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A Note on the KF_q –Cohomology of Lens Space $L_0^n(l^d)$

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The purpose of this paper is to determine the KF_q -cohomology of lens space $L_0^n(l^d)$ in a special case.

§ 1. Introduction

Let l be an odd prime number and $L^n(l^d)$ the standard $(2n+1)$-dimensional lens space $S^{2n+1}/(\boldsymbol{Z}/l^d)$. The lens space $L^n(l^d)$ has the CW-decomposition

$$L^n(l^d) = S^1 \cup e^2 \cup e^3 \cup \cdots \cup e^{2n} \cup e^{2n+1}$$

and we write $L_0^n(l^d)$ for its $2n$-skeleton.

Denote by Λ the ring of l-adic integers $\boldsymbol{Z}_l$. Let $K_\Lambda(-)$ be the l-adic completion of the classical complex K-theory, η the canonical complex line bundle of $L^n(l^d)$ and put $x = \eta - 1$. Then the K_Λ-cohomology of lens space is (see [3])

$$K_\Lambda\ (L^n(l^d)) \cong K_\Lambda\ (L_0^n(l^d)) \cong \Lambda[x]\ /\ ((1+x)^{l^d} - 1,\ x^{n+1}).$$

In [1], for $d = 1$ we showed that there is an element ξ of $\tilde{K}_\Lambda\ (L^n(l))$ such that

$$K_\Lambda\ (L^n(l)) \cong K_\Lambda\ (L_0^n(l)) \cong \Lambda[\xi]\ /\ (\xi^l + l\xi,\ \xi^{n+1}).$$

Let $\boldsymbol{F}_q$ be a finite field of order q and assume that q is prime to l. By $K\boldsymbol{F}_q\ (-)$ we denote the algebraic K-cohomology for $\boldsymbol{F}_q$.

The purpose of the paper is to show the following two theorems :

Theorem 1.1. *There is an element* ξ *of* $\tilde{K}_\Lambda\ (L^n(l^d))$ *and a monic polynomial* $f(X)$ *of degree* $(l^d - 1)\ /\ (l - 1)$ *such that*

(i) $K_\Lambda\ (L^n(l^d)) \cong \Lambda[\xi]\ /\ (f(\xi^{l-1})\xi,\ \xi^{n+1})$ *and*

(ii) *for an integer* k *whose order in* $(\boldsymbol{Z}/l^d)^\times$ *is prime to* l, $\psi^k(\xi) = c_k\xi$ $(c_k \in \Lambda)$, *where* ψ^k *is the Adams operation.*

Theorem 1.2. *Let F_q be a finite field of order q and assume the order of q in $(Z/l^d)^\times$ is prime to l. Let r be the least positive integer such that $q^r \equiv 1 \pmod{l}$. Then the algebraic K-cohomology group of lens space $\tilde{K}_{F_q}(L_0^n(l^d))$ is isomorphic to the torsion subgroup of $\Lambda[\zeta] / (f(\zeta^s)\zeta, \zeta^{m+1})$, where $\zeta = \xi^r$, $s = (l-1)/r$ and $m = [n/r]$.*

§ 2. Splitting of K_Λ-cohomology

Let $\rho \in \Lambda$ be a primitive $(l-1)$-th root of unity. Then for $1 \leqq i \leqq l-1$ we define the Λ-module homomorphism $\Phi_i : K_\Lambda(-) \to K_\Lambda(-)$ by

$$\Phi_i = \frac{1}{l-1} \sum_{m=1}^{l-1} \rho^{-mi} \psi^{\rho^m},$$

where ψ^k is the Adams operation.

Then we have (cf. [2])

Proposition 2.1.

(i) $\Phi_1 + \Phi_2 + \cdots + \Phi_{l-1} = id$,

(ii) $\Phi_i \Phi_j = \begin{cases} \Phi_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

Let k be an element of Λ such that $k \not\equiv 0 \pmod{l}$. Then there exists only one element m of Λ such that $m^{l-1} = 1$ and $m \equiv k \pmod{l}$. We write $\tilde{k}$ for the element m. Then we have

Proposition 2.2. *Let k be an integer whose order in $(Z/l^d)^\times$ is prime to l, then*

$$\psi^k \Phi_i = \Phi_i \psi^k = \tilde{k}^i \Phi_i$$

as Λ-module homomorphism $K_\Lambda(L^n(l^d)) \to K_\Lambda(L^n(l^d))$.

Proof. The commutativity of ψ^k and Φ_i is clear. Since the order of k in $(Z/l^d)^\times$ is prime to l, $\psi^k = \psi^{\tilde{k}}$. From the fact that $\tilde{k}$ is an $(l-1)$-th root of unity, there is an integer s such that $\tilde{k} = \rho^s$. Then

$$\psi^k \Phi_i = \frac{1}{l-1} \sum_{m=1}^{l-1} \rho^{-mi} \psi^{\tilde{k}\rho^m} = \frac{1}{l-1} \sum_{m=1}^{l-1} \rho^{-mi} \psi^{\rho^{m+s}}$$

$$= \frac{1}{l-1} \sum_{m=1}^{l-1} \rho^{si} \rho^{-(m+s)i} \psi^{\rho^{m+s}}$$

$$= \frac{1}{l-1} \rho^{si} \sum_{m=1}^{l-1} \rho^{-mi} \psi^{\rho^m} = \tilde{k}^i \Phi_i.$$

This completes the proof of the proposition.

§ 3. Structure of $K_\Lambda(L^n(l^d))$

In this section we write R for the ring $\Lambda[x] / ((1+x)^{l^d} - 1)$. The ring is a free Λ-module with a basis $\{1, x, x^2, \ldots, x^{l^d-1}\}$. The action of the Adams operation ψ^k to R is $\psi^k(x) = (1+x)^k - 1$.

Put $\xi = \Phi_1(x)$ and $m = l^d - 1$, then we have

Proposition 3.1.
(i) $\{1, \xi, \xi^2, \ldots, \xi^m\}$ *is a basis of the* Λ*-module* R *and*
(ii) *the ideal* (ξ) *of* R *is equal to* (x).

Proof. Since $\{1, x, x^2, \ldots, x^m\}$ is a basis of R, we can write

$$\xi^i = c_{i0} + c_{i1}x + c_{i2}x^2 + \cdots + c_{im}x^m,$$

where $0 \leqq i \leqq m$ and c_{ij} is an element of Λ. And $c_{i0} = c_{0i} = 0$ for $1 \leqq i \leqq m$. Since the mod l reduction of $(1 + x)^{l^d} - 1$ is x^{l^d} and

$$c_{11} \equiv \frac{1}{l-1} \sum_{m=1}^{l-1} \rho^{-m}\rho^m = 1 \pmod{l},$$

we have that $c_{ij} \equiv 0 \pmod{l}$ for $i > j$ and $c_{ii} \equiv 1 \pmod{l}$. Therfore $\det(c_{ij}) \equiv 1 \pmod{l}$, which implies Proposition 3.1.

Proposition 3.2. *For* $k \geqq 0$, *we have*
(i) $\Phi_i(\xi^k) = 0$ *if* $i \not\equiv k \pmod{l-1}$ *and*
(ii) $\Phi_i(\xi^k) = \xi^k$ *if* $i \equiv k \pmod{l-1}$.

Proof. To prove (i), assume $i \not\equiv k \pmod{l-1}$. By Proposition 2.2, $\psi^\rho\Phi_i(\xi^k) = \rho^i\Phi_i(\xi^k)$. On the other hand

$$\psi^\rho\Phi_i(\xi^k) = \Phi_i((\psi^\rho(\xi))^k) = \Phi_i((\psi^\rho\Phi_1(x))^k)$$
$$= \Phi_i((\rho\Phi_1(x))^k) = \rho^k\Phi_i(\xi^k).$$

Since $\rho^i - \rho^k$ is a unit of Λ, $(\rho^i - \rho^k)\,\Phi_i(\xi^k) = 0$ implies $\Phi_i(\xi^k) = 0$. To prove (ii), assume $i \equiv k \pmod{l-1}$. Applying Proposition 2.1 (i) to ξ^k, we have

$$\xi^k = \Phi_1(\xi^k) + \Phi_2(\xi^k) + \cdots + \Phi_{l-1}(\xi^k)$$
$$= \Phi_i(\xi^k).$$

This completes the proof of the proposition.

Proposition 3.3. *There is a monic polynomial* $f(X)$ *of degree* $(l^d - 1) / (l - 1)$ *such that* $R \cong \Lambda[\xi] / (f(\xi^{l-1})\xi)$.

Proof. By Proposition 3.1 there is a relation

$$\xi^{m+1} = a_0 + a_1\xi + a_2\xi^2 + \cdots + a_m\xi^m,$$

where a_i is an element of Λ for $0 \leqq i \leqq m$. From Proposition 3.2, we have

$$\xi^{m+1} = \Phi_1(\xi^{m+1})$$
$$= \Phi_1(a_0 + a_1\xi + a_2\xi^2 + \cdots + a_m\xi^m)$$
$$= a_1\xi + a_l\xi^l + a_{2l-1}\xi^{2l-1} + \cdots + a_{m-l+2}\xi^{m-l+2}.$$

Put

$$f(X) = -a_1 - a_lX - a_{2l-1}X^2 - \cdots - a_{m-l+2}X^{m/(l-1)-1} + X^{m/(l-1)},$$

then we have a relation $f(\xi^{l-1})\xi = 0$ of R. Clearly $\Lambda[\xi] / (f(\xi^{l-1})\xi)$ is a free Λ-module with a basis

$\{1, \xi, \xi^2, \ldots, \xi^m\}$. This implies Proposition 3.3.

Proof of Theorem 1.1. From Proposition 3.1 and Proposition 3.3, we have

$$K_\Lambda(L^n(l^d)) \cong \Lambda[x] / ((1+x)^{l^d} - 1, x^{n+1}) \cong R / (x^{n+1})$$
$$\cong R/(\xi^{n+1}) \cong \Lambda[\xi] / (f(\xi^{l-1})\xi, \xi^{n+1}).$$

Let r be a positive integer such that r divides $l-1$, and put $s = (l-1)/r$. We write R' for the ring $\Lambda[\zeta] / (f(\zeta^s)\zeta)$. Then there is the ring homomorphism $\iota : R' \to R$ such that $\iota(\zeta) = \xi^r$ and $\iota|_\Lambda = id_\Lambda$. Then we have the following proposition :

Proposition 3.4.

(i) ι *is a monomorphism,*

(ii) $\mathrm{Im}(\iota : R' \to R) = \mathrm{Im}(\Phi_r + \Phi_{2r} + \cdots + \Phi_{l-1} : R \to R)$,

(iii) *let* $\pi : R \to R / (\xi^{n+1})$ *be the canonical projection, then* $\mathrm{Ker}(\pi\circ\iota) = (\zeta^{[n/r]+1})$, *and*

(iv) $R' / (\zeta^{[n/r]+1}) \cong \mathrm{Im}(\Phi_r + \Phi_{2r} + \cdots + \Phi_{l-1} : R / (\xi^{n+1}) \to R / (\xi^{n+1}))$.

Proof. Since R' is a free Λ-module with a basis $\{1, \zeta, \zeta^2, \ldots, \zeta^{m/r}\}$ and $\mathrm{Im}(\Phi_r + \Phi_{2r} + \cdots + \Phi_{l-1})$ is a free Λ-module with a basis $\{1, \xi^r, \xi^{2r}, \ldots, \xi^m\}$, (i) and (ii) is clear. To prove (iii), assume that α is an element of $\mathrm{Ker}(\pi\circ\iota)$. Then we can write

$$\iota(\alpha) = \xi^{n+1}(a_0 + a_1\xi + a_2\xi^2 + \cdots + a_m\xi^m)$$
$$= a_0\xi^{n+1} + a_1\xi^{n+2} + a_2\xi^{n+3} + \cdots + a_m\xi^{n+m+1},$$

where $a_i \in \Lambda$ for $0 \leqq i \leqq m$. By Proposition 3.2 we have

$$\iota(\alpha) = (\Phi_r + \Phi_{2r} + \cdots + \Phi_{l-1})(\alpha)$$
$$= \sum_{\substack{0 \leqq i \leqq m \\ n+i+1 \equiv 0 \ (\mathrm{mod}\ r)}} a_i\xi^{n+i+1}$$
$$= \sum_{\substack{0 \leqq i \leqq m \\ n+i+1 \equiv 0 \ (\mathrm{mod}\ r)}} a_i\iota(\zeta^{(n+i+1)/r}).$$

Therefore α is an element of the ideal $(\zeta^{[n/r]+1})$. It is clear that $\iota(\zeta^{[n/r]+1})=0$, so we have $\mathrm{Ker}(\pi\circ\iota) = (\zeta^{[n/r]+1})$. From (i), (ii) and (iii), (iv) follows. This completes the proof of Proposition 3.4.

§ 4. $K\boldsymbol{F}_q$–cohomology of lens space $L_0^n(l^d)$

Let $\boldsymbol{F}_q$ be a finite field of order q and assume that the order of q in $(\boldsymbol{Z}/l^d)^\times$ is prime to l.

Put $M_i = \Phi_i(\tilde{K}_\Lambda(L_0^n(l^d)))$. Then from Proposition 2.1, we have

$$K_\Lambda(L_0^n(l^d)) \cong \Lambda \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_{l-1}.$$

Since $K^{-1}(L_0^n(l^d)) = 0$, there is an exact sequence (cf. [4])

$$0 \to \tilde{K}\boldsymbol{F}_q(L_0^n(l^d)) \to \tilde{K}_\Lambda(L_0^n(l^d)) \xrightarrow{1-\psi^q} \tilde{K}_\Lambda(L_0^n(l^d)).$$

From Proposition 2.2 and the fact that $1-\psi^q$ commutes with Φ_i we have the following proposition :

Proposition 4.1.

(i) *The* $K_{\boldsymbol{F}_q}$*-cohomology of lens space* $L_0^n(l^d)$ *is*

$$K_{\boldsymbol{F}_q}(L_0^n(l^d)) \cong \bigoplus_{i=1}^{l-1} \mathrm{Ker}(1 - \psi^q : M_i \to M_i), \text{ and}$$

(ii) $(1 - \psi^q)(y) = (1 - \tilde{q}^i)y$ *for* $y \in M_i$.

Then the following lemma is well known :

Lemma 4.2. *Let* r *be the least positive integer such that* $q^r \equiv 1 \pmod{l}$. *Then*

(i) $1 - \tilde{q}^i$ *is a unit of* Λ *if* $i \not\equiv 0 \pmod{r}$ *and*

(ii) $1 - \tilde{q}^i = 0$ *if* $i \equiv 0 \pmod{r}$.

Proof of Theorem 1.2. From Proposition 4.1 and Lemma 4.2,

$$\Lambda \oplus \tilde{K}_{\boldsymbol{F}_q}(L_0^n(l^d)) \cong \Lambda \oplus \bigoplus_{i=1}^{(l-1)/r} M_{ir}.$$

Since this Λ-submodule of $K_\Lambda(L_0^n(l^d))$ is generated by 1, ξ^r, ξ^{2r}, ξ^{3r} ,..., we have

$$\Lambda \oplus \tilde{K}_{\boldsymbol{F}_q}(L_0^n(l^d)) \cong \mathrm{Im}(\Phi_r + \Phi_{2r} + \cdots + \Phi_{l-1}).$$

From Proposition 3.4

$$\Lambda \oplus \tilde{K}_{\boldsymbol{F}_q}(L_0^n(l^d)) \cong R' \ / \ (\zeta^{[n/r]+1}),$$

which implies Theorem 1.2.

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