchi and Nonaka 3rd solution (YN3), which is an iterative solution to the Iwagaki equation Eq.(13).

$$\beta_a = \alpha \cdot \coth\left\{\alpha^{1/2}\left(1+\alpha^{1/2}/2\pi\right)\right\} \tag{34}$$

$$\tilde{\varepsilon} = -4.0\times10^{-2}\ (h/L_0 = 0.019) \sim 1.2\times10^{-2}\ (h/L_0 = 0.289)\ \% \tag{35}$$

11-3) The Yamaguchi and Nonaka 4th solution (YN4), which is an iterative solution to the Carvlho 14th equation Eq.(15).

$$\beta_a = \alpha\left(1+\alpha^{-2}\right)^{1/4} \tag{36}$$

$$\tilde{\varepsilon} = -2.9\times10^{-2}\ (h/L_0 = 0.053) \sim 6.7\times10^{-3}\ (h/L_0 = 0.335)\ \% \tag{37}$$

11-4) The Yamaguchi and Nonaka 5th solution (YN5), which is an iterative solution to the modified Fenton and MacKee equation Eq.(19).

$$\beta_a = \alpha\left(\coth\alpha^{m/2}\right)^{1/m} \ ;\ m = 1.434 \tag{38}$$

$$\tilde{\varepsilon} = -4.9\times10^{-3}\ (h/L_0 = 0.036) \sim 4.9\times10^{-3}\ (h/L_0 = 0.296)\ \% \tag{39}$$

When the Fenton and MacKee[6] equation is used as an initial value, the error ranges from -8.5 $\times$ 10^{-3} to 2.3 $\times$ 10^{-3}. The accuracy is worse than this Eq.(38)-based approximation.

11-5) The Yamaguchi and Nonaka 6th solution (YN6), which is an iterative solution to the Carvlho 9th equation Eq.(21).

$$\beta_a = \alpha \cdot \coth\left(\sinh \alpha^{1/2}\right) \tag{40}$$

$$\tilde{\varepsilon} = -4\times10^{-4}\ (h/L_0 = 0.101) \sim 1.4\times10^{-3}\ (h/L_0 = 0.264)\ \% \tag{41}$$

11-6) The Yamaguchi and Nonaka 7th solution (YN7), which is an iterative solution to the modified Guo equation.

$$\beta_a = \alpha\Big/\left\{1-\exp\left(-\alpha^{m/2}\right)\right\}^{1/m} \ ;\ m = 2.445 \tag{42}$$

$$\tilde{\varepsilon} = -1.2\times10^{-3}\ (h/L_0 = 0.030) \sim 1.2\times10^{-3}\ (h/L_0 = 0.278)\ \% \tag{43}$$

11-7) The Yamaguchi and Nonaka 8th solution (YN8), which is an iterative solution to the Yamaguchi and Nonaka 2nd equation Eq.(25).

$$\beta_a = \alpha \cdot \coth\beta_F\,,\quad \beta_F = \alpha\left(\coth\alpha^{m/2}\right)^{1/m} \ ;\ m = 1.310 \tag{44}$$

$$\tilde{\varepsilon} = -9\times10^{-4}\ (h/L_0 = 0.112) \sim 8\times10^{-4}\ (h/L_0 = 0.223)\ \% \tag{45}$$

11-8) The Yamaguchi and Nonaka 9th solution (YN9), which is an iterative solution to the modified Carvlho 5th equation.

$$\beta_a = \alpha \cdot \coth\left(m^{\alpha}\cdot\alpha^{1/2}\right) \ ;\ m = 1.1965 \tag{46}$$

$$\tilde{\varepsilon} = -1.1\times10^{-4}\ (h/L_0 = 0.044) \sim 1.1\times10^{-4}\ (h/L_0 = 0.274)\ \% \tag{47}$$

11-9) The Yamaguchi and Nonaka 10th solution (YN10), which is an iterative solution to the Carvlho 4th equation.

$$\beta_a = \alpha\Big/\left\{\left(\tanh\alpha\right)^{1/4}\left[\tanh\left\{\left(\sinh\alpha\right)^{1/2}\right\}\right]^{1/2}\right\} \tag{48}$$

$$\tilde{\varepsilon} = -7\times10^{-6}\ (h/L_0 = 0.056) \sim 4\times10^{-5}\ (h/L_0 = 0.401)\ \% \tag{49}$$

The power m in 11-4), 11-6), 11-7) and 11-8) is adjusted so that almost the same positive and negative maximum errors (in an absolute sense) is obtained. Naturally, the higher the accuracy of an initial estimate is, the higher the accuracy of the iterative solution becomes.

The second family of this first group, which uses a Padé approximation is based on the equations of Hunt [9].

12) The Hunt[9] 5th order approximate solution(Hunt1)

$$(k_a h)^2 = \alpha\left\{\alpha + \left(1+0.6522\alpha+0.4622\alpha^2+0.0864\alpha^4+0.0675\alpha^5\right)^{-1}\right\} \tag{50}$$

$$\tilde{\varepsilon} = -7.0\times10^{-2}\ (h/L_0 = 0.532) \sim 7.8\times10^{-2}\ (h/L_0 = 0.288)\ \% \tag{51}$$

13) The Hunt [9] 9th order approximate solution(Hunt2)

$$(k_a h)^2 = \alpha\left\{\alpha + \left(1 + 0.66667\alpha + 0.35550\alpha^2 + 0.16084\alpha^3 + 0.06320\alpha^4 + 0.02174\alpha^5 + 0.00654\alpha^6 + 0.00171\alpha^7 + 0.00039\alpha^8 + 0.00011\alpha^9\right)^{-1}\right\} \tag{52}$$

$$\tilde{\varepsilon} = -8.2\times10^{-3}\ (h/L_0 = 0.603) \sim 5.4\times10^{-3}\ (h/L_0 = 0.324)\ \% \tag{53}$$

The maximum errors in the Hunt1 and Hunt2 formulae are around 0.1 % and 0.01% respectively. The accuracy of either of the formulae is not so high compared to any of the iterative solutions by the Newton method given from 11-4) to 11-9).

Each solution from 11-1) to 13) converges to the exact solution in both limits of h/L_0 as well as the individual case from 1) to 10). **Fig. 3** illustrates the relation between the relative error $\tilde{\varepsilon}$ and h/L_0 for the AESs 11-1), 11-2), 11-3), 11-4), 12) and 13), and **Fig. 4** for the high-accuracy AESs from 11-5)to 11-9).

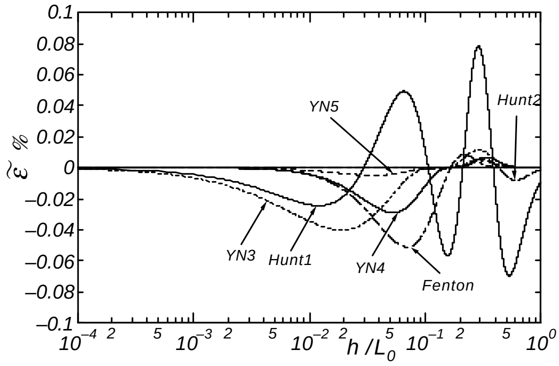

Fig. 3 Relation between relative error and h/L_0 (3).

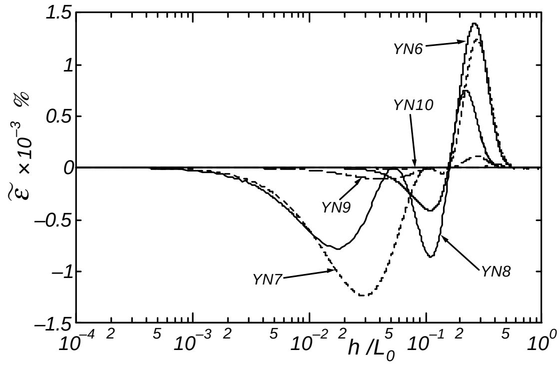

Fig. 4 Relation between relative error and h/L_0 (4).

Taking account of the positive-negative symmetry of error, accuracy and the compactness of an initial solution, we may say that the iterative solution to the Yamaguchi and Nonaka 9th solution (YN9), or the modified Carvlho 5th equation is the best of all the candidate AESs for wave length computation.

The AESs of group II (valid for limited ranges of water depth) and their relative error ranges are indicated below. Each of these AESs satisfies the long wave condition in the limit ($h/L_0 \to 0$). A common characteristic is that with h/L_0 increasing from zero to higher values, the error of each of these AESs either increases to a maximum, followed by a drastic decrease, or (the mirror image, see **Fig. 5**) decreases to a minimum, followed by a drastic increase. The value of h/L_0 where the maximum (or minimum) error is attained again in the drastic decrease (or increase) is indicated with an asterisk (h/L_0^*).

For the sub-group II①(i) (simple, low accuracy, valid for shallower water), the results are given below.

14) The Nielsen[10] 1st solution(Niel1)

$$k_a h = \alpha^{1/2}\{1 + (5/8\pi)\alpha\},\quad h/L_0^* \le 0.192 \tag{54}$$

$$\tilde{\varepsilon} = -0.74\ (h/L_0 = 0.075) \sim 0.74\ (h/L_0^* = 0.192)\ \% \tag{55}$$

15) The Nielsen[10] 2nd solution(Niel2)

$$k_a h = \alpha^{1/2}\{1 + (1/6)\alpha + (11/360)\alpha^2\},\quad h/L_0^* \le 0.401 \tag{56}$$

$$\tilde{\varepsilon} = -0.44\ (h/L_0^* = 0.401) \sim 0.44\ (h/L_0 = 0.268)\ \% \tag{57}$$

16) The Venezian [11] 1st solution(Vene1)

$$k_a h = \alpha^{1/2}/(1 - \alpha/6),\quad h/L_0^* \le 0.165 \tag{58}$$

$$\tilde{\varepsilon} = -4.8\times10^{-2}\ (h/L_0^* = 0.165) \sim 4.8\times10^{-2}\ (h/L_0 = 0.104)\ \% \tag{59}$$

17) The Wu and Thornton [12] 1st solution(WT1)

$$k_a h = \alpha^{1/2}\left\{1+\frac{\alpha}{6}\left(1+\frac{\alpha}{5}\right)\right\},\quad h/L_0^* \le 0.219 \tag{60}$$

$$\tilde{\varepsilon} = -3.4\times10^{-2}\ (h/L_0^*=0.219) \sim 0\ (h/L_0 \to 0)\ \% \tag{61}$$

The accuracy of the Vene1 or WT1 AESs is fairly good in spite of the simple form of the expressions. **Fig. 5** shows the relation between the relative error $\tilde{\varepsilon}$ of these above 4 AESs and h/L_0. The rapid increase or decrease of the error above the critical h/L_0^* value is obvious.

The simple AESs with low accuracy belonging to group II①(ii) which are applicable only to deeper water are written as follows.

18) The Nielsen [13] 3rd solution (Niel3)

$$k_a h = \alpha\{1+2\exp(-2\alpha)\},\quad h/L_0^* \ge 0.300 \tag{62}$$

$$\tilde{\varepsilon} = -0.55\ (h/L_0^*=0.300) \sim 0\ (h/L_0 \to \infty)\ \% \tag{63}$$

The limiting condition is taken as h/L_0 >0.3. The error drastically increases below the range(h/L_0 <0.3) .

19) The Wu and Thornton [12] 2nd solution(WT2)

$$k_a h = \alpha\{1+2t(1+t)\}\quad,\quad t=\exp\{-2\alpha(1+1.26e^{-1.84\alpha})\}\quad,\quad h/L_0^* \ge 0.195 \tag{64}$$

$$\tilde{\varepsilon} = -2.5\times10^{-2}\ (h/L_0=0.252) \sim 2.5\times10^{-2}\ (h/L_0^*=0.195)\ \% \tag{65}$$

The accuracy of the AES is rather high but it has a slightly more complicated form than the other AES.

Fig. 6 shows the relation between the relative error $\tilde{\varepsilon}$ of the above AESs and h/L_0. In shallower water, a rapid increase of negative error can be seen.

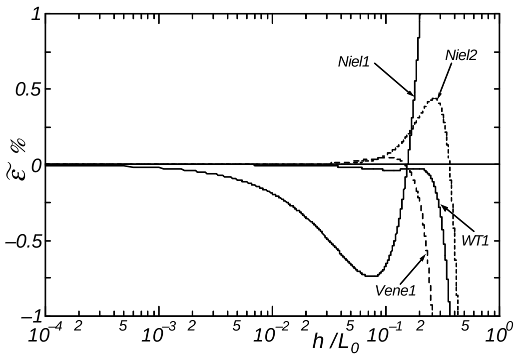

Fig. 5 Relation between relative error and h/L_0 (5).

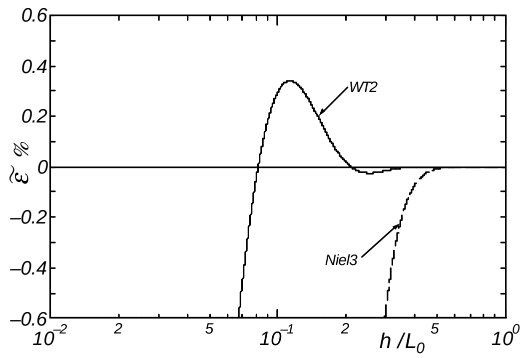

Fig. 6 Relation between relative error and h/L_0 (6).

For the sub-group II②(i) (complex or lengthy expression, high accuracy, valid for shallower water), the results are given below.

20) The You [13] solution (You)

$$k_a h = \alpha^{1/2}\{1+(1/3)\alpha+(4/45)\alpha^2+(16/945)\alpha^3\}^{1/2},\quad h/L_0^* \le 0.179 \tag{66}$$

$$\tilde{\varepsilon} = -5.4\times10^{-3}\ (h/L_0^*=0.179) \sim 5.4\times10^{-3}\ (h/L_0=0.133)\ \% \tag{67}$$

21) The Olson[14] solution (Olson)

$$k_a h = \alpha^{1/2}\{1-(1/3)\alpha+(1/45)\alpha^2+(1/189)\alpha^3+0.000776014\alpha^4-0.000044892\alpha^5 -0.000071391\alpha^6-0.000022654\alpha^7\}^{-1/2},\ h/L_0^* \le 0.186 \tag{68}$$

$$\tilde{\varepsilon} = -3\times10^{-5}\ (h/L_0^*=0.186) \sim 3\times10^{-5}\ (h/L_0=0.162)\ \% \tag{69}$$

This solution includes the Niel2 solution Eq.(56) and the You solution Eq.(66) in the sense that it adds higher order term (in α) to these AESs.

22) The Venezian[11] 2nd solution(Vene2)

$$\left.\begin{aligned} &k_a h = \alpha^{1/2}\left(1+p_1\alpha+p_2\alpha^2+p_3\alpha^3\right)/\left(1+q_1\alpha+q_2\alpha^2+q_3\alpha^3\right), h/L_0^* \le 0.159\ (\alpha^* \le 1) \\ &p_1=-0.42886826,\quad p_2=0.0939283,\quad p_3=-0.00269417, \\ &q_1=-0.59553493,\quad q_2=0.16262861,\quad q_3=-0.01497505 \end{aligned}\right\} \tag{70}$$

$$\tilde{\varepsilon} = -2\times10^{-4}\ (h/L_0^*=0.159) \sim 6\times10^{-6}\ (h/L_0=0.030)\ \% \tag{71}$$

Eq.(70) with its complicated form, has a very high accuracy within its applicability range. If a limit of -0.005 % is taken as a critical value of the error, the applicability range of Eq.(70) becomes h/L_0 <0.235, which is fairly wide.

Fig. 7 shows the relation between relative error $\tilde{\varepsilon}$ based on each of 20), 21) and 22) and h/L_0. Similar behavior as in **Fig. 5** is found.

The combination of one equation in 14) to 17) or in 20) to 22) and any equation of 18) and 19) yields an approximate solution applicable for wave length computation for all values of h/L_0. Examples are the suggestions of Wu and Thornton [12] and You [13]. Wu and Thornton [12] use Eqs.(60) and (64) but at the critical condition of h/L_0 =0.2, a discontinuity in the error appears at a discontinuity of the estimated wave length. This discontinuity problem at the connection value of h/L_0 is not solved and more attention should be paid to combining formula without such discontinuity.

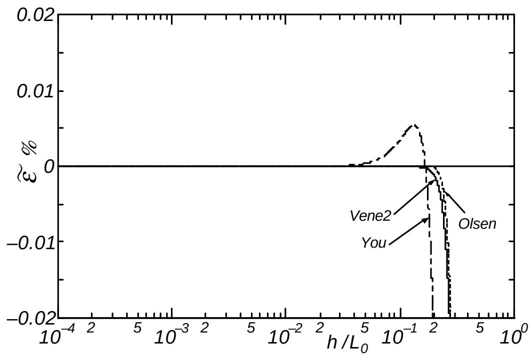

Fig. 7 Relation between relative error and h/L_0 (7).

Table 2 gives a summarized list of relative error range for each of the investigated approximate solutions.

4. Conclusions

The conclusions of the present study of 30 AESs to the dispersion relation, may be summarized as follows. Of the 21 AESs valid for all values of water depth,

① the YN9 equation, which is an iterative solution to the modified Carvlho 5th equation

$$k_a h = \left\{\alpha + \beta_a^2\left(1-\tanh^2\beta_a\right)\right\}/\left\{\tanh\beta_a + \beta_a\left(1-\tanh^2\beta_a\right)\right\}$$

$$\beta_a = \alpha\cdot\coth\left(m^\alpha\cdot\alpha^{1/2}\right)\ ;\ m=1.1965,\quad \alpha = 2\pi h/L_0$$

yields a very highly accurate estimate of wave length with an error less than 0.0001% and

② the Carvlho solution Eq.(27)

$$k_a h = \alpha\cdot\coth\left(1.2^\alpha\cdot\alpha^{1/2}\right),\quad \alpha = 2\pi h/L_0$$

is a simple equation for a quick estimate of wave length with an error from -0.21 % to 0.27 %.

The study of the remaining 9 AESs for either shallower water or deeper water, indicates the following results.

① Each of the solutions with complicated or lengthy terms like the Venezian 2nd solution (applicable only in shallower water) has generally a very high accuracy within its range of applicability but beyond this range, the accuracy deteriorates rapidly.

② It is not easy to find a consistent formula with high accuracy for all water depths by combining two of these AESs (for shallower and deeper water). The problem is the appearance of a discontinuity at the connection point of

h/L_0 (both in the wave length and the error). For practical applications, such a single solution valid for all depths would be preferable.

Table 2 Error range of the investigated approximate solutions.

No.	formula	relative error (%)	No.	formula	relative error (%)
1)	Eckart	0～5.24	11-6)	YN7	-1.2×10^{-3}～1.2×10^{-3}
2)	Iwagaki	−3.05～3.14	11-7)	YN8	-9×10^{-4}～8×10^{-4}
3)	Carv14	−2.45～3.28	11-8)	YN9	-1.1×10^{-4}～1.1×10^{-4}
4)	FM	−1.39～1.66	11-9)	YN10	-7×10^{-6}～4×10^{-5}
5)	YN1	−1.52～1.55	12)	Hunt1	-7.0×10^{-2}～7.8×10^{-2}
6)	Carv9	−1.12～0	13)	Hunt2	-8.2×10^{-3}～5.4×10^{-3}
7)	Guo	−0.75～0.75	14)	Niel1	±0.74 ($h/L_0^* \leqq 0.192$)
8)	YN2	−0.73～0.73	15)	Niel2	±0.44 ($h/L_0^* \leqq 0.401$)
9)	Carv5	−0.21～0.27	16)	Vene1	$\pm4.8\times10^{-2}$ ($h/L_0^* \leqq 0.165$)
10)	Carv4	−0.12～0.20	17)	WT1	-3.4×10^{-2}～0 ($h/L_0^* \leqq 0.219$)
11-1)	Fenton	-5.1×10^{-2}～8.4×10^{-3}	18)	Niel3	−0.55～0 ($h/L_0^* \geqq 0.300$)
11-2)	YN3	-4.0×10^{-2}～1.2×10^{-2}	19)	WT2	$\pm2.5\times10^{-2}$ ($h/L_0^* \geqq 0.195$)
11-3)	YN4	-2.9×10^{-2}～6.7×10^{-3}	20)	You	$\pm5.4\times10^{-3}$ ($h/L_0^* \leqq 0.179$)
11-4)	YN5	-4.9×10^{-3}～4.9×10^{-3}	21)	Olson	$\pm3\times10^{-5}$ ($h/L_0^* \leqq 0.186$)
11-5)	YN6	-4×10^{-4}～1.4×10^{-3}	22)	Vene2	-2×10^{-4}～6×10^{-6} ($h/L_0^* \leqq 0.159$)

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