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On the Splitting of K–Cohomology of Lens Space $L^n(p^d)$

Dedicated to Professor Hirosi Toda on his 60th birthday

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The purpose of the paper is to define a ring G and to determine the K–cohomology of lens space as a G–module.

§ 1. Introduction

Let p be an odd prime number and $L^n(p^d)$ the standard $(2n+1)$–dimensional lens space $S^{2n+1}/(\mathbf{Z}/p^d)$.

Denote by Λ the ring of p–adic integers Z_p. Let $ku^*_\Lambda(-)$ be the p–adic completion of (-1)–connected complex K–theory. In [1], we showed that there is an element ξ of $\tilde{ku}^0_\Lambda(L^n(p^d))$ and a monic polynomial $f_d(X)$ of degree $(p^d-1)/(p-1)$ such that

$$ku^0_\Lambda(L^n(p^d)) = \Lambda[\xi]/(\xi f_d(\xi^{p-1}), \xi^{n+1}).$$

Let G be the ring $\Lambda[t]/(f_d(t))$, where the polynomial $f_d(X)$ is as above. The purpose of the paper is to determine $\tilde{ku}^0_\Lambda(L^n(p^d))$ as a G–module and to state some results about the polynomial $f_d(X)$.

The main theorems are Theorem 2.7, Theorem 3.1 and Theorem 3.2.

§ 2. Splitting of $ku^*_\Lambda(L^n(p^d))$

Let $ku^*(-)$ be the (-1)–connected complex K–theory, η the canonical complex line bundle of $L^n(p^d)$, and put $x = \eta - 1$.

Then the ku^0-cohomology of lens spaces is as follows (see [2,3]):

Theorem 2.1. *We have*

$$ku^0\,(L^n(p^d)) \;\cong\; \boldsymbol{Z}\,[x]\,/\,((1+x)^{p^d}-1,\ x^{n+1}).$$

Denote by Λ the ring of p-adic integers $\boldsymbol{Z}_p$. Let $ku^*_\Lambda\,(-)$ be the p-adic completion of $ku^*\,(-)$. Let $\rho \in \Lambda$ be a primitive $(p-1)$-th root of unity, and put $\xi = \sum_{m=1}^{p-1} \rho^{-m}\psi^{\rho^m}(x)$, where ψ^k is the Adams operation.

Then we have

Theorem 2.2. *There is a monic polynomial $f_d\,(X) \in \Lambda\,[X]$ of degree $(p^d-1)\,/\,(p-1)$ such that*

$$ku^0_\Lambda\,(L^n(p^d)) \;\cong\; \Lambda\,[\xi]\,/\,(\xi f_d\,(\xi^{p-1}),\ \xi^{n+1}).$$

Proof. Put $\bar{\xi} = \dfrac{1}{p-1}\xi$. From [1], there is a monic polynomial $\bar{f}_d\,(X)$ of degree $(p^d-1)\,/\,(p-1)$ such that

$$ku^0_\Lambda\,(L^n(p^d)) \;\cong\; \Lambda\,[\bar{\xi}]\,/\,(\bar{\xi}\bar{f}_d\,(\bar{\xi}^{p-1}),\ \bar{\xi}^{n+1}).$$

Since $p-1$ is a unit element of Λ, if we put

$$f_d\,(X) \;=\; (p-1)^{p^d-1}\bar{f}_d\,((p-1)^{-(p-1)}X),$$

then we have

$$ku^0_\Lambda\,(L^n(p^d)) \;\cong\; \Lambda\,[\xi]\,/\,(\xi f_d\,(\xi^{p-1}),\ \xi^{n+1}).$$

For the polynomial $f_d\,(X)$ we have the following lemma :

Lemma 2.3. *The constant term of $f_d\,(X)$ is $p^d u$, where u is a unit element of Λ.*

Proof. Let c be the constant term of $f_d\,(X)$. Then,

$$ku^0_\Lambda\,(L^1(p^d)) \;\cong\; \Lambda\,[\xi]\,/\,(\xi f_d\,(\xi^{p-1}),\ \xi^2) \;\cong\; \Lambda\,[\xi]\,/\,(c\xi,\ \xi^2)$$
$$\cong\; \Lambda \oplus \Lambda\,/\,(c).$$

On the other hand,

$$ku^0_\Lambda\,(L^1(p^d)) \;\cong\; \Lambda\,[x]\,/\,((1+x)^{p^d}-1,\ x^2) \;\cong\; \Lambda\,[x]\,/\,(p^d x,\ x^2)$$
$$\cong\; \Lambda \oplus \Lambda\,/\,(p^d),$$

which implies Lemma 2.3.

Let G be the ring $\Lambda\,[t]\,/\,(f_d\,(t))$. Then, $\widetilde{ku}^0_\Lambda\,(L^n(p^d))$ is a G-module by $g\,(t)\cdot a = g\,(\xi^{p-1})a$, where $g\,(t) \in G$ and $a \in \widetilde{ku}^0_\Lambda\,(L^n(p^d))$.

Let G_n be the G-module $G/(t^n)$, and put $a(i,n) = [\dfrac{n-i}{p-1}]+1$.

Since $\tilde{ku}_{\Lambda}^{0}\ (L^n(p^d))$ is a Λ–module generated by ξ, ξ^2, . . . , and ξ^{p^d-1}, $\tilde{ku}_{\Lambda}^{0}\ (L^n(p^d))$ is a G–module generated by ξ, ξ^2, . . . , and ξ^{p-1}. Then we have easily the following theorem :

Theorem 2.4. *As a G–module, there is an isomorphism*

$$\tilde{ku}_{\Lambda}^{0}\ (L^n(p^d)) \cong \bigoplus_{i=1}^{p-1} G_{a\,(i,n)}\,\xi^i.$$

To determine $\tilde{ku}_{\Lambda}^{2k}\ (L^n(p^d))$, we define the filtration $F^k\tilde{ku}_{\Lambda}^{0}\ (L^n(p^d))$ of $\tilde{ku}_{\Lambda}^{0}\ (L^n(p^d))$ as follows :

Definition 2.5. Let $L^k\ (p^d) \longrightarrow L^n\ (p^d)$ $(0 \leqq k \leqq n)$ be the canonical inclusion, then we put

$$F^k\tilde{ku}_{\Lambda}^{0}\ (L^n(p^d)) = \begin{cases} \tilde{ku}_{\Lambda}^{0}\ (L^n(p^d)) & \text{if } k \leqq 0, \\ \operatorname{Ker}\ (\tilde{ku}_{\Lambda}^{0}\ (L^n(p^d)) \longrightarrow \tilde{ku}_{\Lambda}^{0}\ (L^k(p^d))) & \text{if } 0 < k < n, \\ 0 & \text{if } k \geqq n. \end{cases}$$

Since Atiyah–Hirzebruch spectral sequence of $ku^*\ (L^n(p^d))$ collapses, we have :

Proposition 2.6. *The ku_{Λ}–cohomology of lens space is*

$$\tilde{ku}_{\Lambda}^{2k}\ (L^n(p^d)) \cong F^k\tilde{ku}_{\Lambda}^{0}\ (L^n(p^d)).$$

Theorem 2.7. *The ku_{Λ}–cohomology of lens space is*

$$\tilde{ku}_{\Lambda}^{2k}\ (L^n(p^d)) \cong \begin{cases} \bigoplus_{i=1}^{p-1} G_{a\,(i,n)}\xi^i & \text{if } k \leqq 0, \\ \bigoplus_{i=1}^{p-1} G_{a(i,n)-a(i,k)}\xi^{i+(p-1)a(i,k)} & \text{if } 0 < k < n, \\ 0 & \text{if } k \geqq n. \end{cases}$$

Proof. It is clear for $k \leqq 0$ and for $k \geqq n$. For $0 < k < n$, we have

$$\begin{aligned} \tilde{ku}_{\Lambda}^{2k}\ (L^n(p^d)) &\cong \bigoplus_{i=1}^{p-1} \operatorname{Ker}\ (G_{a\,(i,n)} \longrightarrow G_{a\,(i,k)}) \\ &\cong \bigoplus_{i=1}^{p-1} t^{a\,(i,k)}\, G_{a\,(i,n)}. \end{aligned}$$

By Lemma 2.3, we have G–module isomorphism $t^lG_m \cong G_{m-l}$. This completes the proof of the theorem.

§ 3. Polynomial $f_d(X)$

Denote by $g_d(X)$ the polynomial $Xf_d(X^{p-1})$. In this section we state some results about $g_d(X)$.

Let R be the ring $\Lambda[x]/((1+x)^{p^d}-1) \cong \Lambda[\eta]/(\eta^{p^d}-1)$. First we recall that to prove $g_d(X) = X^{p^d}+a_{p^d-1}X^{p^d-1}+ \cdots +a_1X+a_0$ we need to find a relation $g_d(\xi) = 0$ in R.

For $p = 3$, we have

Theorem 3.1. *If $p = 3$, then we have $g_1(X) = X^3+3X$ and $g_d(X) = g_{d-1}(X)^3+3g_{d-1}(X)$ for $d>1$.*

Proof. Since $\rho = -1$, $\xi = \eta-\eta^{-1}$. Then, we have

$$\begin{aligned}
g_1(\xi) &= \xi^3+3\xi = (\eta-\eta^{-1})^3+3(\eta-\eta^{-1}) \\
&= \eta^3-\eta^{-3}. \\
g_2(\xi) &= g_1(\xi)^3+3g_1(\xi) = (\eta^3-\eta^{-3})^3+3(\eta^3-\eta^{-3}) \\
&= \eta^{3^2}-\eta^{-3^2}. \\
&\cdots \\
g_d(\xi) &= g_{d-1}(\xi)^3+3g_{d-1}(\xi) \\
&= (\eta^{3^{d-1}}-\eta^{-3^{d-1}})^3+3(\eta^{3^{d-1}}-\eta^{-3^{d-1}}) \\
&= \eta^{3^d}-\eta^{-3^d}=0.
\end{aligned}$$

This completes the proof of the theorem.

Denote by $M(x_0, x_1, \ldots, x_{N-1})$ the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \\ x_{N-1} & x_0 & \cdots & x_{N-2} \\ & & \cdots & \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix}.$$

Then we have

Theorem 3.2. *If we write $\xi = c_0+c_1\eta+ \cdots +c_{p^d-1}\eta^{p^d-1}$ $(c_i \in \Lambda)$, Then we have*
$$g_d(X) = \det M(X-c_0, -c_1, -c_2, \ldots, -c_{p^d-1}).$$

Proof. Put $m=p^d-1$. Since R is a free Λ-module with a basis $\{\eta^m, \eta^{m-1}, \ldots, 1\}$, denote an element $a_0\eta^m+a_1\eta^{m-1}+ \cdots +a_m$ of R by $(a_0, a_1, \ldots, a_m)$. Let A be the matrix $M(c_0, c_1, \ldots, c_m)$, then we have $\xi^i = A^ie$ where $e = (0, 0, \ldots, 0, 1)$. Since $g_d(X)$ is the characteristic polynomial of A, we have $g_d(A) = 0$. Therefore $g_d(\xi) = g_d(A)e = 0$.

We have some calculations of $g_d(X)$:

For $p=5$ and $\rho\equiv 2 \pmod{p}$, we have

$g_1(X)=X^5+(15+20\rho)X$, and

$g_2(X)=g_1(X)(X^{20}+(60-20\rho)X^{16}+(200-1000\rho)X^{12}$
$+(-12625-18500\rho)X^8+(-73750+95000\rho)X^4$
$+(61815+247420\rho))$.

For $p=7$ and $\rho\equiv 3 \pmod{p}$, we have

$g_1(X)=X^7+(385-273\rho)X$, and

$g_2(X)=g_1(X)(X^{42}+(105+273\rho)X^{36}+(2063537-3426717\rho)X^{30}$
$+(1343153896-318232215\rho)X^{24}$
$+(-171515355277+231015496383\rho)X^{18}$
$+(-15283832013744+12511015521816\rho)X^{12}$
$+(2217175867312-4115711963349\rho)X^6$
$+(-71626902095+68439131943\rho))$.

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