工学ジャーナル　第13巻
2014年3月

Comparative Study of Explicit Solutions to Wave Dispersion Relationship (2)

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Yamaguchi and Nonaka[1] clarified relative accuracy of several approximate and explicit solutions(AESs) to the transcendental dispersion equation in the Airy wave theory with small amplitude, in cases where a numerically exact solution is obtained by Newton's iterative method. This paper reinforces the previous results with new investigations for accuracy of 4 recently-proposed AESs[2],[3] (Beji-1, Beji-2, Vata-1, Vata-2) and some of AESs[4]~[8] (from Cham-2 to Cham-7 etc.) non-discussed in Yamaguchi and Nonaka[1], and their Newton method-based solutions. The main findings are as follows: (1) The 4 AESs show higher accuracy than the previous AESs, in cases where use of Beji-2 or Vata-1 is recommended from the view point of a balance between accuracy and compactness (computational efficiency) of the equation; (2) The 4 AESs(Cham-4, Cham-5, Cham-6, Cham-7) of the 7 AESs proposed by Chamberlain and Porter[5] have small maximum relative errors ranging from 0.16 % to 0.0035 %, which show increasing accuracy with increased sophistication of the formula; (3) The Padé approximation-based Hunt[4] AES with the 9th order gives a high accuracy but less accuracy compared to Vata-2 or Cham-7; (4) The 1st iteration solution by Newton's method for an initial value based on each of the AESs provides much higher accuracy than the original AES, in cases where any of the Vata-2, Hunt-9 and Cham-7-based solutions corresponds to a numerically exact solution to the best degree; (5) Each of the You[6]~[8]-type Piecewise AESs(PAESs) applicable only to a shallow water area indicates reasonable accuracy within its effective range, which means the usefulness within a limited condition in shallow water from a point of compactness of the expression; (6) A modified version for the Combined Piecewise AES(CPAES) proposed by Newman[9] which is applicable to a full range of water depth condition has the highest accuracy among not only various CPAESs but also the investigated AESs.

Key Words；dispersion relationship, Airy wave theory, recently-proposed/previously overlooked AESs, Newton method-based AESs and exact solutions

1.　Introduction

The dispersion relationship in shallow water based on the small amplitude wave theory constitutes a transcendental equation with respect to wave length and this property makes it impossible to derive the analytical solution. For this reason, many kinds of approximate and explicit solution(AES) have been proposed and their error characteristics have been investigated through the comparison with a numerically exact solution. In 2007, Yamaguchi and Nonaka[1] classified most of previously-proposed 30 AESs including the authors-modified versions and made clear an error range for each of the AESs. But concerted efforts for developing new AESs are continuing.

By taking this situation into account, this paper enphances the previous results with new investigations for

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原稿受理 平成25年10月5日

accuracy of 4 recently-proposed AESs[2],[3] and some old AESs[4]~[8] non-discussed in Yamaguchi and Nonaka[1], and their Newton method-based solutions. The AESs investigated here are ①2 expressions by Beji[2](Beji-1, Beji-2), ②2 expressions by Vatankhah and Aghashariatmadari[3](Vata-1, Vata-2), ③4 expressions with several orders of relative water to depth-deep water wave length parameter α by Hunt[4](Hunt-4, Hunt-5, Hunt-6, Hunt-9), ④6 expressions by Chamberlain and Porter[5](Cham-2, Cham-3, Cham-4, Cham-5, Cham-6, Cham-7), ⑤a set of the 1st iteration solution(2-step AES) by Newton's method associated with an initial value calculated using each of the above-mentioned AESs. Also the You[6]~[8]-type PAESs applicable only to shallow water area and the PAESs by Chamberlain and Porter[5] and Newman[9] application-limited to either of shallow water area or deeper water area are added to the candidates to be investigated. Then characteristics of 2 Combined PAESs(CPAESs) consisting of the 2 types of PAES mentioned above are discussed.

2. Dispersion relationship and numerically exact solution

The dispersion relationship based on the small amplitude wave theory(the Airy wave theory) on the constant water depth is expressed as

$$\alpha = \beta \cdot \tanh\beta \ , \quad \alpha = k_0 h \ , \quad \beta = kh \tag{1}$$

where h is the water depth, $k_0 = 2\pi/L_0$ the wave number in deep water, L_0 the wave length in deep water, $k = 2\pi/L$ the wave number in shallow water and L the wave length in shallow water. Eq.(1) is a typical transcendental equation with the unknown variable of β. Numerical computation of Eq.(1) is made using Newton's method and a numerical solution with the relative error less than 10^{-10} obtained through iterative computations is regarded as a numerically exact solution of Eq.(1) in this study. An initial value in the computation β_a is due to either of the following expressions.

$$\left.\begin{array}{l} \alpha \ge 1 \ \text{ or } \ h/L_0 \ge 1/2\pi \ : \ \beta_a = \alpha = 2\pi(h/L_0) \\ \alpha < 1 \ \text{ or } \ h/L_0 < 1/2\pi \ : \ \beta_a = \alpha^{1/2} = \{2\pi(h/L_0)\}^{1/2} \end{array}\right\} \tag{2}$$

The number of iterations reaching a numerically exact solution is only from 2 to 4, because Newton's method has a property of quadratic convergence.

3. Approximate and explicit solutions(AESs) for computation of wave length and their accuracy

3.1 Classification of AESs[1]

AESs published so far may be classified into 2 main groups, I : AESs applicable to a full range of relative water depth h/L_0 of 0 to ∞ and II : AESs valid for a limited range of h/L_0. Each group may be sub-classified into ① AESs with simple form but lower accuracy, ②AESs with complicated or lengthy form but higher accuracy and ③2-step AESs with high accuracy. AESs of group II-① may be separated into (i)shallower water use and (ii)deeper water use.

Table 1 re-summarizes the above-mentioned description which has a slightly different form from Table 1 in Yamaguchi and Nonaka[1]. AESs classified into I-① are the 1st and 2nd equations by Beji[2](Beji-1, Beji-2), the 1st equation by Vatankhah and Aghashariatmadari[3](Vata-1), the Padé approximation-based Hunt 4th order and 5th order solutions(Hunt-4, Hunt-5), and the 1st 4 equations by Chamberlain and Porter[5](Cham-2, Cham-3, Cham-4, Cham-5). Then AESs classified into I-② are the 2nd equation by Vatankhah and Aghashariatmadari[3](Vata-2), the Hunt 6th order and 9th order solutions (Hunt-6, Hunt-9) and the last 2

Table 1 Grouping of AESs.

Group classification	Simple, low or moderate accuracy①		Complicated or lengthy, high accuracy②		2-step AES high accuracy③
Full range of water depth I	Beji-1, Beji-2, Vata-1,Hunt-4, Hunt-5, Cham-2, Cham-3, Cham-4, Cham-5		Vata-2, Hunt-6, Hunt-9 Cham-6, Cham-7		Beji-1N ~ Cham-7N
Limited range of water depth II	Shallower water(i)	Deeper water(ii)	Shallower water(i)	Deeper water(ii)	
	You-1, You-4, Cham-6L, Cham-7L, CP-1	CP-2, New-2S	New-1	New-2, New-3	

equations by Chamberlain and Porter[5](Cham-6, Cham-7). A set of AESs classified into I-③ is the 1st iteration solution (2-step AES) by Newton's method taking each of the AESs as an initial value, in cases where ' N ' is added to each of the notations such as Beji-1N.

Moreover AESs classified into II-①(i) are the n-th order($n \leq 3$) equations given by You[6]~[8](You-1, You-2/WT1, You-3/Niel2, You-4, You-5/You), the modified versions(Cham-6L/Olson3, Cham-7L/YNH) and the Chamberlain and Porter[5] equation(CP-1) applicable only to a shallow water condition. An AES sub-grouped into II-②(i)(shallower water) is the Newman[9] 8th order equation(New-1) and an AES sub-grouped into II-①(ii)(deeper water) is the Chamberlain and Porter[5] equation(CP-2). Either the Newman[9] 5th order solution(New-2) or 4th order solution(New-3) belongs to II-②(ii)(deeper water). A simplified New-2 solution with the 2nd order(New-2S) is classified into II-①(ii)(deeper water). The other AESs grouped in category II)-① or II-② were discussed in Yamaguchi and Nonaka[1].

Numerical computations are conducted for a range of $h/L_0 = 10^{-5}$ - 1 with an increment of $\Delta(h/L_0) = 10^{-5}$. The error of each AES relative to the exact solution ε is defined as :

$$\varepsilon = \left(L_{app}/L_{exac} - 1\right) \times 100\ \% \tag{3}$$

where the subscript '*app*' means an approximated wave length and the subscript '*exac*' means the exact wave length computed with an relative error less than 10^{-10} using Newton's method.

3.2 AESs applicable to a full range of relative water depth h/L_0

Each of the 2 AESs proposed by Beji[2], its classification, its abbreviated notation and the maximum relative error ε_{max} indicated in his paper are written as :

$$\beta = \alpha\left[1 + \alpha^{1.3}\exp\{-(1.1 + 2.0\alpha)\}\right]/(\tanh\alpha)^{1/2}\ ,\ \ [\text{ I -①, Beji-1, } \varepsilon_{max} = 0.187\%] \tag{4}$$

$$\beta = \alpha\left[1 + \alpha^{1.09}\exp\left\{-\left(1.55 + 1.30\alpha + 0.216\alpha^2\right)\right\}\right]/(\tanh\alpha)^{1/2}\ ,\ \ [\text{ I -①, Beji-2, } \varepsilon_{max} = -0.044\%] \tag{5}$$

Also, those by Vatankhah and Aghashariatmadari[3] are similarly expressed as

$$\beta = \alpha\left[1 + \alpha\cdot\exp\left\{-\left(1.835 + 1.225\alpha^{1.35}\right)\right\}\right]/(\tanh\alpha)^{1/2}\ ,\ \ [\text{ I -①, Vata-1, } \varepsilon_{max} = 0.019\%] \tag{6}$$

$$\beta = \frac{\alpha\left[1+\alpha\cdot\exp\left\{-\left(3.2+\alpha^{1.65}\right)\right\}\right]}{(\tanh\alpha)^{1/2}} + \alpha\left\{1-\exp\left(-\alpha^{0.132}\right)\right\}^{\left(5.0532+2.158\alpha^{1.505}\right)},\quad [\,\text{I -②},\ \text{Vata-2},\ \varepsilon_{max}=0.001\%] \tag{7}$$

Both Vata-1 and Vata-2 may correspond to improved versions of Beji-1 and Beji-2 respectively. But Vata-2 has a rather sophisticated form compared to either Beji-1 or Beji-2.

Fig. 1 shows the relation between relative error ε and relative water depth h/L_0 for not only Eqs. Beji-1, Beji-2, Vata-1 and Vata-2 but also the 5th equation by Carvlho (Carv5) given in Yamaguchi and Nonaka[1]. The characteristics of Carv5 is described as

$$\beta = \alpha/\tanh\left(1.2^{\alpha}\alpha^{1/2}\right),\quad [\,\text{I -①},\ \text{Carv5},\ \varepsilon_{max}=0.27\%] \tag{8}$$

Relative error ε in any of the AESs yields a positive or negative behavior with respect to change of h/L_0, which finally approaches zero with either infinitesimal decrease or infinite increase of h/L_0.

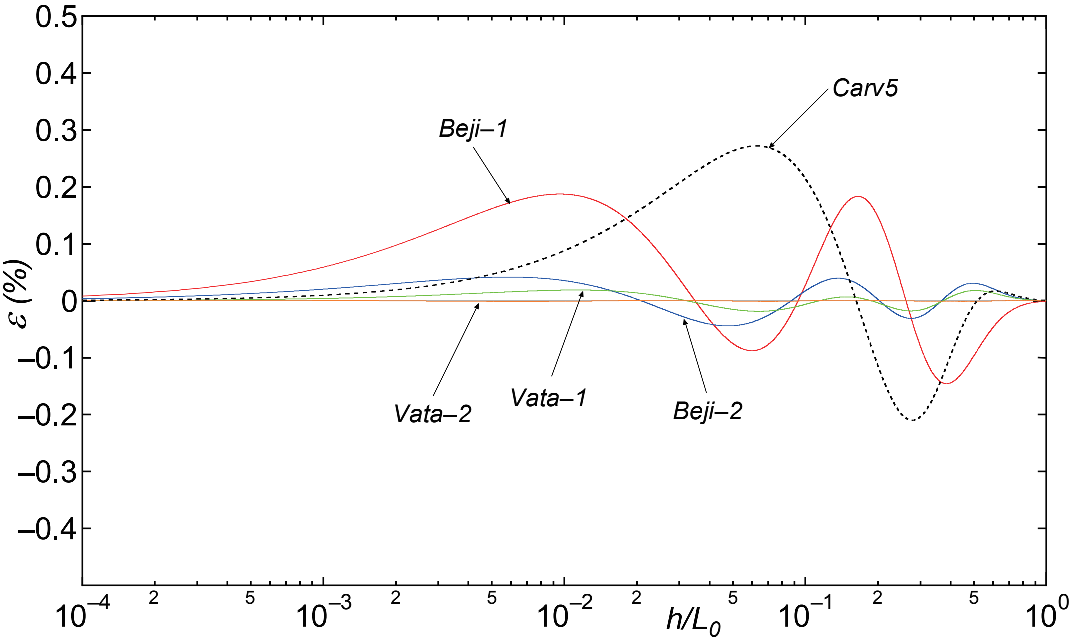

Fig. 1 Relation between relative error ε and relative water depth h/L_0 for any of Beji-1, Beji-2, Vata-1, Vata-2 and Carv5.

A range of the negative maximum and positive maximum relative error ε_{max} and its corresponding h/L_0 for any of Beji-1, Beji-2, Vata-1, Vata-2 and Carv5 are collectively written as :

23) Beji-1 : -0.15% (h/L_0 =0.385) ～0.19% (h/L_0 =0.010) (9)

24) Beji-2 : -0.044% (h/L_0 =0.048) ～0.042% (h/L_0 =0.006) (10)

25) Vata-1 : -0.019% (h/L_0 =0.064) ～0.019% (h/L_0 =0.011) (11)

26) Vata-2 : -0.0012% (h/L_0 =0.006) ～0.0012% (h/L_0 =0.149) (12)

9) Carv5 : -0.21% (h/L_0 =0.278) ～0.27% (h/L_0 =0.063) (13)

The leading number such as 23) in Eq.(9) indicates the number connecting to the number provided in Table 2 by Yamaguchi and Nonaka[1] and the same number such as 9) is given for the same case. Positive or negative maximum value of ε in Eq.(9) to Eq.(12) coincides with the one given in either of Beji[2] or Vatankhah and Aghashariatmadari[3]. In short, the maximum relative error ε_{max} is 0.2 % for Beji-1, (-)0.04 % for Beji-2,

± 0.02 % for Vata-1 and ± 0.001 % for Vata-2 respectively, as shown in each paper. Comparing with Carv5 which yields the highest accuracy in the previous single expression-based AESs grouped in I-①, Beji-1 gives a comparable accuracy and any of Beji-2, Vata-1 and Vata-2 produces a higher accuracy in this order. It should be emphasized that Vata-2 consisting of 2 terms has too complicated a form. In addition, the reason why Beji-1 as well as the below-mentioned Cham-3 leaves a significant relative error greater than the other AESs even in a very small h/L_0 case such as $h/L_0 = 10^{-4}$ is analytically and numerically investigated in the Appendix.

Next, the Padé-approximation-based Hunt[4] equation with the 6th order of relative water depth of $\alpha(=2\pi h/L_0)$ is given as :

$$\beta = \alpha^{1/2}\left\{\alpha + \left(1 + D_1\alpha + D_2\alpha^2 + D_3\alpha^3 + D_4\alpha^4 + D_5\alpha^5 + D_6\alpha^6\right)^{-1}\right\}^{1/2}, \quad [\text{ I -②}, \quad \text{Hunt-6}] \tag{14}$$

$$D_1 = 2/3 \approx 0.6666666666\,, \quad D_2 = 16/45 \approx 0.3555555555\,,$$
$$D_3 = 152/945 \approx 0.1608465608\,, \quad D_4 = 896/14175 \approx 0.0632098765\,,$$
$$D_5 = 3392/155925 \approx 0.0217540484\,, \quad D_6 = 9792/1497065 \approx 0.0065407982 \tag{15}$$

The fraction-used expressions for the coefficients D_5 and D_6 in Eq.(14) are made in this study due to their lack in the Hunt[4] paper. The numeric figure of D_5 with 10 digits are in perfect agreement with the one by Hunt[4] and then it should be noted that the figure for D_6 is 10^{-10}, which differs from the one by Hunt[4] ($D_6 = 0.0065407983$). Although the cause is not clear, the effect of the difference on relative error is substantially zero. In addition, the Hunt[4] 4th order solution, its classification, the abbreviated notation and the maximum relative error ε_{max} are as follows :

$$\beta = \alpha^{1/2}\left\{\alpha + \left(1 + 0.666\alpha + 0.445\alpha^2 - 0.105\alpha^3 + 0.272\alpha^4\right)^{-1}\right\}^{1/2}, \quad [\text{ I -①}, \quad \text{Hunt-4}, \quad \varepsilon_{max} = 0.2\%] \tag{16}$$

Fig. 2 indicates the relation between ε and h/L_0 for not only Hunt-4 and Hunt-6 but also Hunt-5 and Hunt-9 investigated in Yamaguchi and Nonaka[1]. While Hunt-6 has a single negative peak of ε with h/L_0, the other 3 AESs yield 2 positive and negative peaks. Hunt-6 shows a different behavior of ε from any of the other AESs.

Similar to the former cases, a range of the negative maximum and positive maximum relative error and its

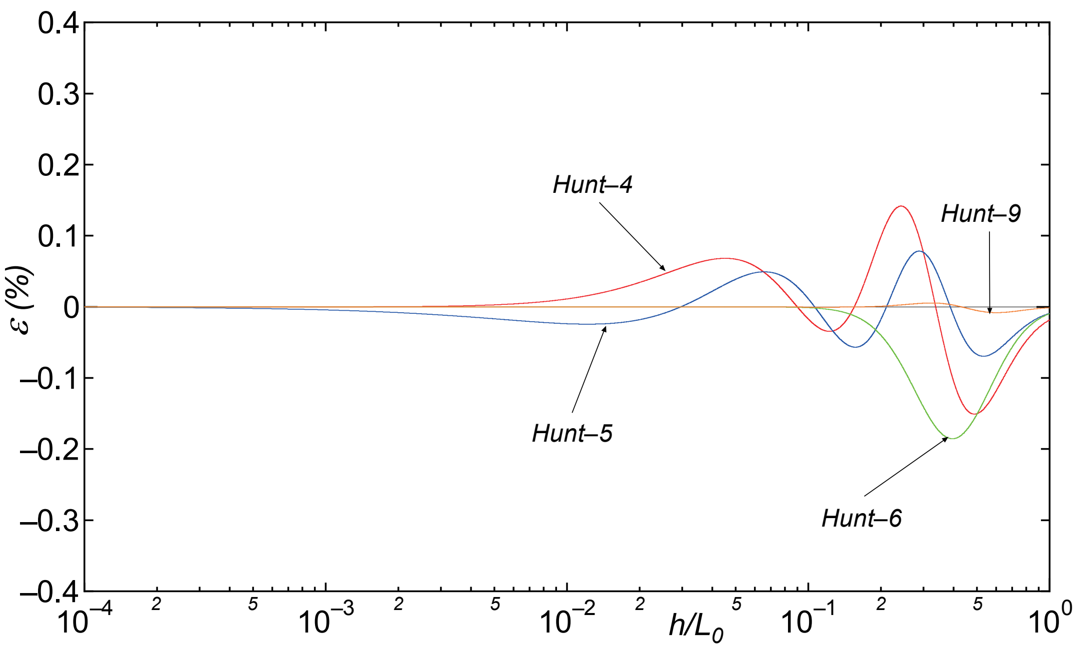

Fig. 2 Relation between relative error ε and relative water depth h/L_0 for any of Hunt-4, Hunt-5, Hunt-6 and Hunt-9.

corresponding h/L_0 for any of Hunt-4, Hunt-5, Hunt-6 and Hunt-9 are collected as

27) Hunt-4 : -0.15%（h/L_0 =0.488）～0.14%（h/L_0 =0.242） (17)

12) Hunt-5 : -0.070%（h/L_0 =0.532）～0.078%（h/L_0 =0.288） (18)

28) Hunt-6 : -0.19%（h/L_0 =0.395）～0.0%（$h/L_0 \to 0, \ \infty$） (19)

13) Hunt-9 : -0.0082%（h/L_0 =0.603）～0.0054%（h/L_0 =0.324） (20)

The results for Hunt-6 and Hunt-9 were provided by Yamaguchi and Nonaka[1]. The maximum relative error ε_{max} is (-)0.15 % for Hunt-4, 0.08 % for Hunt-5 and (-)0.008% for Hunt-9, which reveals decrease of the error associated with a higher order approximation. The error of (-)0.19 % for Hunt-6 is greater in absolute value than that for Hunt-4. Adjustment or tuning for the coefficients $D_1 \sim D_6$ in Eq. (14) may be required in order to make Hunt-6 more practical. Moreover, the accuracy of Hunt-9 with the maximum error of (-)0.008 % is lower that that of Vata-2 with an error of 0.0012 %. Vata-2 uses both exponential function with argument of real number power of α and hyperbolic function. This may correspond to making use of an infinite series with integer number power terms of variable α. On the other hand, Hunt-9 is expressed by a polynomial of degree 9 in variable α. Although Hunt-9 appears to have a lengthy form, Hunt-9 may be regarded as an AES with shorter and more compact form compared to AESs using exponential and hyperbolic functions such as Vata-1 and Vata-2.

Next, each of the 6 AESs proposed by Chamberlain and Porter[5] but overlooked in Yamaguchi and Nonaka[1], its classification, its abbreviated notation and the maximum relative error ε_{max} indicated in their paper are written in order as :

$$\beta = \alpha / \left\{ 1 - \alpha / \left(\mathrm{e}^{\alpha} \sinh\alpha \right) \right\}^{1/2} , \ [\ \mathrm{I}\text{-①},\ \text{Cham-2},\ \ \varepsilon_{max} = 0.747\%] \tag{21}$$

$$\beta = \alpha \left\{ \left(4\cosh^2\alpha - \sinh 2\alpha + 2\alpha \right) / \left(\sinh 2\alpha + 2\alpha \right) \right\}^{1/2} , \ [\ \mathrm{I}\text{-①},\ \text{Cham-3},\ \ \varepsilon_{max} = 2.73\%] \tag{22}$$

$$\beta = \alpha \left\{ \left(\sinh 2\alpha + 2\alpha \right) / \left(4\sinh^2\alpha - \sinh 2\alpha + 2\alpha \right) \right\}^{1/2} , \ [\ \mathrm{I}\text{-①},\ \text{Cham-4},\ \ \varepsilon_{max} = 0.163\%] \tag{23}$$

$$\beta = \alpha \left[1 - \frac{4\left\{ 1 - (1+\alpha)\,\mathrm{e}^{-2\alpha} \right\}}{\sinh 2\alpha + 2\alpha} \right]^{-1/4} , \ [\ \mathrm{I}\text{-①},\ \text{Cham-5},\ \ \varepsilon_{max} = 0.0710\%] \tag{24}$$

$$\beta = \alpha \left(1 - \frac{2\left\{ 3 - \left(3 + 6\alpha + 6\alpha^2 - 2\alpha^3 \right) \mathrm{e}^{-2\alpha} \right\}}{3\left[\sinh 2\alpha + 2\alpha - 4\left\{ 1 - (1+\alpha)\,\mathrm{e}^{-2\alpha} \right\} \right]} \right)^{-1/2} , \ [\ \mathrm{I}\text{-②},\ \text{Cham-6},\ \ \varepsilon_{max} = 0.0126\%] \tag{25}$$

$$\beta = \alpha \left(1 - \frac{4\left\{ 15 - \left(15 + 30\alpha + 30\alpha^2 + 5\alpha^3 - 10\alpha^4 + 2\alpha^5 \right) \mathrm{e}^{-2\alpha} \right\}}{15\left[\sinh 2\alpha + 2\alpha - 4\left\{ 1 - (1+\alpha)\,\mathrm{e}^{-2\alpha} \right\} \right]} \right)^{-1/4} ,$$
$$[\ \mathrm{I}\text{-②},\ \text{Cham-7},\ \ \varepsilon_{max} = 0.00351\%] \tag{26}$$

Their 1st AES is neglected, because it coincides with the well-known Eckart equation described in Yamaguchi and Nonaka[1].

Fig. 3 illustrates the relation between ε and h/L_0 for each of the 6 equations from Eq.(21) to Eq.(26). Eq.(26)-based ε has a positive peak and a negative peak with h/L_0 variation and the other equation-based ε takes a single negative peak respectively. The ε in any of the equations approaches zero with either decreasing or increasing h/L_0. The positive maximum or negative maximum relative error and its corresponding h/L_0 are given as follows :

29)Cham-2 : -0.742 %（h/L_0 =0.2496）～0.0%（$h/L_0 \to 0, \ \infty$） (27)

30)Cham-3 : 2.805 %（h/L_0 =0.0642）～0.0%（$h/L_0 \to 0, \ \infty$） (28)

31)Cham-4 : -0.162 %（h/L_0 =0.1343）～0.0%（$h/L_0 \to 0, \ \infty$） (29)

32)Cham-5 : -0.0710 %（h/L_0 =0.1246）～0.0%（$h/L_0 \to 0, \ \infty$） (30)

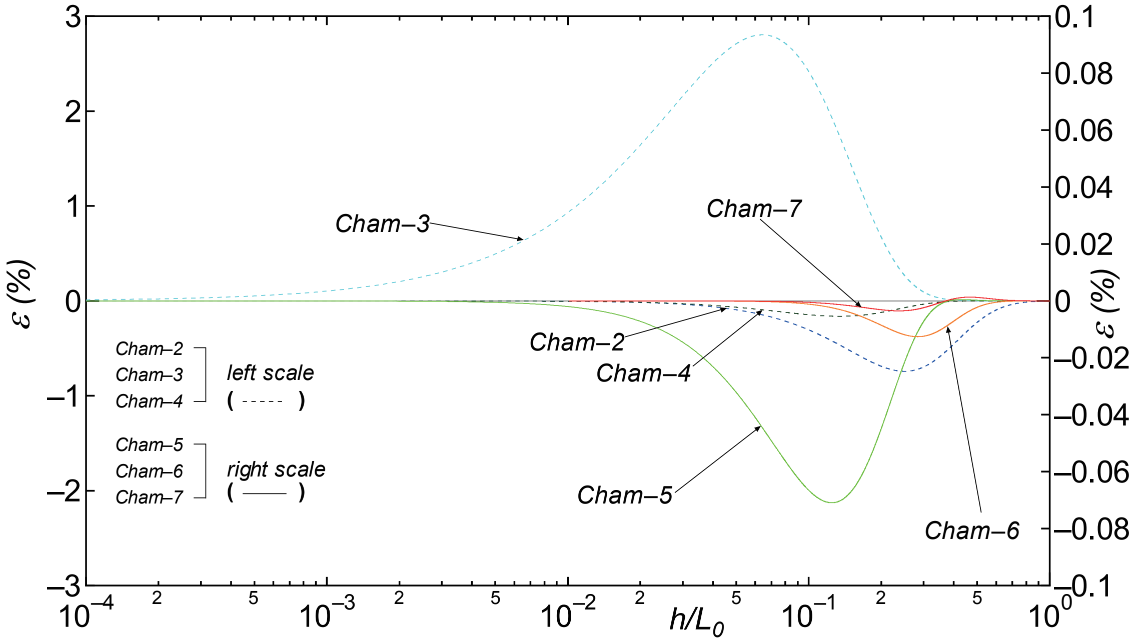

Fig. 3 Relation between relative error ε and relative water depth h/L_0 for any of Cham-2, Cham-3, Cham-4, Cham-5, Cham-6 and Cham-7.

33)Cham-6 : -0.0126 % (h/L_0 =0.2830) ~0.0% (h/L_0 →0, ∞) (31)

34)Cham-7 : -0.00351 % (h/L_0 =0.2368) ~0.00132% (h/L_0 =0.4662) (32)

The maximum relative error (absolute value) ε_{max} in each equation is nearly identical with the value by Chamberlain and Porter[5] indicated in Eq.(21) to Eq.(26) respectively. A negligibly small deviation may be due to some difference of the computation conditions. Looking at the accuracy of the equations in turn, the absolute ε_{max} in Eq.(21)(Cham-2) is (-)0.74 %, which is almost equal to either 0.75 % in the Guo[1] equation as :

$$\beta = \alpha / \left\{ 1 - \exp\left(- \alpha^{m/2} \right) \right\}, \quad m = 2.4901 \tag{33}$$

or 0.73 % in the 2nd equation by Yamaguchi and Nonaka[1] as :

$$\beta = \alpha \coth\left\{ \alpha \left(\coth \alpha^{m/2} \right)^{1/m} \right\}, \quad m = 1.378 \tag{34}$$

The ε_{max} in Eq.(22)(Cham-3) is 2.81 %, which means insufficiently low accuracy for a sophisticated formulation of the equation. The ε_{max} in Eq.(23)(Cham-4) is (-)0.162 %, which is comparable to either 0.19 % in Eq.(4)(Beji-1) or 0.15 % in Eq.(10) (Hunt-4). The ε_{max} in Eq.(24)(Cham-5) is 0.071 %, which is close to 0.078 % in the Hunt-5 equation and greater than (-)0.044 % in Eq.(5)(Beji-2). The ε_{max} in Eq.(25)(Cham-6) is (-)0.0126 %, which is slightly smaller than 0.019 % in Eq.(6)(Vata-1). The ε_{max} in Eq.(26)(Cham-7) is (-)0.0035 %, which is greater than 0.0012 % in Eq.(7)(Vata-2) and less than 0.0082 % in the Hunt-9 equation.

It should be noted that a numerical computation using either Eq.(25)(Cham-6) or Eq.(26)(Cham-7) may require special consideration because of round-off error produced even in the double precision computations. That is, α -related expansion of either Eq.(25) or Eq.(26) under the assumption of $\alpha \le 1$ yields the following equation to the order of α^3 with its classification and abbreviated notation including long wave approximation-based 'L' respectively as :

$$\beta = \alpha^{1/2} \left\{ 1 - (1/3)\alpha + (1/45)\alpha^2 + (1/189)\alpha^3 \right\}^{-1/2}, \quad [\text{II-①(i)}, \ \text{Cham-6L/Olson3}] \tag{35}$$

$$\beta = \alpha^{1/2}\left\{1-(2/3)\alpha+(7/45)\alpha^2-(4/945)\alpha^3\right\}^{-1/4}, \quad [\text{II-①(i), Cham-7L/YNH}] \tag{36}$$

in cases where the notation 'YNH' in Eq.(36), which comes from the initials of the present authors (Yamaguchi, Nonaka and Hatada). Eq.(25)(Cham-6)-based Eq.(35) coincides with either the Eq.(58)(You-5/You) mentioned below or the 7th order Olson equation of α to the 3rd order. Eq.(36) was derived in this study. Use of Eq.(35) for Eq.(25) and Eq.(36) for Eq.(26) is recommended in the case of $h/L_0 \le 2\times10^{-3}$. In short, the Chamberlain and Porter[5] equation excluding Eq.(22) indicates more improved accuracy with increasing degree of approximation. Eq.(22) is not good for practical use due to its relatively low accuracy.

3.3 2-step AESs with high accuracy applicable to a full range of relative water depth h/L_0

Highly accurate AESs may be derived by making use of the 1st iteration solution for the dispersion relationship Eq.(1) based on Newton's method as :

$$\beta = \frac{\alpha+\beta_a^2\mathrm{sech}^2\beta_a}{\tanh\beta_a+\beta_a\mathrm{sech}^2\beta_a} = \frac{\alpha+\beta_a^2\left(1-\tanh^2\beta_a\right)}{\tanh\beta_a+\beta_a\left(1-\tanh^2\beta_a\right)} \tag{37}$$

in cases where any of Eqs.(4), (5), (6) and (7) are given as an initial value β_a. The furthest right side term is expressed as a function of $\tanh\beta_a$ only, by taking computational efficiency into account. The 2-step solution is denoted as Beji-1N, Beji-2N, Vata-1N and Vata-2N in order, by adding ' N ' to each notation. These solutions are classified into I-③.

Fig. 4 describes the relation between relative error ε and relative water depth h/L_0 for any of Beji-1N, Beji-2N, Vata-1N, Vata-2N and YN9. The YN9 is the result obtained under an initial value by use of a modified version of the Carvlho 4th AES(Yamaguchi and Nonaka[1]. The absolute value of ε associated with oscillating change is quite small. A range of the relative error and the corresponding h/L_0 is collectively written as

35) Beji-1N : -1.6×10^{-4} % (h/L_0 =0.009) ～2.1×10^{-4} % (h/L_0 =0.367) (38)

36) Beji-2N : -8.2×10^{-6} % (h/L_0 =0.006) ～1.1×10^{-6} % (h/L_0 =0.276) (39)

37) Vata-1N : -1.6×10^{-6} % (h/L_0 =0.011) ～3.7×10^{-7} % (h/L_0 =0.277) (40)

38) Vata-2N : -6.3×10^{-7} % (h/L_0 =0.006) ～8.1×10^{-10} % (h/L_0 =0.436) (41)

11-8) YN9 : -1.1×10^{-4} % (h/L_0 =0.044) ～1.1×10^{-4} % (h/L_0 =0.274) (42)

The relative error in this case takes a place corresponding to the accuracy of an initial value. In particular, the solution by Vata-2 may be regarded as a numerically quasi-exact solution for the sake of extremely small relative error of 10^{-6} to 10^{-9}. Also, YN-9 gives about 2 times higher accuracy than Beji-1N but more than 2 order magnitude-lower accuracy than the other 3 AESs. It may be said that use of an over-complicated AES for an initial value does not yield an efficient estimate, because 2 to 4 times iteration of Eq.(37) under the initial condition by Eq.(2) gives a numerically exact solution. In the very latest publication, Simarro and Orfila[10] indicates that the 1st iteration solution of Newton's method with use of an initial estimate by Beji-2 gives the maximum relative error(absolute value) of 8.2×10^{-6} %, the same value in Eq.(39), and that a higher accuracy is attained by using an initial value based on Vata-2.

Next, the 1st iteration solution by Newton's method using Eq.(37) under initial value by any of Hunt 4th, 5th, 6th and 9th order solutions is obtained in succession. These belong to the group classification of I-③. Fig. 5 shows the relation between ε and h/L_0 for any of Hunt-4N, Hunt-5N, Hunt-6N and Hunt-9N, in cases where ' N ' is added to each notation. Any solution provides asymptotically zero-approaching behavior with

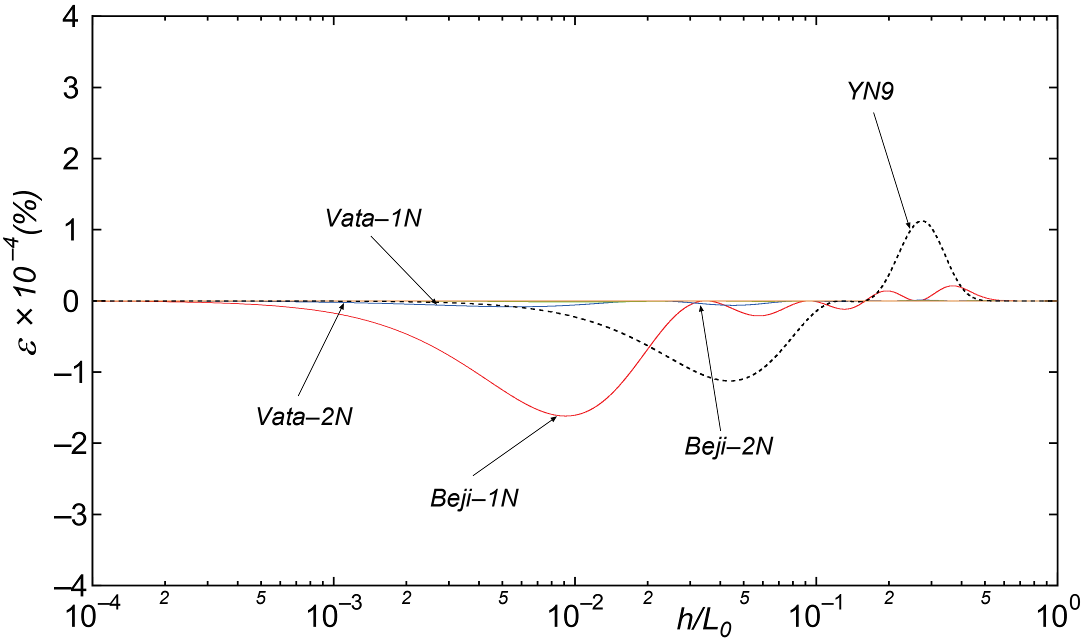

Fig. 4 Relation between relative error ε and relative water depth h/L_0 for any of Beji-1N, Beji-2N, Vata-1N, Vata-2N and YN9.

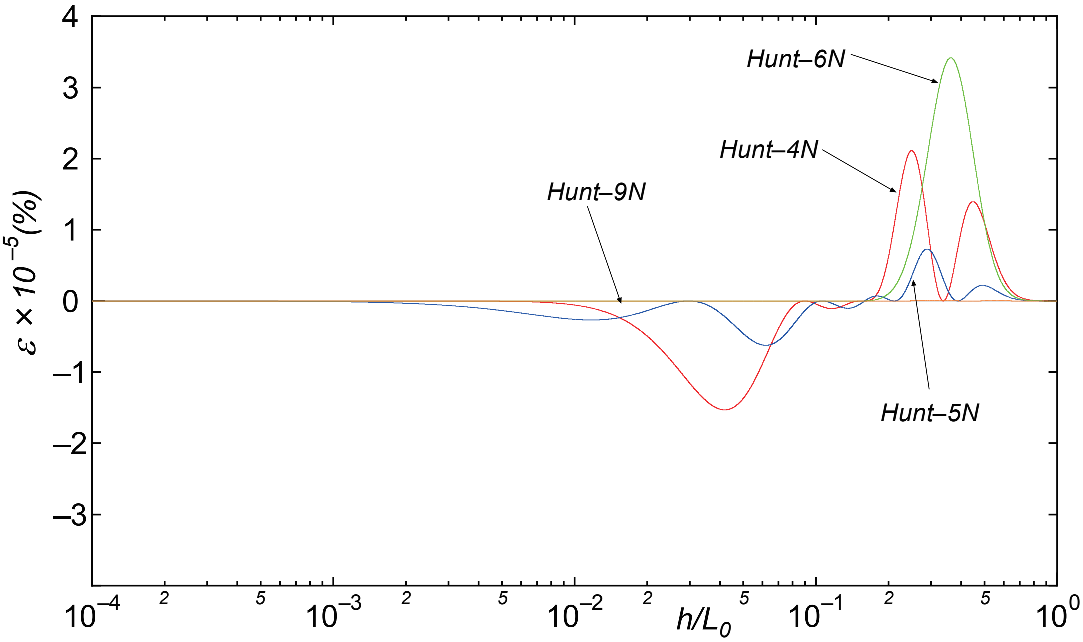

Fig. 5 Relation between relative error ε and relative water depth h/L_0 for any of Hunt-4N, Hunt-5N, Hunt-6N and Hunt-9N.

either infinitesimal h/L_0 ($h/L_0 \to 0$) or infinite h/L_0 ($h/L_0 \to \infty$) and positive/ negative change or change with seemingly only one positive hump(Hunt-6N) in a middle range of h/L_0. A range of respective relative error and the corresponding h/L_0 is written as follows :

39) Hunt-4N : -1.5×10^{-5} % (h/L_0 =0.042) ～2.1×10^{-5} % (h/L_0 =0.249) (43)

40) Hunt-5N : -6.2×10^{-6} % (h/L_0 =0.062) ～7.3×10^{-6} % (h/L_0 =0.289) (44)

41) Hunt-6N : -2.4×10^{-8} % (h/L_0 =0.144) ~3.4×10^{-5} % (h/L_0 =0.362) (45)

42) Hunt-9N : -1.8×10^{-10} % (h/L_0 =0.095) ~3.4×10^{-8} % (h/L_0 =0.320) (46)

Hunt-6N yields a positive value-biased change of ε with h/L_0, which reflects the error characteristics of Hunt-6. The relative error ε becomes smaller with increasing order in the Hunt equations except for Hunt-6N. Hunt-9N yields a smaller relative error than Vata-2N and results in a closer estimate to the exact solution. It may be concluded that for a practical use, Hunt-4N, the 1st iteration solution by eq.(37) under an initial value by Hunt-4 produces a satisfactory estimate for wave length, with a relative error within $\pm 2\times10^{-5}$ %.

In addition, Fig. 6 indicates the relation between ε and h/L_0 for any of Cham-2N, Cham-3N, Cham-4N, Cham-5N, Cham-6N and Cham-7N, each of which is obtained from the Ist iteration solution of Eq.(37) under the initial value by any of Eq.(21) to Eq.(26). These are classified into I-③. Any ε changes with h/L_0, taking a positive peak and a negative peak, when the difference is prominent. The absolute value of ε is very small. A range of the relative error and the corresponding h/L_0 for each solution are given in succession as follows :

43) Cham-2N : -1.42×10^{-4} % (h/L_0 =0.1130) ~6.31×10^{-4} % (h/L_0 =0.2703) (47)

44) Cham-3N : -2.21×10^{-2} % (h/L_0 =0.0490) ~3.07×10^{-4} % (h/L_0 =0.1880) (48)

45) Cham-4N : -3.11×10^{-4} % (h/L_0 =0.00915) ~9.56×10^{-6} % (h/L_0 =0.2035) (49)

46) Cham-5N : -6.65×10^{-6} % (h/L_0 =0.0886) ~1.31×10^{-6} % (h/L_0 =0.1961) (50)

47) Cham-6N : -4.83×10^{-9} % (h/L_0 =0.1351) ~1.87×10^{-7} % (h/L_0 =0.2870) (51)

48) Cham-7N : -8.94×10^{-10} % (h/L_0 =0.1332) ~1.28×10^{-8} % (h/L_0 =0.2500) (52)

As indicated above, the absolute value of ε_{max} decreases with more sophisticated AES-based initial value except for the Cham-3N case. In particular, ε_{max} is (-)6.7×10^{-6} % for Cham-5N, 1.9×10^{-7} % for Cham-6N and 1.3×10^{-8} % for Cham-7N. Any of these solutions corresponds to a numerically quasi-exact solution. In a practical sense, Cham-2N yields a satisfactory estimate and Cham-5N may be preferable for more accurate

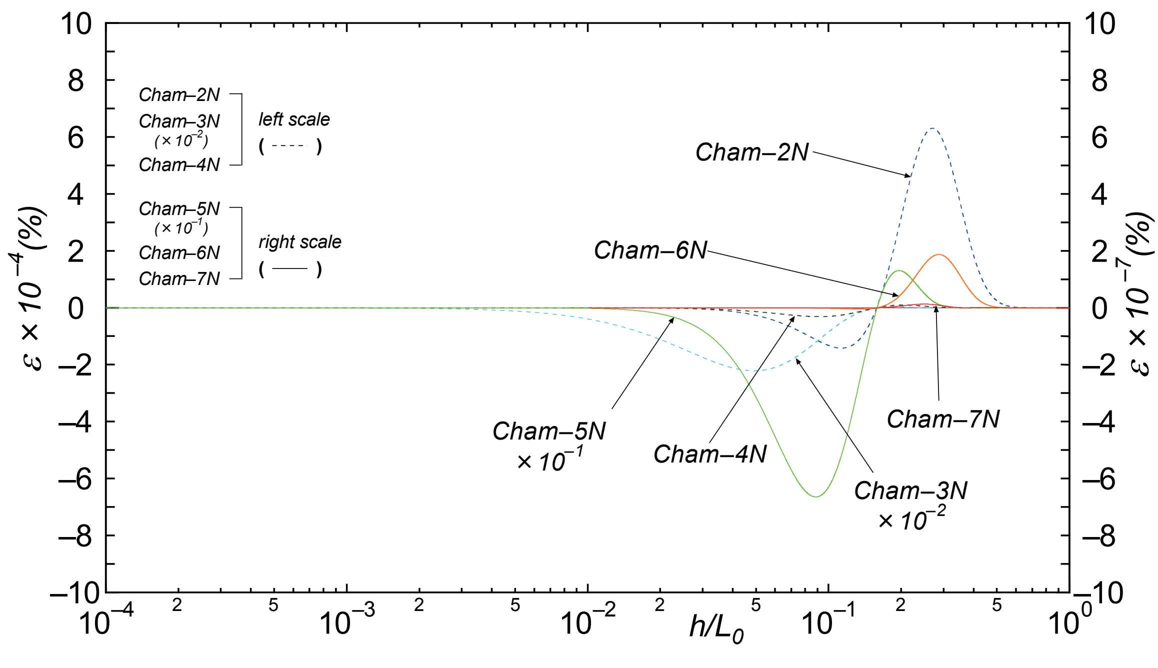

Fig. 6 Relation between relative error ε and relative water depth h/L_0 for any of Cham-2N, Cham-3N, Cham-4N, Cham-5N, Cham-6N and Cham-7N.

evaluation. Also, the accuracy of Cham-3N is relatively lower than the other cases, reflecting the accuracy of Cham-3 used for the initial value computation.

3.4 Application-limited PAESs to shallow water condition or deeper water condition

In the first part of this section, characteristics of the relative errors associated with the 5 PAESs including the Wu and Thornton[1] PAES(WT1) introduced in You[6]~[8] and the Chamberlain and Porter[5] PAES(CP-1), any of which is applicable only to shallow water conditions, are discussed. These PAESs are respectively expressed as :

$$\beta = \alpha^{1/2}\{1+(1/6)\alpha\}, \quad [\text{II-①(i)}, \text{ You-1}] \quad (53)$$

$$\beta = \alpha^{1/2}\{1+(1/6)\alpha+(1/30)\alpha^2\}, \quad [\text{II-①(i)}, \text{ You-2/WT1}^{[1]}] \quad (54)$$

$$\beta = \alpha^{1/2}\{1+(1/6)\alpha+(11/360)\alpha^2\}, \quad [\text{II-①(i)}, \text{ You-3/Niel2}^{[1]}] \quad (55)$$

$$\beta = \alpha^{1/2}\{1+(1/6)\alpha+(13/360)\alpha^2\}, \quad [\text{II-①(i)}, \text{ CP-1}^{[6]}] \quad (56)$$

$$\beta = \alpha^{1/2}\{1+(1/6)\alpha+(11/360)\alpha^2+(17/5040)\alpha^3\}, \quad [\text{II-①(i)}, \text{ You-4}] \quad (57)$$

$$\beta = \alpha^{1/2}\{1+(1/3)\alpha+(4/45)\alpha^2+(16/945)\alpha^3\}^{1/2}, \quad [\text{II-①(i)}, \text{ You-5/You}^{[1]}] \quad (58)$$

Eq.(54) is the Wu and Thornton PAES(WT1) proposed in 1986, Eq.(55) the Nielsen 2nd PAES(Niel2) in 1982 and Eq.(58) the You[6] PAES in 2003. Eq.(58) is in agreement with Eq.(35) or the Olson[1] equation to $O(\alpha^3)$. Accuracy of these PAESs is given in Yamaguchi and Nonaka[1] and in the table mentioned below. Also Eq.(56) is the Chamberlain and Porter[5] PAES in 1999. Eqs.(53) to Eq.(58) are classified into II-①. Each equation asymptotically approaches a long wave theory-based $\alpha^{1/2}$ and then zero with decreasing h/L_0. Also, α-expanded equation of Eq.(58) to $O(\alpha^3)$ is in complete agreement with Eq.(57), as described in You[7]. Moreover, Eq.(35) and Eq.(36) are added in the following investigation.

Fig. 7 illustrates the relation between ε and h/L_0 for any of the above-mentioned 8 PAESs. With augmentation of h/L_0, Eq.(53)(You-1)-based ε increasingly deviates from nearly zero to the positive side and Eq.(34)(You-4)-based ε from nearly zero to the negative side. Any of the remaining 3 PAESs, Eq.(54)(You-2/WT1), Eq.(55)(You-3/Niel2) and Eq.(58)(You-5/You) has a peak of ε with respect to h/L_0 and then shows a rapid fall of ε to the negative side. It should be added that Eq.(53)(You-1)-based ε takes a peak value at a large h/L_0 of 0.955.

Table 2 lists the peak value of relative error ε_{peak} with respect to h/L_0 and the corresponding $(h/L_0)_{peak}$, and $(h/L_0)_r$ yielding any of the reference relative errors ε_r of 1 %, 0.5 %, 0.1 %, 0.05 % and 0.01 % in cases where the notation '+' is given for positive error in the parenthesis and the notation '-' for negative error. A smaller $(h/L_0)_r$ is adopted in the multi-$(h/L_0)_r$ cases. Absolute value of relative error ε becomes smaller than the reference relative error ε_r in a range of h/L_0 less than an indicated $(h/L_0)_r$ value. The following description may be made from the table :

(1) An applicability region of each equation naturally becomes narrower with decrease of the reference relative error ε_r.

(2) The applicability region of Eq.(53)(You-1) is too narrow to be available for a practical use.

(3) The applicability region of Eq.(54)(You-2/WT1) is somewhat wider than that of Eq.(58)(You-5/You) for each of the reference relative errors except for ε_r=0.01 % case. The region of Eq.(57)(You-4) is relatively narrow.

(4) The applicability region of Eq.(55)(You-3/Niel2) is wide in the case of a larger reference relative error, but

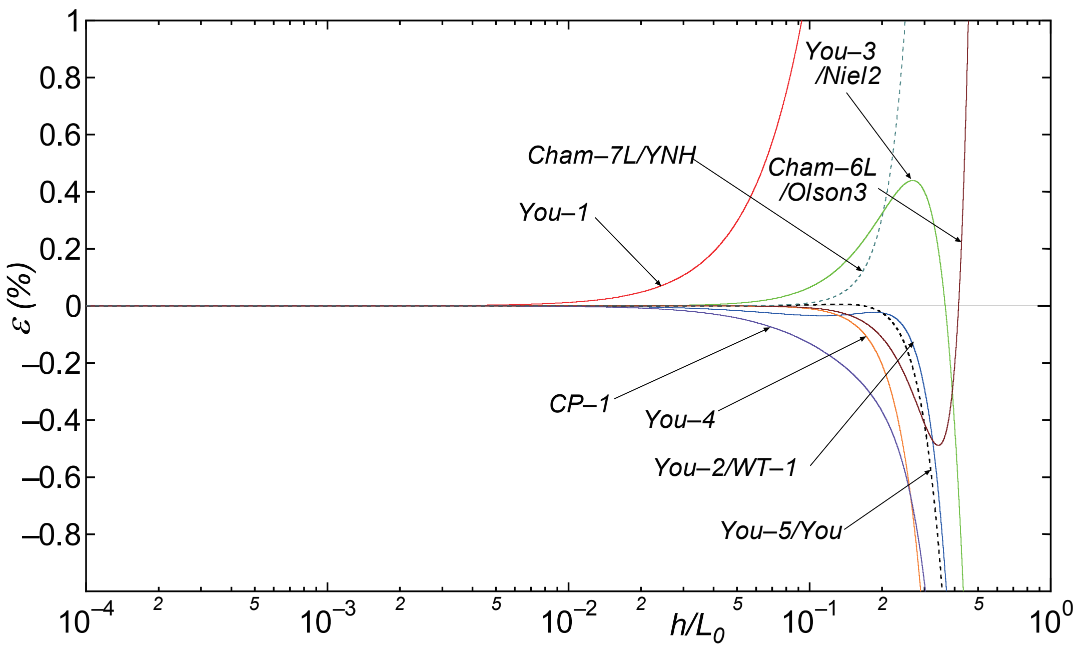

Fig. 7 Relation between relative error ε and relative water depth h/L_0 for any of You-1, You-2, You-3, You-4 and You-5.

Table 2 peak value ε_{peak} , corresponding $(h/L_0)_{peak}$ and reference value ε_r -based $(h/L_0)_r$ for the shallow water case(1).

Name, eq.	ε_{peak} ・ $(h/L_0)_{peak}$	$(h/L_0)_r$ $(\varepsilon_r = 1\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.5\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.1\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.05\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.01\%)$
You-1, (53)	22.48% ・ 0.955	0.093(+)	0.065(+)	0.029(+)	0.020(+)	0.009(+)
You-2, (54)	-0.034% ・ 0.113	0.369(-)	0.327(-)	0.257(-)	0.233(-)	0.036(-)
You-3, (55)	0.44% ・ 0.268	0.434(-)	0.405(-)	0.118(+)	0.091(+)	0.051(+)
CP-1, (56)	— —	0.303(-)	0.233(-)	0.084(-)	0.055(-)	0.023(-)
You-4, (57)	— —	0.289(-)	0.245(-)	0.169(-)	0.144(-)	0.100(-)
You-5, (58)	0.0054% ・ 0.133	0.355(-)	0.310(-)	0.238(-)	0.217(-)	0.185(-)
Cham-6L(35)	-0.489% ・ 0.342	0.455(+)	0.438(+)	0.194(-)	0.163(-)	0.109(-)
Cham-7L(36)	— —	0.249(+)	0.219(+)	0.160(+)	0.139(+)	0.099(+)

rapidly becomes narrower in case of a smaller reference relative error.

(5) The reference relative error of $\varepsilon_r = \pm 0.01$ % gives fairly small $(h/L_0)_r$ for any of Eq.(53) to Eq.(56). But in case of Eq.(58)(You-5/You), $(h/L_0)_a$ takes a relatively large value of 0.185. In short, within an applicability region of each equation, Eq.(54)(You-2/WT1) may be applicable for a reasonable estimation of wave length and use of Eq.(58)(You-5/You) may be recommended and for more accurate estimation.

(6) The applicability region of either Eq.(35)(Cham-6L) or Eq.(36)(Cham-7L) is narrower than that of Eq.(58)(You-5/You).

In short, within an applicability region of each equation, Eq.(54)(You-2/WT1) may be available for a reasonable estimation of wave length and use of Eq.(58)(You-5/You) may be recommended for more accurate estimation.

A range of relative error and the corresponding h/L_0 for any of Eq.(53)(You-1), Eq.(56)(CP-1), Eq.(57)(You-4), Eq.(35)(Cham-6L/Olson3) and Eq.(36)(Cham-7L/YNH) are written in order as :

49) You-1 : 1%（h/L_0 =0.093）～0%（h/L_0 →0） (59)

50) CP-1 : -0.1%（h/L_0 =0.084）～0%（h/L_0 →0） (60)

51) You-4 : -0.01%（h/L_0 =0.100）～0%（h/L_0 →0） (61)

52) Cham-6L/Olson3 : -0.01%（h/L_0 =0.109）～0%（h/L_0 →0） (62)

53) Cham-7L/YNH : 0.01%（h/L_0 =0.099）～0%（h/L_0 →0） (63)

A similar investigation is conducted for the Nielsen 1st PAES(Niel1, group II-①(i)), the Venezian 1st PAES(Vene1, group II-①(i)), the Venezian 2nd PAES(Vene2, group II-②(i)) and the 7th order Olson PAES(Olson7, group II-②(i)), in cases where the error characteristics were discussed in Yamaguchi and Nonaka[1]. Table 3 lists the peak value of relative error ε_{peak} with respect to h/L_0 and the corresponding $(h/L_0)_{peak}$, and $(h/L_0)_r$ yielding any of the reference relative errors ε_r of 1 %, 0.5 %, 0.1 %, 0.05 % and 0.01 % for the above-mentioned 4 PAESs as well as Table 2. Fig. 8 shows the relation between ε and h/L_0 for each PAES. The following feature is indicated from the table and the figure :

(1) The critical value $(h/L_0)_r$ for ε_r becomes naturally smaller with the decrease of ε_r. The degree of decrease is greater in NIel1 with low approximation. This tendency is also observed in Vene1.

(2) The critical value $(h/L_0)_r$ for either Olson7 or Vene2 is relatively large and the reduction rate of $(h/L_0)_r$ value associated with the lowering of ε_r is moderate. This means a wider applicability of either Olson7 or Vene2.

Next, the Nielsen[1] 3rd PAES(Niel3) and the Wu and Thornton[1] 2nd PAES(WT2) applicable only to deeper water cases are written with classification and abbreviated name respectively as :

$$\beta = \alpha\{1 + 2\exp(-2\alpha)\}, \qquad \text{[II-①(ii), Niel3]} \qquad (64)$$

$$\beta = \alpha\{1 + 2t(1+t)\}, \quad t = \exp\{-2\alpha(1 + 1.26\mathrm{e}^{-1.84\alpha})\}, \qquad \text{[II-①(ii), WT2]} \qquad (65)$$

Error characteristics of these PAESs were discussed in Yamaguchi and Nonaka[1]. Also, the newly-investigated 2 PAESs in this study, that is the Chamberlain and Porter[5] 2nd PAES(CP-2) and the simplified Newman[9] PAES(New-2S) with classification and abbreviated name are given in order as :

$$\beta = \alpha\{1 + 2\alpha\mathrm{e}^{-2\alpha} + 2(4\alpha - 5\alpha^2)\mathrm{e}^{-4\alpha}\}, \qquad \text{[II-①, CP-2]} \qquad (66)$$

$$\beta = \alpha + 0.00005 + 1.9738\alpha\mathrm{e}^{-2\alpha} - 5.26\alpha^2\mathrm{e}^{-4\alpha}, \qquad \text{[II-①(ii), New-2S]} \qquad (67)$$

Fig. 9 describes the relation between ε and h/L_0 for any of Niel3, WT2, CP-2 and New-2S.

Table 3 peak value ε_{peak}, corresponding $(h/L_0)_{peak}$ and reference ε_r-based $(h/L_0)_r$ for shallow water case(2).

Name	ε_{peak} ・ $(h/L_0)_{peak}$	$(h/L_0)_r$ $(\varepsilon_r = 1\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.5\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.1\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.05\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.01\%)$
Niel1	-0.74%・0.075	0.156(+)	0.032(-)	0.005(-)	0.0025(-)	0.0005(-)
Vene1	0.048%・0.104	0.258(-)	0.225(-)	0.177(-)	0.166(-)	0.033(+)
Vene2	―・―	0.427(-)	0.393(-)	0.326(-)	0.302(-)	0.253(-)
Olson7	―・―	0.401(-)	0.375(-)	0.323(-)	0.303(-)	0.264(-)

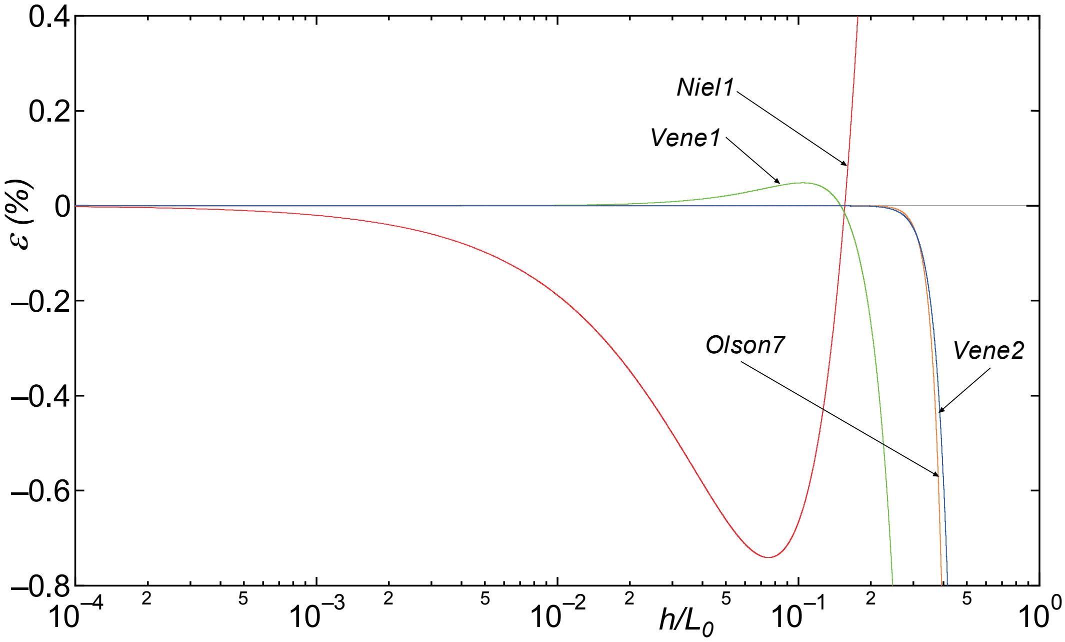

Fig. 8 Relation between relative error ε and relative water depth h/L_0 for any of Niel1, Vene1, Vene2 and Olson7.

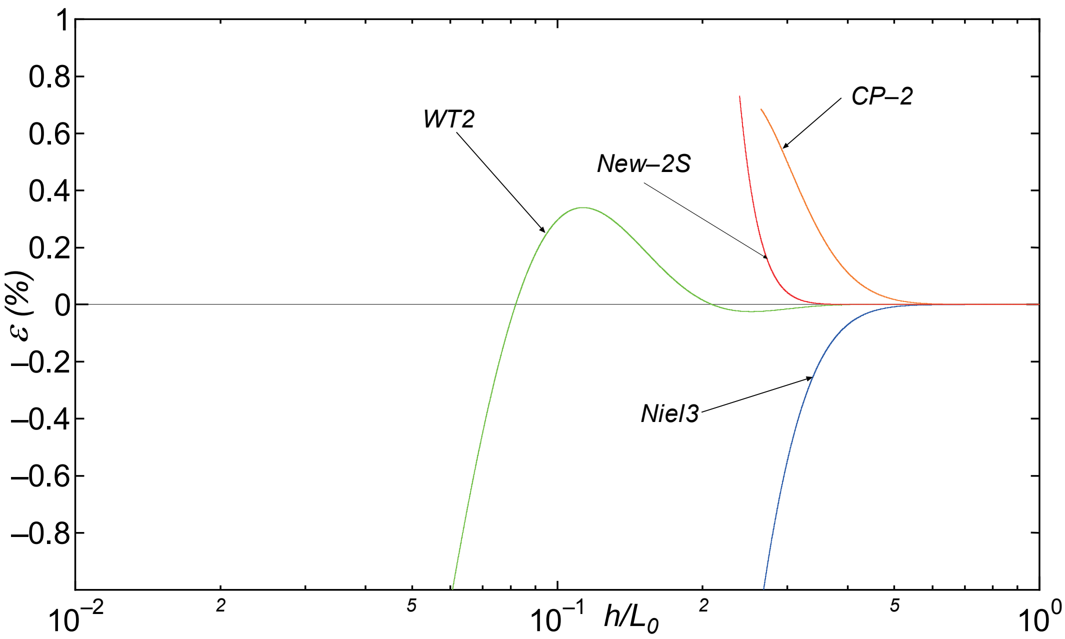

Fig. 9 Relation between relative error ε and relative water depth h/L_0 for any of Niel3, WT2, CP-2 and New-2S.

CP-2-based ε rapidly decreases from a positive value toward zero with increasing h/L_0. A similar behavior seems to be observed for New-2S-based ε. Exactly speaking, New-2S-based ε approaches nearly zero taking negative and positive peak with increasing h/L_0, as shown below in Fig. 11. The constant term 0.00005 in Eq.(67)(New-2S) is a kind of error-adjusting factor. As a matter of fact, removal of this constant term makes the maximum error twice and yields an improvement of accuracy in a larger region of h/L_0 . Relative error statistics for both CP-2 and New-2S are summarized as follows :

54) CP-2 : 0.686% (h/L_0 =0.2642) ~0% ($h/L_0 \to \infty$) (68)

55) New-2S : 0.0020% (h/L_0 =0.3183) ~0% ($h/L_0 \to \infty$) (69)

Table 4 gives a list of the results for any of Niel3, WT2, CP-2 and New-2S as well as Table 3. In a deeper water application case, a smaller $(h/L_0)_r$ signifies a wider application range of the PAES. In this sense, an application range of WT2 is rather wide in $\varepsilon_r \geq 0.05$ % case, while that of either Niel3 or CP-2 is narrower. New-2S has a high accuracy of ε_{max} =(-)0.002 %, but the application range is not so wide as that of WT2. You[7] states in his paper that Bagatur[11] provides an approximate solution method based on the Newton-Raphson method. You[7],[8] discusses characteristics of relative error with any of $\beta = \alpha^{1/2}$ and the above-mentioned Eq.(53) to Eq.(58) excluding Eq.(56). Also You[7],[8] investigates relative error with Newton's method-based solution giving each of them as an initial value and proposes a preferable use of Eq.(37) associated with Eq.(55). But usage of this method may not be recommended, because the maximum relative error of 7×10^{-3} does not necessarily suggest a high accuracy for the application of Newton's method.

Table 4 peak value ε_{peak} , corresponding $(h/L_0)_{peak}$ and reference value ε_r -based $(h/L_0)_r$ for deeper water cases.

Name, eq.	ε_{peak} ・ $(h/L_0)_{peak}$	$(h/L_0)_r$ $(\varepsilon_r = 1\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.5\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.1\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.05\%)$	$(h/L_0)_r$ $(\varepsilon_r = 0.01\%)$
Niel3, (64)	― ・ ―	0.268(+)	0.304(+)	0.384(+)	0.416(+)	0.489(+)
WT2, (65)	-0.025% ・ 0.252			0.171(+)	0.186(+)	0.321(-)
CP-2, (66)	― ・ ―		0.300(+)	0.468(+)	0.445(+)	0.529(+)
New-2S, (67)	-0.002% ・ 0.349			0.260(+)	0.276(+)	0.303(+)

3.5 CPAESs applicable to a full range of relative water depth h/L_0

Combining a shallow water-limited PAES with a deeper water-limited PAES may yield a CPAES applicable to a full range of water depth conditions. These trials were conducted by Wu and Thornton[1], You[6], Chamberlain and Porter[5] and Newman[9]. But the CPAES by Wu and Thornton[1] yields discontinuity of relative error at critical h/L_0 corresponding to an application limit of each PAES, which may not be a reasonable behavior. The CPAES by Chamberlain and Porter[5] indicates the same property.

First, we investigated the error characteristics of 4 CPAESs such as ①the Chamberlain and Porter[5] CPAES, ② CPAES consisting of You-3(Niel2) and WT-2(Eq.(55)+Eq.(65)), ③ CPAES combining YOU-3(Niel2) and Niel3(Eq.(55) +Eq.(64)), ④CPAES combining You-5(You) and Niel3(Eq.(58)+Eq.(64)). The 2nd CPAES of the 4 CPAESs is made in this study and the latter 2 CPAESs were proposed by You[6].

The CPAES by Chamberlain and Porter[5], its classification, abbreviated name and maximum relative error(absolute value) ε_{max} are expressed as follows :

$$\beta=\begin{cases}\alpha^{1/2}\left\{1+(1/6)\alpha+(13/360)\alpha^2\right\}, \quad \alpha\le 1.66(h/L_0\le 0.264) \\ \qquad\qquad (\text{II-①(i)},\ \text{CP-1},\ \varepsilon_{max}=(-)0.683\%) & (70) \\ \alpha+2\left(\alpha e^{-2\alpha}\right)+2(4-5\alpha)\left(\alpha e^{-2\alpha}\right)^2, \quad \alpha>1.66(h/L_0>0.264) \\ \qquad\qquad (\text{II-①(ii)},\ \text{CP-2},\ \varepsilon_{max}=0.683\%) & (71)\end{cases}$$

Eq.(70)(CP-1) is the same form as Eq.(56) and Eq.(71)(CP-2) is the same form as Eq.(66). Their error characteristics have been discussed above. Relative errors by both PAESs at the connection point of h/L_0 =0.264 have completely reversed positive and negative signs. Chamberlain and Porter[5] says that transformation into the PAESs from the original AES Eq.(24)(Cham-5) made a simplification of the formula possible but yielded loss of the high accuracy. In fact, the accuracy of the CPAES becomes 10 times lower than that of the original AES.

Fig. 10 shows the relation between ε and h/L_0 for any of the above-mentioned 4 CPAESs. In the Chamberlain and Porter[5] CPAES, Eq.(70)(CP-1)-based ε becomes greater toward the side with increase of h/L_0 and takes -0.678 % at h/L_0 =0.2642. On the other hand, Eq.(71)(CP-2)-based ε gives 0.686 % at h/L_0 =0.2642 and rapidly decreases toward zero from this point with an increase of h/L_0 . Accordingly, the Chamberlain and Porter[5] CPAES yields the maximum relative error of $\varepsilon=\pm$ 0.68 % at h/L_0 =0.2642, which means a significant relative error on the periphery of this critical h/L_0 value. As a final result, we may not be able to say from this property that the Chamberlain and Porter[5] CPAES is a practically useful CPAES, although it has a simplified formulation. In the cases of the remaining 3 CPAESs, the 2 relative error curves cross each other at a certain value of h/L_0 respectively, as shown in Fig. 10. The relative error at a cross point is written in succession as :

56) CPAES consisting of You-3(Niel2) and WT2

0.186% (h/L_0 =0.152) →0% (h/L_0 →0, ∞) (72)

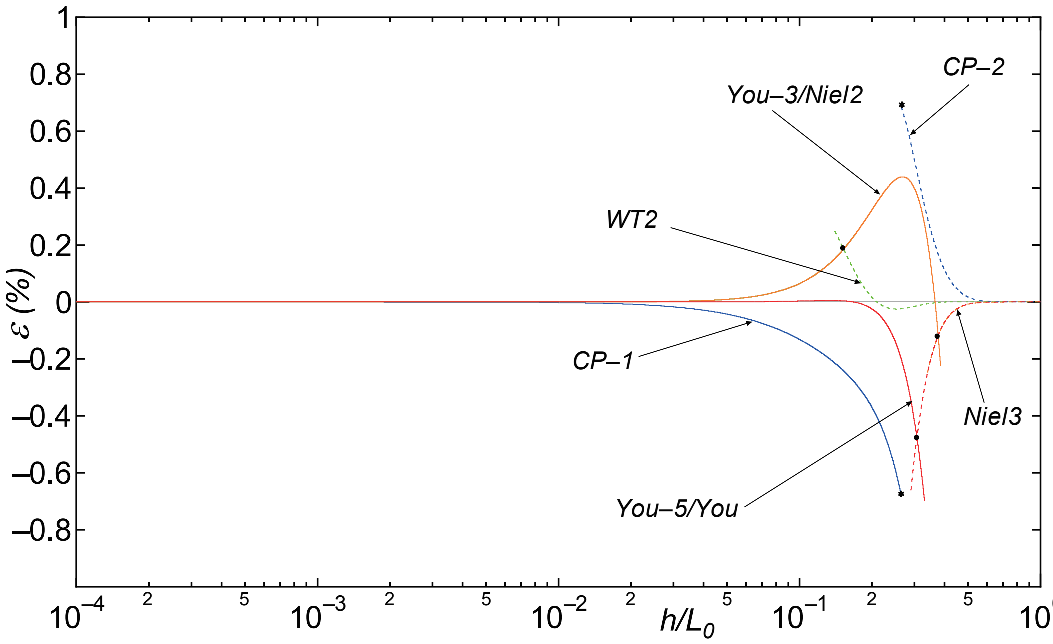

Fig. 10 Relation between relative error ε and relative water depth h/L_0 for any of Niel3, WT2, CP-2 and New-2S.

57) CPAES consisting of You-3(Niel2) and Niel3

$$-0.118\% \ (h/L_0=0.376) \rightarrow 0\% \ (h/L_0 \rightarrow 0,\ \infty) \quad (73)$$

58) CPAES consisting of You-5(You) and Niel3

$$-0.476\% \ (h/L_0=0.307) \rightarrow 0\% \ (h/L_0 \rightarrow 0,\ \infty) \quad (74)$$

Eq.(73) yields the least relative error(-0.118 %) in the above 3 CPAESs, but as can be seen in Fig. 10 and Table 2, You-3/Niel2 gives the peak relative error of 0.44 % at h/L_0 =0.268 situating in the applicability range and then results in a large relative error throughout a full range of h/L_0. Eq.(72), CPAES consisting of You-3/Niel2 and WT2 is the most proper CPAES of the 3 CPAESs. Remarks are added that the value of h/L_0 =0.376 in Eq.(73) nearly corresponds to that of α =2.37 given in You[6] and that the value of h/L_0 =0.307 in Eq.(74) to that of α =1.94 in You[6].

Next, the Newman[9] PAESs, its classification and abbreviated notation are expressed in order as follows :

a) $\alpha \le 2(h/L_0 \le 0.3183)$;

$$\beta = \alpha^{1/2} \Big/ \Big\{ 1.00000000 - 0.33333372(\alpha/2) - 0.01109668(\alpha/2)^2 + 0.01726435(\alpha/2)^3 + 0.01325580(\alpha/2)^4 - 0.00116594(\alpha/2)^5 + 0.00829006(\alpha/2)^6 - 0.01252603(\alpha/2)^7 + 0.00404923(\alpha/2)^8 \Big\}$$

（Ⅱ-②(i), New-1） (75)

b) $\alpha > 2(h/L_0 > 0.3183)$; $z = (1/2)\alpha e^{4-2\alpha}$,

$$\beta = \alpha + 0.000000122 + 0.073250017z - 0.009899981z^2 + 0.002640863z^3 - 0.000829239z^4 - 0.000176411z^5$$

（Ⅱ-②(ii), New-2） (76)

and New-3 is given by Eq.(76) to $O(z^4)$ as

$$\beta = \alpha + 0.000000122 + 0.073250017z - 0.009899981z^2 + 0.002640863z^3 - 0.000829239z^4$$

（Ⅱ-②(ii), New-3） (77)

Fig. 11 illustrates the relation between relative error ε and relative wave depth h/L_0 for any of Eq.(75)(New-1), Eq.(76)(New-2), Eq.(77)(New-3) and Eq.(67)(New-2S). The following features are indicated from this figure.

(1) Eq.(75)(New-1)-based ε is very small for h/L_0 <0.34 as indicated by Newman[9], but rapidly increases for h/L_0 >0.34. Looking into ε with a steep rise indicates $\varepsilon = 1.63\times10^{-6}$ % at h/L_0 =0.3183(α =2), $\varepsilon = 2.23\times10^{-5}$ % at h/L_0 =0.3300, $\varepsilon = 8.24\times10^{-5}$ % at h/L_0 =0.3400, $\varepsilon = 5.42\times10^{-4}$ % at h/L_0 =0.3600 and $\varepsilon = 1.15\times10^{-3}$ % at h/L_0 =0.3700.

(2) Eq.(76)(New-2)-based ε rapidly decreases for h/L_0 >0.31 and approaches nearly zero($\varepsilon = 1.2\times10^{-7}$ %) with increasing h/L_0. The behavior is represented by $\varepsilon = 1.71\times10^{-2}$ % at the above-mentioned separation point of h/L_0 =0.3183(α =2), $\varepsilon = 2.06\times10^{-3}$ % at h/L_0 =0.3600, $\varepsilon = 7.33\times10^{-4}$ % at h/L_0 =0.3800, $\varepsilon = 2.59\times10^{-4}$ % at h/L_0 =0.400 and $\varepsilon = 1.51\times10^{-4}$ % at h/L_0 =0.4100.

(3) The h/L_0 - ε curves based on both Eq.(75)(New-1) and Eq.(76)(New-2) cross at h/L_0 =0.3705, where ε is equal to=0.0012%. Each equation gives a rather small relative error ε within its effective range of h/L_0, but as both ε curves intersect at h/L_0 situating outside of their effective range of h/L_0, the relative error around the crossing point becomes somewhat greater in either case. For example, Eq.(75)(New-1)-based relative error ranges from $\varepsilon = 2.30\times10^{-4}$ % at h/L_0 =0.35 to $\varepsilon = 1.20\times10^{-3}$ % and Eq.(76)(New-2)-based relative error from $\varepsilon = 1.20\times10^{-3}$ % at h/L_0 =0.3705 to $\varepsilon = 4.36\times10^{-4}$ % at h/L_0 =0.3900. If an application limit of Eq.(75)(New-1) and Eq.(76)(New-2) is taken at $\alpha = 2$(h/L_0 =0.3183), as was done by Newman[9],

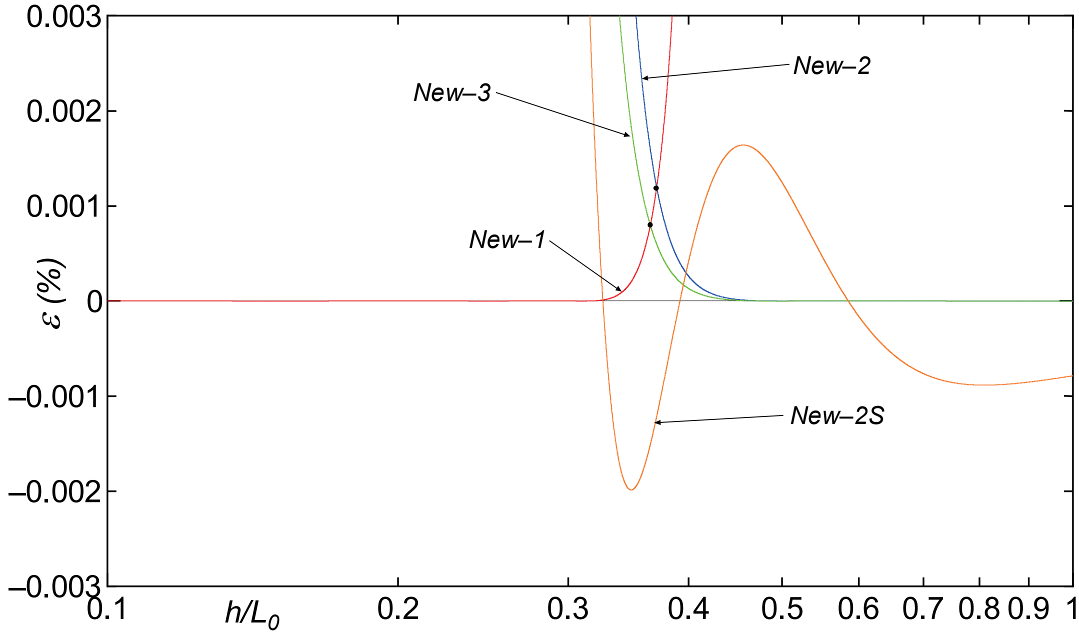

Fig. 11 Relation between relative error ε and relative water depth h/L_0 for any of New-1, New-2, New-3 and New-2S.

Eq.(75)(New-1) yields $\varepsilon=1.63\times10^{-6}$ % and Eq.(76)(New-2) $\varepsilon=1.71\times10^{-2}$ % respectively. The relative error at the connection point of $\alpha=2(h/L_0=0.3183)$ becomes discontinuous. Specifically Eq.(76)(New-2) gives large relative error around the connection point and introduces significant decrease of accuracy in Eq.(76)(New-2)-based CPAES with the connection point of $\alpha=2(h/L_0=0.3183)$.

(4) The 4th order Eq.(77)(New-3)-based ε decreases more rapidly with increasing h/L_0 than the 5th order Eq.(76)(New-2)-based ε and approaches nearly zero($\varepsilon=1.2\times10^{-7}$ %). Eq.(77)(New-3) has a little higher accuracy compared to Eq.(76)(New-2). In this context, Eq.(77)(New-3) provides $\varepsilon=1.31\times10^{-4}$ % at $h/L_0=0.400$($\varepsilon=2.59\times10^{-4}$ % for Eq.(76)(New-2)) and $\varepsilon=7.75\times10^{-5}$ % at $h/L_0=0.4100$($\varepsilon=1.51\times10^{-4}$ % for Eq.(76)(New-2)).

(5) Eq.(75)(New-1)-based relative error curve crosses the 4th order Eq.(77)(New-3)-based relative error curve at the value of $h/L_0=0.3650$, where ε takes 8.0×10^{-4} %. That is, CPAES combining Eq.(75)(New-1) with Eq.(77)(New-3)in place of Eq.(76)(New-2) yields better accuracy.

The summary of the above discussion is described as follows :

59) CPAES of New-1($h/L_0<0.3705$) and New-2($h/L_0\geq0.3705$)

$$0\%\ (h/L_0\to0)\ \sim1.20\times10^{-3}\%\ (h/L_0=0.3705)\ \sim0\%\ (h/L_0\to\infty) \tag{78}$$

60) CPAES of New-1($h/L_0<0.3650$) and New-3($h/L_0\geq0.3605$)

$$0\%\ (h/L_0\to0)\ \sim8.0\times10^{-4}\%\ (h/L_0=0.3650)\ \sim0\%\ (h/L_0\to\infty) \tag{79}$$

It should be added that the effect of the constant term of 1.22×10^{-7} in either Eq.(76)(New-2) or Eq.(77)(New-3) on the relative error is extremely small and that its removal from each equation may be possible. The PAES for either shallow water region or deeper water region proposed by Newman[9] has very high accuracy within each application range but its accuracy is lowered around an actual application limit for h/L_0 which is situated outside of the recommended application range. Even in this case, the accuracy of the above-mentioned CPAES with the maximum relative error of around 0.001% is rather high.

Table 5 indicates a list of relative error range of the investigated solutions, in cases where Table 2 in Yamaguchi and Nonaka[1] is reinforced by the results of this study such as No. 23) to 60) and the notations, 12)Hunt1 and 13) Hunt2 in Table 2 are rewritten as 12) Hunt-5 and 13) Hunt-9 respectively.

4. Conclusions

This study investigates accuracy of approximate and explicit equations (AESs) for the Airy wave theory based-wave length presented after the publication of the Yamaguchi and Nonaka[1] report in 2007 and several previous AESs non-discussed in the report, and draws out the following results :

(1) The 2 AESs by Beji[2](Beji-1, Beji-2) and those by Vatankhah and Aghashariatmadari[3](Vata-1,Vata-2) successively published in 2013 yield higher accuracy compared to the previous AESs respectively. In particular, the accuracy of Vata-2 consisting of 2 terms is very high, with an absolute value of the maximum relative error of 0.001 %. Practically, use of either Beji-2 with the maximum relative error of 0.04 % or Vata-1 with the maximum relative error of 0.02 % may be recommended for the sake of its comparatively compact form.

(2) Any of the Padé approximation-based Hunt[4] AESs expressed by a power series of h/L_0 gives relatively high accuracy, corresponding to the used order number. Respective maximum relative error(absolute value) is 0.15 % for Hunt-4, 0.08 % for Hunt-5, 0.19 % for Hunt-6 and 0.008 % for Hunt-9. Improvement of Hunt-6 is required. The Hunt AESs have a more regular form compared to the AESs expressed by real number-powered variables, exponential functions and/or hyperbolic functions, although they seem to be a lengthy expression associated with the use of a polynomial. But, accuracy of Hunt-9(maximum relative error 0.008 %) does not exceed that of Vata-2 with 2 complicated terms(0.001%).

(3) The maximum relative error(absolute value) associated with each of the Chamberlain and Porter[5] AESs is 0.16 % for Cham-4, 0.07 % for Cham-5, 0.013 % for Cham-6 and 0.0035 % for Cham-7. Sophistication of the AES gives rise to not only higher order accuracy but also more complicated formulation. As for the accuracy, Cham-7 is higher than Hunt-9 but lower than Vata-2.

(4) Maximum relative error(absolute value) with Newton's method-based 1st iteration solution which gives any of Beji-1, Beji-2, Vata-1 and Vata-2 as an initial estimate is very small such as 2×10^{-4} %, 8×10^{-6} %, 2×10^{-6} % and 6×10^{-7} % in this order. In particular, a range of relative error in the Vata-2 case is from -6×10^{-7} % 7 to 8×10^{-10} % and the solution with Vata-2 corresponds to a numerically exact solution. This is brought about by the high accuracy of Vata-2 used for an initial guess. The maximum relative error is more than 2 orders of magnitude smaller than that of the previous best estimate.

(5) Maximum relative error(absolute value) of Newton's method-based 1st iteration solution which gives any of Hunt-4, Hunt-5, Hunt-6 and Hunt-9 as an initial estimate is 2×10^{-5} %, 7×10^{-6} %, 3×10^{-5} % and 3×10^{-8} % in this order. The magnitude is comparable to the result of (3) or slightly smaller.

(6) Maximum relative error(absolute value) of Newton's method-based 1st iteration solution which gives any of Cham-2, Cham-3, Cham-4, Cham-5, Cham-6 and Cham-7 as an initial estimate is 6×10^{-4} %, 3×10^{-4} %, 7×10^{-6} %, 2×10^{-7} % and 1×10^{-8} % in succession. The magnitude becomes smaller according to this order, reflecting the accuracy of an initial estimate. Cham-5-based 1st iteration solution of Newton's method gives a fairly satisfactory evaluation with the maximum relative error of 7×10^{-6} %, even when a highly accurate estimate would be requested.

Table 5 Summary of error range of the investigated approximate solutions.

No.	formula	relative error（%）	No.	formula	relative error（%）
1)	Eckart	0～5.24	11-6)	YN7	−0.0012～0.0012
2)	Iwagaki	−3.05～3.14	11-7)	YN8	-9×10^{-4}～8×10^{-4}
3)	Carv14	−2.45～3.28	11-8)	YN9	-1.1×10^{-4}～1.1×10^{-4}
4)	FM	−1.39～1.66	11-9)	YN10	-7×10^{-6}～4×10^{-5}
5)	YN1	−1.52～1.55	12)	Hunt-5	−0.070～0.078
6)	Carv9	−1.12～0	13)	Hunt-9	−0.0082～0.0054
7)	Guo	−0.75～0.75	14)	Niel1	±0.74（$h/L_0^* \leqq 0.192$）
8)	YN2	−0.73～0.73	15)	Niel2	±0.44（$h/L_0^* \leqq 0.401$）
9)	Carv5	−0.21～0.27	16)	Vene1	±0.048（$h/L_0^* \leqq 0.165$）
10)	Carv4	−0.12～0.20	17)	WT1	−0.034～0（$h/L_0^* \leqq 0.219$）
11-1)	Fenton	−0.051～0.0084	18)	Niel3	−0.55～0（$h/L_0^* \geqq 0.300$）
11-2)	YN3	−0.0040～0.012	19)	WT2	±0.025（$h/L_0^* \geqq 0.195$）
11-3)	YN4	—0.029～0.0067	20)	You	±0.0054（$h/L_0^* \leqq 0.179$）
11-4)	YN5	−0.0049～0.0049	21)	Olson	$\pm3\times10^{-5}$（$h/L_0^* \leqq 0.186$）
11-5)	YN6	-4×10^{-4}～1.4×10^{-3}	22)	Vene2	-2×10^{-4}～6×10^{-6}（$h/L_0^* \leqq 0.159$）
23)	Beji-1	−0.15～0.19	24)	Beji-2	−0.044～0.042
25)	Vata-1	−0.019～0.019	26)	Vata-2	−0.0012～0.0012
27)	Hunt-4	−0.15～0.14	28)	Hunt-6	−0.19～0
29)	Cham-2	−0.742～0	30)	Cham-3	0.0～2.81
31)	Cham-4	−0.162～0	32)	Cham-5	−0.071～0
33)	Cham-6	−0.013～0	34)	Cham-7	−0.0035～0.0013
35)	Beji-1N	-1.6×10^{-4}～2.1×10^{-5}	36)	Beji-2N	-8.2×10^{-6}～1.1×10^{-6}
37)	Vata-1N	-1.6×10^{-6}～3.7×10^{-7}	38)	Vata-2N	-6.3×10^{-7}～8.1×10^{-10}
39)	Hunt-4N	-1.5×10^{-5}～2.1×10^{-5}	40)	Hunt-5N	-6.2×10^{-6}～7.3×10^{-6}
41)	Hunt-6N	-2.4×10^{-8}～3.4×10^{-5}	42)	Hunt-9N	-1.8×10^{-10}～3.4×10^{-8}
43)	Cham-2N	-1.4×10^{-4}～6.3×10^{-4}	44)	Cham-3N	-2.2×10^{-2}～3.1×10^{-4}
45)	Cham-4N	-3.1×10^{-4}～9.6×10^{-6}	46)	Cham-5N	-6.7×10^{-6}～1.3×10^{-6}
47)	Cham-6N	-4.8×10^{-9}～1.9×10^{-7}	48)	Cham-7N	-8.9×10^{-10}～1.3×10^{-8}
49)	You-1	0～1（$h/L_0^* \leqq 0.093$）	50)	CP-1	-0.68～0（$h/L_0^* \leqq 0.264$）
51)	You-4	−0.01～0（$h/L_0^* \leqq 0.100$）	52)	Cham-6L	-0.01～0（$h/L_0^* \leqq 0.109$）
53)	Cham-7L	0～0.01（$h/L_0^* \leqq 0.099$）	54)	CP-2	0.69（$h/L_0^* \geqq 0.264$）
55)	New-2S	0.0020～0（$h/L_0^* \geqq 0.3186$） ; $(h/L_0)_c = h/L_0^*$			
56)	You-3（Niel2）+WT2	0（$h/L_0 \to 0$）～0.186（h/L_0=0.152）～0（$h/L_0 \to \infty$）			
57)	You-3（Niel2）+Niel3	0（$h/L_0 \to 0$）～0.118（h/L_0=0.376）～0（$h/L_0 \to \infty$）			
58)	You-5（You）+Niel3	0（$h/L_0 \to 0$）～-0.476（h/L_0=0.307）～0（$h/L_0 \to \infty$）			
59)	New-1+New-2	0（$h/L_0 \to 0$）～0.0012（h/L_0=0.3705）～0（$h/L_0 \to \infty$）			
60)	New1+New-3	0（$h/L_0 \to 0$）～0.0008（h/L_0=0.3650）～0（$h/L_0 \to \infty$）			

(7) The You[6]~[8] equations with several orders applicable only to shallow water conditions and their equivalences transformable to each other to $O(\alpha^3)$ such as Cham-6L and Cham-7L, yield relatively high accuracy within their limiting ranges. Compactness and accuracy of any of the You[6]~[8] equations suggests an efficiency for practical use within its application range. In the case of reference relative error of 0.05 %, the limiting range of α^2-based Eq.(54)(You-2/WT1) is h/L_0 <0.233 and that of α^3-based Eq.(58)(You-5/You) is h/L_0 <0.217. Eq.(54)(You-2/WT1) has a wider application range than Eq.(58)(You-5/You), but in the 0.01 % relative error case, the α^2-based Eq.(54)(You-2/WT1) does not have any application range.

(8) A limited accuracy and application range of the Chamberlain and Porter[5] CP-2 PAES for deeper water conditions are not beneficial to its use. The Newman[9] New-2S PAES has a high accuracy but a narrow application range of larger h/L_0 value. This suggests its incompatibility with CPAES.

(9) CPAES tends to yield a decrease of its accuracy at around a critical h/L_0 value connecting a PAES for shallow water use with a PAES for deeper water use. For this reason, overall accuracy of CPAES is lowered, even if each PAES gives high accuracy within the application range.

(10) CPAES consisting of You-3(Niel2) and WT2 with the maximum relative error of 0.118 % at the connection of h/L_0 =0.152 is available for a moderately accurate application and CPAES consisting of New-1 and New-3 with the maximum relative error of 0.0008 % at the connection point of h/L_0 =0.365 is recommended for a highly accurate application. The latter CPAES with 0.0008 % error indicates a further higher accuracy than a single expression-based Vata-2 with 0.0012 % error. On the other hand, both the Wu and Thornton[1] CPAES composed of WT1 and WT2 and the Chamberlain and Porter[5] CPAES composed of CP-1 and CP-2 give unfavorable discontinuity of the relative error at the connection point of h/L_0 respectively.

(11) In summary, it may be said that Vata-1 with the maximum relative error of 0.02 % makes it possible to easily estimate the wave length with a satisfactory accuracy for a full range of h/L_0 . Needless to say either any single expression-based AES of Cham-7, Vata-2 and Hunt-9 or New-1 and New-3-composed CPAES is available for more accurate estimation and thus, Newton’s method-based 1st iteration solution with an initial guess based on one of these AESs is applicable for extremely high accuracy estimation.

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Appendix A

1. PAESs for wave length computation based on long wave approximation

Expanding the dispersion relationship based on the Airy wave theory by fixing the power number in the total form of a polynomial under the assumption of $\alpha < 1$ yields each of the following PAESs to $O(\alpha^3)$ in succession :

$$\beta = \alpha^{1/2}\{1 + (1/6)\alpha + (11/360)\alpha^2 + (17/5040)\alpha^3\}, \quad \text{[You-4]} \tag{A-1}$$

$$\beta = \alpha^{1/2}\{1 + (1/3)\alpha + (4/45)\alpha^2 + (16/945)\alpha^3\}^{1/2}, \quad \text{[You-5/You]} \tag{A-2}$$

$$\beta = \alpha^{1/2}\{1 - (1/3)\alpha + (1/45)\alpha^2 + (1/189)\alpha^3\}^{-1/2}, \quad \text{[Cham-6L/Olson3]} \tag{A-3}$$

$$\beta = \alpha^{1/2}\{1 - (2/3)\alpha + (7/45)\alpha^2 - (4/945)\alpha^3\}^{-1/4}, \quad \text{[Cham-7L/YNH]} \tag{A-4}$$

Eq.(A-1) is equal to the You-4 given in Eq.(57), Eq.(A-2) the You-5/You given in Eq.(58), Eq.(A-3) the Cham-6L/Olson3 corresponding to the 3rd order Olson[1] PAES and Eq.(A-4) is a PAES derived newly in this study. These PAESs are transformable to each other to $O(\alpha^3)$. As shown in Fig. A-1, the accuracy obtained by computation is ranked with very small difference from lowest Eq.(A-1)(You-4) to highest Eq.(A-3)(Cham-6L/Olson3) throughout Eq.(A-4)(Cham-7L/YNH) and Eq.(A-2)(You-5/You) in its order, in cases where the number enclosed in the parentheses of the notation in the figure indicates the order number of the total polynomial. Each PAES provides high accuracy for a range of h/L_0 less than 0.02 and approaches the exact solution with decreasing α. The relative error (absolute value) of each PAES at h/L_0 =0.02 is 1.25 $\times 10^{-4}$ % for Eq.(A-1)(You-4), 1.24 $\times 10^{-4}$ % for Eq.(A-2)(You-5/You), 1.00 $\times 10^{-4}$ % for Eq.(A-3)(Cham-6L/Olson3) and 1.04 $\times 10^{-4}$ % for Eq.(A-4)(Cham-7L/YNH) respectively, that is around 10^{-4} %. When the level of the relative error(absolute value) is taken as either 0.01% or 0.1%, which is much higher than the 10^{-4} % level, the corresponding h/L_0 is 0.100(0.169 in the 0.1% case) for Eq.(A-1) (You-4), 0.185(0.238) for Eq.(A-2)(You-5/You), 0.109(0.194) for Eq.(A-3)(Cham-6L/Olson3) and 0.099(0.160) for

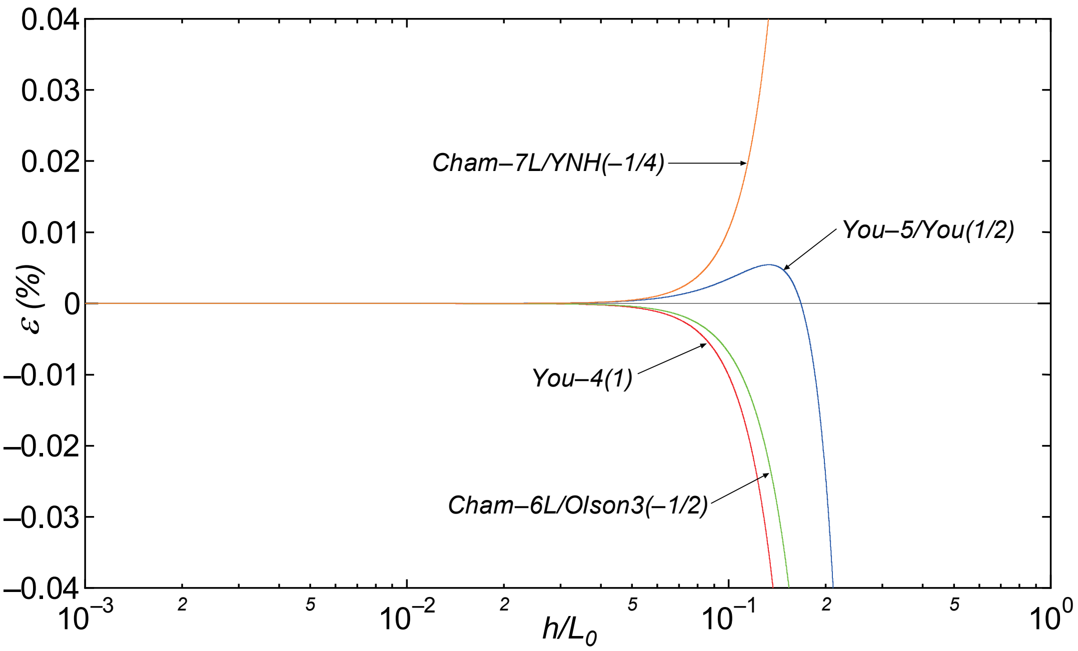

Fig. A-1 Relation between relative error ε and relative water depth h/L_0 for any of You-4, You-5/You, Cham-6L/Olson3 and Cham-7L/YNH.

Eq.(A-4)(Cham-7L/YNH).

In the following sections, the error behavior of Beji-nL , Vata-nL, Hunt-nL and Cham-nL PAESs respectively reduced to long wave conditions is investigated by taking Eq.(A-1)(You-4) with the simplest form for a standard PAES in the long wave region, in cases where n is a specified number of each PAES and ' L ' is added to describe long wave conditions.

2. Long wave approximation to Beji-n and Vata-n AESs

Long wave approximation(LWA)-based PAES for any of Eq.(4)(Beji-1), Eq.(5)(Beji-2), Eq.(6)(Vata-1) and Eq.(7)(Vata-2) β_l and the residual expression from Eq.(A-1)(You-4) to $O(\alpha^2)$ $\Delta\beta$ denoted by addition of 'R', as exemplified by Beji-1LR with underline in succession as :

$$\beta_l = \alpha^{1/2}\left(1+0.3328711\alpha^{1.3}+(1/6)\alpha^2-0.6657422\alpha^{2.3}+0.7212207\alpha^{3.3}\right),\ \text{[Beji-1L]} \tag{A-5}$$

$$\Delta\beta = \alpha^{1/2}\left\{-(1/6)\alpha+0.332871\alpha^{1.3}+(49/360)\alpha^2\right\},\ \underline{\text{[Beji-1LR]}} \tag{A-6}$$

$$\beta_l = \alpha^{1/2}\left(1+0.212248\alpha^{1.09}+(1/6)\alpha^2-0.2759224\alpha^{2.09}+0.1688787\alpha^{3.09}\right),\ \text{[Beji-2L]} \tag{A-7}$$

$$\Delta\beta = \alpha^{1/2}\left\{-(1/6)\alpha+0.212248\alpha^{1.09}+(49/360)\alpha^2\right\},\ \underline{\text{[Beji-2LR]}} \tag{A-8}$$

$$\beta_l = \alpha^{1/2}\left(1+0.1596135\alpha+(1/6)\alpha^2-0.195526537\alpha^{2.35}+0.02660225\alpha^3\right),\ \text{[Vata-1L]} \tag{A-9}$$

$$\Delta\beta = \alpha^{1/2}\left\{-0.007054\alpha+(49/360)\alpha^2\right\},\ \underline{\text{[Vata-1LR]}} \tag{A-10}$$

$$\begin{aligned}\beta_l &= \alpha^{1/2}\left(1+0.0407622\alpha+(1/6)\alpha^2-0.04076221\alpha^{2.65}+0.00679370\alpha^3\right)+A,\\ A &= \alpha^{1.66702}\left(1-2.526233z+3.392794z^2-3.133044z^3+2.024330z^4-0.7059084z^5\right),\\ &\qquad z=\alpha^{0.132},\ \text{[Vata-2L]}\end{aligned} \tag{A-11}$$

$$\begin{aligned}\Delta\beta = \alpha^{1/2}\{&-0.125904\alpha+1.0\alpha^{1.16702}-2.526233\alpha^{1.29902}+3.392794\alpha^{1.431022}-3.133044\alpha^{1.563022}\\ &+2.024330\alpha^{1.695022}-0.7059084\alpha^{1.827022}+O(\alpha^{1.959022})\},\ \underline{\text{[Vata - 2LR]}}\end{aligned} \tag{A - 12}$$

in cases where subscript ' l ' is added to β in order to distinguish the LWA-based PAES from the original AES. The 1st term and 2nd term(A term) in Eq.(A-11) are called Vata-21L and Vata-22L respectively. As the above-mentioned LWA-based PAESs include the α^a terms with a different real power number a, their mutual relationship is not so clear in the derived form.

Also, the 2nd term in Eq.(7)(Vata-2) named Vata-22 is transformed into

$$\begin{aligned}\text{Vata - 22} &= \alpha\times\{1-\exp(\text{-z})\}^{(5.0532+2.158\alpha^{1.505})} = \alpha\times z^{(5.0532+2.158\alpha^{1.505})}\times\left\{\frac{1-\exp(-z)}{z}\right\}^{(5.0532+2.158\alpha^{1.505})}\\ &= \alpha\times\alpha^{\{0.132\times(5.0532+2.158\alpha^{1.505})\}}\times\left\{\frac{1-\exp(-z)}{z}\right\}^{(5.0532+2.158\alpha^{1.505})}\\ &= \alpha\times\alpha^{0.6670224}\times\alpha^{0.284856\alpha^{1.505}}\times\left\{\frac{1-\exp(-z)}{z}\right\}^{(5.0532+2.158\alpha^{1.505})}\\ &= \alpha^{1.6670224}\times\alpha^{0.284856\alpha^{1.505}}\times\left\{\frac{1-\exp(-z)}{z}\right\}^{(5.0532+2.158\alpha^{1.505})} = ①\times②\times③\end{aligned} \tag{A - 13}$$

The A term(Vata-22L) in the above-mentioned Eq.(A-11) valid for $\alpha < 0.01$ is obtained by taking notice of ② $\alpha^{0.284856\alpha^{1.505}} \approx 1$ which ranges from 0.99872 for α =0.01 to nearly 1 for $\alpha \approx 0$ such as 0.9999399 for α =0.001 and by approximating ③the 3rd term with use of the 5th order polynomial of z. Accuracy of the polynomial with an absolute error less than 0.001 is very high for a range of $\alpha < 0.01$.

Fig. A-2 indicates the relation between any of various kinds of Vata-2-based wave number characteristics and relative water depth h/L_0 . These are ①Vata-21- and Vata-22-based dimensionless wave numbers β_1

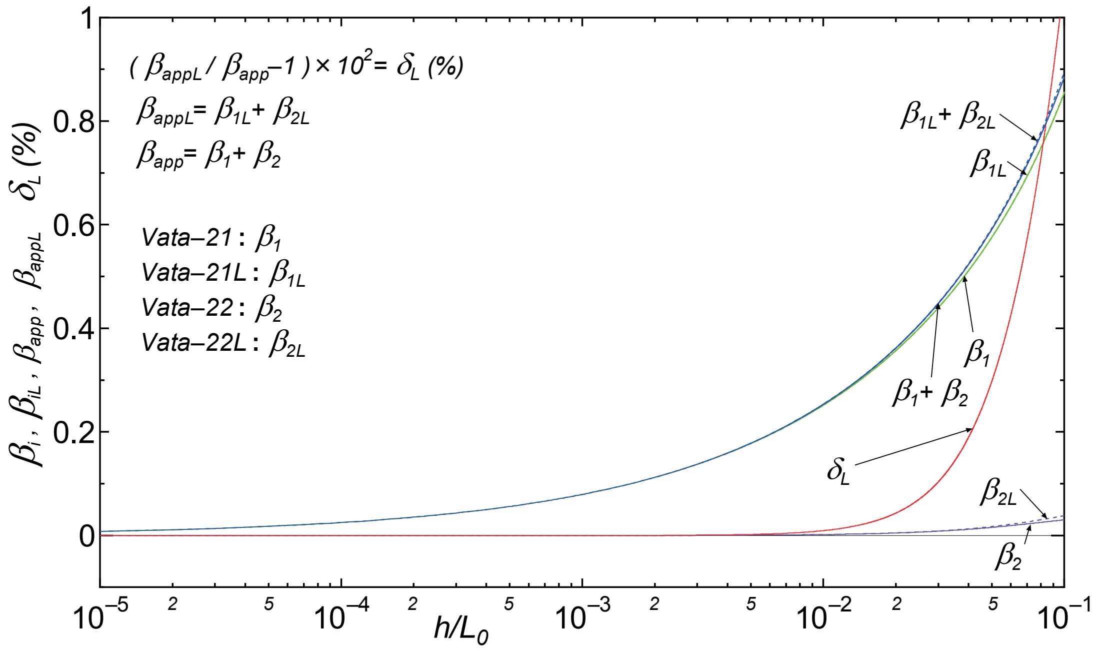

Fig. A-2 Change of any of dimensionless wave numbers β_1, β_2, β_{1L}, β_{2L}, β_{app}, β_{appL} and relative difference of wave number δ_L associated with increase of relative water depth h/L_0 in the case of Vata-2.

and β_2, ②Vata-21L- and Vata-22L-based dimensionless wave numbers β_{1L} and β_{2L}, ③the sums such as $\beta_{app}(=\beta_1+\beta_2)$ and $\beta_{appL}(=\beta_{1L}+\beta_{2L})$, and ④percentage of relative difference between β_{appL} and β_{app} defined by $\delta_L=(\beta_{appL}/\beta_{app}-1)\times 10^2$. Since accuracy of Vata-2-based β_{app} is very high, δ_L would be nearly regarded as the exact solution(β_{exac})-based $(\beta_{appL}/\beta_{exac}-1)\times 10^2$, that is a relative error of LWA-based β_{appL}. It is observed from this figure that Vata-21-based β_1 is an order of magnitude greater than Vata-22-based β_2 and that β_2 plays the role of correction term to β_1. While the effect of LWA on β_1 or β_2 becomes slightly greater with increasing h/L_0, it may not be as significant within the range of h/L_0 given in the figure. This results in a gross agreement between β_1 and β_{1L} and between $\beta_1+\beta_2$ and $\beta_{1L}+\beta_{2L}$ respectively. But, Vata-2L-based relative difference of wave number δ_L rises gradually and then rapidly with increasing h/L_0 for a range of h/L_0 >0.01. In this connection, δ_L takes 0.0095% at h/L_0 =0.01, 0.30% at h/L_0 =0.05 and 1.08% at h/L_0 =0.1. If a critical value for relative wave number difference is given as 0.1%, then an effective range of LWA for Vata-2 would be h/L_0 <0.03. Of course, the effective range depends on a selected critical value. The critical value of h/L_0 for a 1% relative wave number difference becomes somewhat larger such as 0.0957~0.1.

Fig. A-3 illustrates the relation between relative difference of wave number δ_L and relative water depth h/L_0 for any of Beji-1L/Beji-1, Beji-2L/Beji-2, Vata-1L/Vata-1 and Vata-2L/Vata-2, in cases where LWA-based dimensionless wave number β_{appL} and the original equation-based dimensionless wave number β_{app} are used. Each relative difference δ_L rapidly develops toward the positive or negative side with augmentation of h/L_0 in a range of h/L_0 >0.005~0.01. If a reference level to δ_L is taken as either 1 % or 0.1%, the critical h/L_0 is 0.0281 (0.00974 in the case of 0.1 %) for Beji-1, 0.0980(0.0541) for Beji-2, 0.102(0.0496) for Vata-1 and 0.0957(0.0295) for Vata-2. For Beji-1, the critical h/L_0 takes a smaller value. For any of Beji-2, Vata-1 and Vata-2, it gives a comparable value of 0.01 for δ_L =1% and nearly twice the difference such as from 0.03 to 0.05 for δ_L =0.1%.

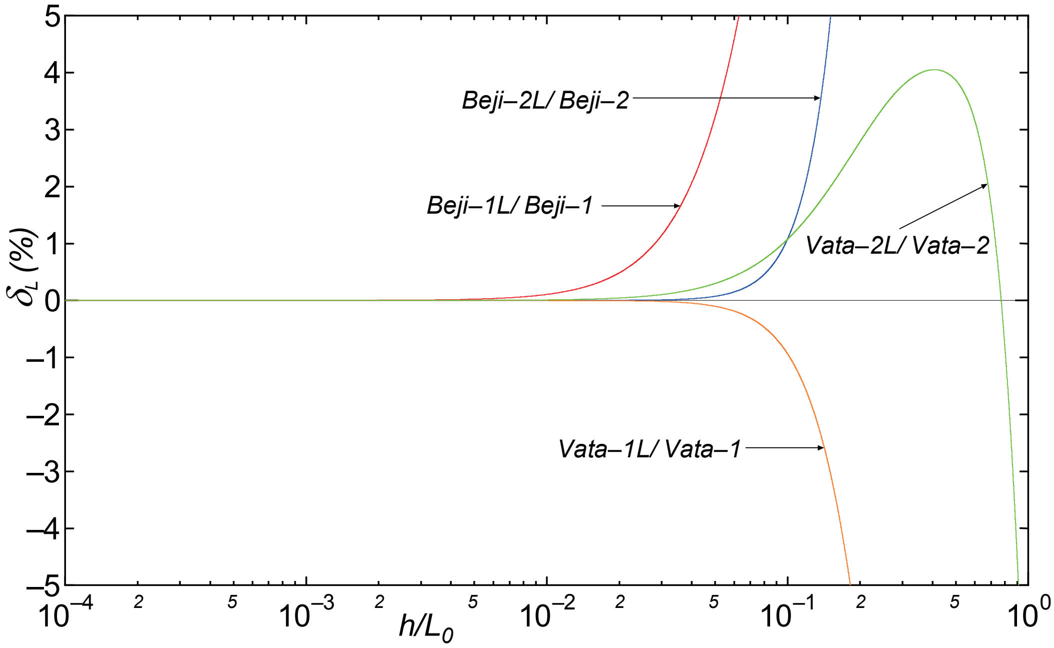

Fig. A-3 Relation between relative difference of wave number δ_L and relative water depth h/L_0 for any of Beji-1L/Beji-1, Beji-2L/Beji-2, Vata-1L/Vata-1 and Vata-2L/Vata-2.

3. Long wave approximation to Hunt-n AESs

The LWA-based expression for Hunt-4 β_l, β_a to $O(\alpha^2)$ and residual of β_a to $O(\alpha^2)$ from Eq.(A-1)(You-4) $\Delta\beta$ are written as

$$\beta_l = \alpha^{1/2}\left(1+0.334\alpha-0.001444\alpha^2+0.40233\alpha^3-0.060924\alpha^4\right)^{1/2}, \text{ [Hunt-4L]} \tag{A-14}$$

$$\beta_a = \alpha^{1/2}\left\{1+0.167\alpha-0.014665\alpha^2\right\} \tag{A-15}$$

$$\Delta\beta = \alpha^{1/2}\times\left(-0.045221\alpha^2\right), \text{ \underline{[Hunt-4LR]}} \tag{A-16}$$

in cases where β_l is obtaied by inverting the denominator in Hunt-4 and its truncation to $O(\alpha^4)$, and β_a is derived by transformation of the whole polynomial with 1/2 power into a usual polynomial. For any of Hunt-5, Hunt-6 and Hunt-9, β_l, β_a and $\Delta\beta$ are described as follows respectively :

$$\beta_l = \alpha^{1/2}\left(1+0.3478\alpha-0.036835\alpha^2+0.32547\alpha^3-0.28165\alpha^4+0.022108\alpha^5\right)^{1/2}, \text{ [Hunt-5L]} \tag{A-17}$$

$$\beta_a = \alpha^{1/2}\left(1+0.1739\alpha-0.033538\alpha^2\right) \tag{A-18}$$

$$\Delta\beta = \alpha^{1/2}\left\{0.00723\alpha-0.064094\alpha^2\right\}, \text{ \underline{[Hunt-5LR]}} \tag{A-19}$$

$$\beta_l = \alpha^{1/2}\left\{1+\left(\frac{1}{3}\right)\alpha+\left(\frac{4}{45}\right)\alpha^2+\left(\frac{16}{945}\right)\alpha^3+\left(\frac{16}{14175}\right)\alpha^4-\left(\frac{320}{467775}\right)\alpha^5-\left(\frac{64}{5}\right)\left(\frac{106}{218295}-\frac{153}{299413}\right)\alpha^6\right\}^{1/2},$$

$$\text{[Hunt-6L]} \tag{A-20}$$

$$\beta_a = \alpha^{1/2}\left\{1+(1/6)\alpha+(11/360)\alpha^2\right\} \tag{A-21}$$

$$\Delta\beta = \alpha^{1/2}\times O\left(\alpha^3\right), \text{ \underline{[Hunt-6LR]}} \tag{A-22}$$

$$\beta_l = \alpha^{1/2}\left(1+0.33333\alpha+0.088949\alpha^2+0.016862\alpha^3+0.0011647\alpha^4-0.0006938\alpha^5\right.$$
$$\left.-0.00032505\alpha^6-0.00007426\alpha^7+0.00000458\alpha^8-0.00001823\alpha^9\right)^{1/2}, \text{[Hunt - 9L]} \quad \text{(A - 23)}$$
$$\beta_a = \alpha^{1/2}\left(1+0.16667\alpha+0.030586\alpha^2\right) \quad \text{(A-24)}$$
$$\Delta\beta = \alpha^{1/2}\times\left(0.000030\alpha^2\right), \quad \underline{\text{[Hunt-9LR]}} \quad \text{(A-25)}$$

It is noted that the decimal equal to the fraction in the last term of Eq.(A-20) is 0.00032535.

Fig. A-4 shows the relation between relative difference of number δ_L and relative water depth h/L_0 for any of Hunt-4L/Hunt-4, Hunt-5/Hunt-5, Hunt-6L/Hunt-6 and Hunt-7L/Hunt-7. The critical value of h/L_0 for a reference level(absolute value) of δ_L with 1% or 0.1% is 0.081(0.042 in the case of 0.1%) for Hunt-4, 0.135(0.086) for Hunt-5, 0.449(0.276) for Hunt-6 and 0.367(0.279) for Hunt-9. The critical h/L_0 becomes greater with the improved degree of the approximation in Hunt-*n* AESs. One exception is that the critical h/L_0 for Hunt-6 is larger than that for Hunt-9 at the 1% level of δ_L. The critical h/L_0 for each of Hunt-*n* AESs tends to take a generally larger value compared to that for any of the other AESs, because the number of terms in the LWA-based Hunt-*n*L equation is greater than at least 4.

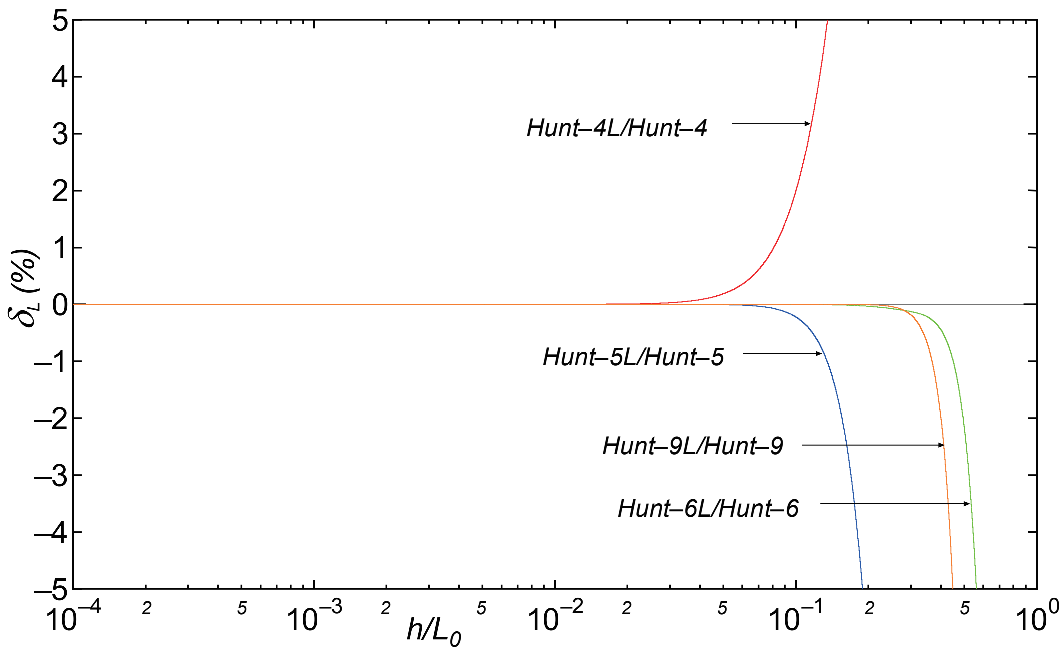

Fig. A-4 Relation between relative difference of wave number δ_L and relative water depth h/L_0 for any of Hunt-4L/Hunt-4, Hunt-5L/Hunt-5, Hunt-6L/Hunt-6 and Hunt-9L/Hunt-9.

4. Long wave approximation to Cham-*n* AESs

For any of Cham-*n* AESs (*n*=2, 3, 4, 5), LWA-based β_l, whole polynomial with usual form to $O(\alpha^2)$ β_a and residual of β_a to $O(\alpha^2)$ from Eq.(A-1)(You-4) $\Delta\beta$ are written in succession as follows :

$$\beta_l = \alpha^{1/2}\left\{1-(1/3)\alpha+(1/45)\alpha^3\right\}^{-1/2}, \quad \text{[Cham-2L]} \quad \text{(A-26)}$$
$$\beta_a = \alpha^{1/2}\left\{1+(1/6)\alpha+(1/24)\alpha^2\right\} \quad \text{(A-27)}$$
$$\Delta\beta = \alpha^{1/2}\times\left\{(1/90)\alpha^2\right\}, \quad \underline{\text{[Cham-2LR]}} \quad \text{(A-28)}$$

$$\beta_l = \alpha^{1/2}\left\{1 + (2/3)\alpha^2 - (1/3)\alpha^3\right\}^{1/2}, \quad \text{[Cham-3L]} \tag{A-29}$$
$$\beta_a = \alpha^{1/2}\left\{1 + (1/3)\alpha^2\right\} \tag{A-30}$$

$$\Delta\beta = \alpha^{1/2}\left\{-(1/6)\alpha + (109/360)\alpha^2\right\}, \quad \underline{\text{[Cham-3LR]}} \tag{A-31}$$

$$\beta_l = \alpha^{1/2}\left\{1 + (1/3)\alpha + (1/9)\alpha^2 - (1/135)\alpha^3\right\}^{1/2}, \quad \text{[Cham-4L]} \tag{A-32}$$
$$\beta_a = \alpha^{1/2}\left\{1 + (1/6)\alpha + (1/24)\alpha^2\right\} \tag{A-33}$$
$$\Delta\beta = \alpha^{1/2} \times (1/90)\alpha^2, \quad \underline{\text{[Cham-4LR]}} \tag{A-34}$$

$$\beta_l = \alpha^{1/2}\left\{1 - (2/3)\alpha + (2/15)\alpha^2 + (2/45)\alpha^3\right\}^{-1/4}, \quad \text{[Cham-5L]} \tag{A-35}$$
$$\beta_a = \alpha^{1/2}\left\{1 + (1/6)\alpha - (7/360)\alpha^2\right\} \tag{A-36}$$
$$\Delta\beta = \alpha^{1/2} \times (-1/20)\alpha^2, \quad \underline{\text{[Cham-5LR]}} \tag{A-37}$$

It should be noted that Eq.(A-26) lacks the $\mathrm{O}(\alpha^2)$ term and that Eq.(A-29) the $\mathrm{O}(\alpha)$ term. In addition, LWA-based PAES for Cham-6(Cham-6L) and LWA-based PAES for Cham-7(Cham-7L) are given by Eq.(35) and Eq.(36) in this text respectively.

Fig. A-5 indicates the relation between relative difference of wave number δ_L and relative water depth h/L_0 for any of Cham-nL/Cham-n(n=2, 3, 4, 5, 6 and 7). In the case of reference absolute value of δ_L=1% or 0.1%, critical h/L_0 is 0.232(0.149 for 0.1% level) for Cham-2, 0.125(0.070) for Cham-3, 0.248(0.127) for Cham-4, 0.145(0.084) for Cham-5, 0.455(0.198) for Cham-6 and 0.249(0.159) for Cham-7 in this order. The critical h/L_0 for Cham-3 or Cham-5 is relatively small and the critical h/L_0 is rather large for Cham-6. Any of Cham-2, Cham-4 and Cham-7 takes middle value between the above-mentioned 2 h/L_0 values respectively.

As a compendium of Sections 2., 3. and 4., Table A-1 indicates a list of h/L_0 yielding a relative difference of wave number δ_L of either 0.1% or 1 % level for each of the investigated AESs. Critical value of

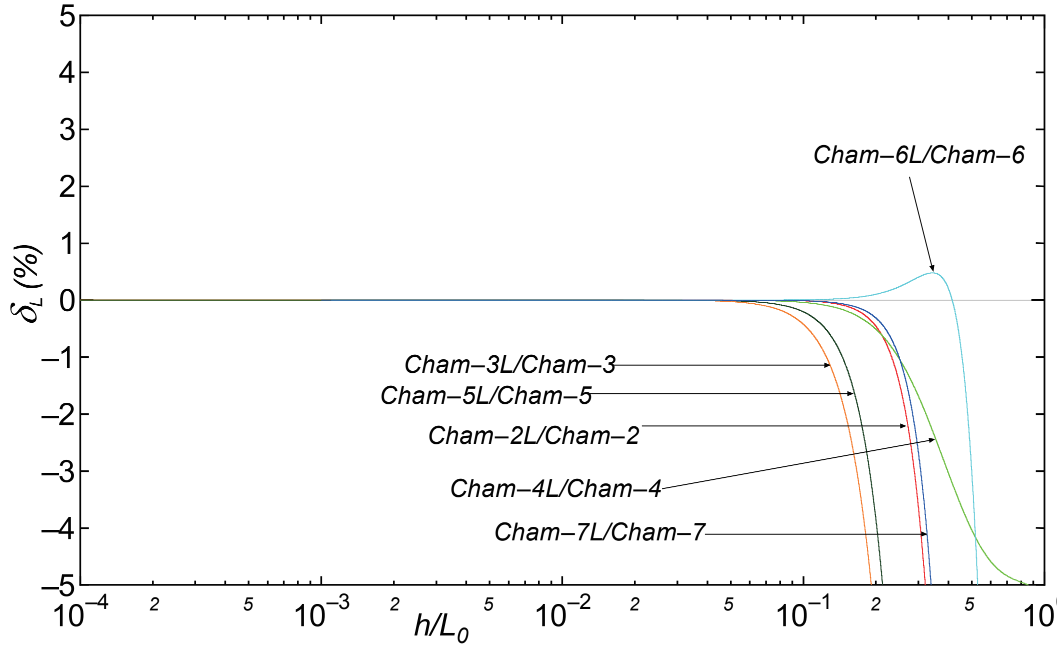

Fig. A-5 Relation between relative difference of wave number δ_L and relative water depth h/L_0 for any of Cham-2L/Cham-2, Cham-3L/Cham-3, Cham-4L/Cham-4, Cham-5L/Cham-5, Cham-6L/Cham-6 and Cham-7L/Cham-7.

Table A-1 List of critical h/L_0 yielding relative wave number difference of 0.1 % or 1 % level by long wave approximation.

App. eq.	h/L_0 (δ_L %)		App. eq.	h/L_0 (δ_L %)		App. eq.	h/L_0 (δ_L %)	
	0.1%	1%		0.1%	1%		0.1%	1%
Beji-1	0.010	0.028	Hunt-4	0.042	0.081	Cham-2	0.149	0.232
Beji-2	0.054	0.098	Hunt-5	0.086	0.135	Cham-3	0.070	0.125
Vata-1	0.050	0.102	Hunt-6	0.276	0.449	Cham-4	0.127	0.248
Vata-2	0.030	0.096	Hunt-9	0.279	0.367	Cham-5	0.084	0.145
						Cham-6	0.198	0.455
						Cham-7	0.159	0.249

h/L_0 where the effect of LWA starts emerging in the approximate computation of wave number significantly changes depending on the kind and the approximation order of AESs. The critical value of h/L_0 is greater in the cases of higher order Hunt-*n* and Cham-*n* AESs and smaller in the cases of Beji-*n* and Vata-*n* AESs.

5. Comparison of residual terms in LWA solutions

Residual terms with a different power number of α in LWA solutions which were described in Sections 2, 3 and 4 make mutual comparison in analytical form difficult. In this section, the magnitude of residual in each LWA-based PAES is investigated by taking Eq.(A-1)(You-4)-based β as a reference value, in cases where β may be close to the exact solution with relative error less than 1.3×10^{-4} % for a range of h/L_0 <0.02.

Table A-2 summarizes the coefficients of the residual terms to $O(\alpha^2)$, percentage of relative residual $\Delta\beta$ to Eq.(A-1)(You-4)-based β ; $\Delta\beta/\beta$ for any of $h/L_0=10^{-3}$, 5×10^{-4} and 10^{-4} and its order placed from the largest $\Delta\beta/\beta$ in the 12 LWA-based PAESs. Both Cham-6 and Cham-7 are excluded, because either of their LWA-based PAESs, that is Eq.(35)(Cham-6L/Olson3) and Eq.(36)(Cham-7L/YNH) coincides with Eq.(A-1)(You-4) to $O(\alpha^3)$. The order in any of the 3 h/L_0 groups is the same except for the 4th, 5th and 6th $\Delta\beta/\beta$. The residual equations such as Cham-3LR, Beji-1LR and Beji-2LR give the 1st, 2nd and 3rd largest residual in this order. Each residual equation keeps $O(\alpha)$ term with the coefficient of 1/6 in its expression of $O(\alpha^2)$. Also, either Vata-11LR or Hunt-5LR keeps $O(\alpha)$ term. But, the contribution of the $O(\alpha)$ term to the total residual is not significant within an indicated h/L_0 range, because magnitude of the coefficient in $O(\alpha)$ term is about 1/25 compared to that(1/6) in any of Cham-3LR, Beji-1LR and Beji-2LR. Since the residual equation Vata-2LR consists of 7 α^a terms with a gradually increasing power number *a* between $O(\alpha)$ and $O(\alpha^2)$, relative estimation of the contribution rate of each term is a hard task. All terms-based contribution, which takes the 4th largest place for the $h/L_0=\times10^{-4}$ case and the 6th largest place for both the $h/L_0=5\times10^{-4}$ and 10^{-3} cases is not classified into a group with the highest rank.

Fig. A-6 shows the relation between the relative residual of wave number $\Delta\beta/\beta$ and h/L_0 for the 1st largest to the 6th largest residual case such as Cham-3LR, Beji-1LR, Beji-2LR, Vata-2LR, Hunt-5LR and Vata-1LR. As indicated in Table A-2, the residuals computed by Cham-3LR and Beji-1LR increase toward the negative side more rapidly compared to the residuals of the other expressions.

Table A-2 Coefficients of residual terms and relative residual with its order.

Name of approx.	Coeff. of residual terms			Percentage of residual to original eq. $\Delta\beta/\beta(\text{You-4})$% and order			
	α	α^2	α^a	$h/L_0=10^{-3}$	$h/L_0=5\times10^{-4}$	$h/L_0=10^{-4}$	order
Beji-1LR	-1/6	49/360	0.332871 $\alpha^{1.3}$	-0.058	-0.034	-0.0082	2
Beji-2LR	-1/6	49/360	0.212248 $\alpha^{1.09}$	-0.020	-0.013	-0.0036	3
Vata-1LR	-0.007054	49/360		-0.0039 *5	-0.0021 *5	-0.00044	6
Vata-2LR	-0.125905		sum of 6 terms	-0.0011 *6	-0.00099 *6	-0.00064	4
Hunt-4LR		-0.044522		-1.8×10^{-4}	-4.5×10^{-5}	-1.8×10^{-6}	8
Hunt-5LR	0.00723	-0.064094		0.0043 *4	0.0022 *4	0.00045	5
Hunt-6LR	0	0					
Hunt-9LR		0.000030		1.2×10^{-7}	3.0×10^{-8}	1.2×10^{-9}	11
Cham-2LR		1/90		4.4×10^{-5}	1.1×10^{-5}	4.4×10^{-7}	9
Cham-3LR	-1/6	109/360		-0.10	-0.052	-0.010	1
Cham-4LR		1/90		4.4×10^{-5}	1.1×10^{-5}	4.4×10^{-7}	9
Cham-5LR		-1/20		-2.0×10^{-4}	-4.9×10^{-5}	-2.0×10^{-6}	7

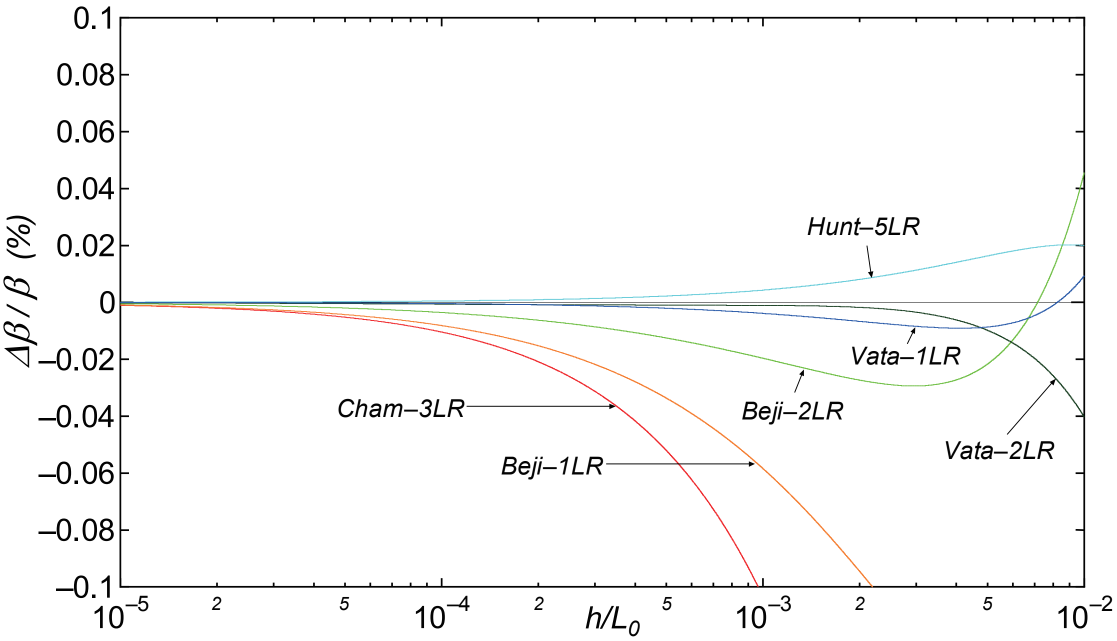

Fig. A-6 Relation between percentage of residual wave number against You-4 PAES-based wave number $\Delta\beta/\beta$ and relative water depth h/L_0 for any of Beji-1LR, Beji-2LR, Vata-1LR, Vata-2LR, Hunt-5LR and Cham-3LR.

6. Summary

The above discussion leads us to the following summary :

1) The critical value of h/L_0 where the effect of long wave approximation(LWA) appears in the AES-based wave number computation depends significantly on the kind and degree of approximation of AES to be applied. The critical value is comparatively larger in the case of any of the higher-order Hunt-*n* and Cham-*n* AESs, while it is smaller in the case of any of Beji-*n* and Vata-*n* AESs.

2) Either Cham-3- or Beji-1-based wave number computation yields larger relative error compared to the other AESs-based computations even in a range of h/L_0 <0.001 where long wave approximation valid for h/L_0 <0.02 condition works with satisfactory accuracy. This is due to their basic characteristics that either of the AESs deviates from long wave theory-based AES with a magnitude of $\mathrm{O}(\alpha)$ in a long wave approximation range.

Appendix B

Accuracy of Newton's method-based 2-step AES

Table B-1 shows the order descending from the largest absolute value of the maximum relative error indicated with a bold-faced figure $\varepsilon_{max}^{(1)}$, the AES name and a range of relative error $\varepsilon^{(1)}$ for each of 25 kinds of single expression-based (1-step) AESs in the left column and the order of the maximum relative error in bold-faced type $\varepsilon_{max}^{(2)}$, the name, the range of relative error $\varepsilon^{(2)}$ for each of 25 kinds of Newton's method-based 2-step AESs associated with an initial value estimated by the 1-step AES on the same line and the the maximum relative error ratio γ defined by $\varepsilon_{max}^{(2)}/\varepsilon_{max}^{(1)}$ in the right column. This table is made from Table 5 in this text by re-arrangement of AESs on the basis of the magnitude of the maximum relative error $\varepsilon_{max}^{(1)}$. Each column in the list includes an AES(Vata-0 or Vata-0N) to be newly discussed in Appendix C. The 25 kinds of 1-step AESs are classified into 7 groups according to the magnitude of the maximum relative error $\varepsilon_{max}^{(1)}$ at the first stage to make the discussion easier.

In the table, the order of 2-step AES-based the maximum relative error $\varepsilon_{max}^{(2)}$ roughly corresponds to the order of 1-step AES-based maximum relative error $\varepsilon_{max}^{(1)}$. A range of the maximum relative error ratio γ is between 10^{-2} and 4×10^{-6}. The tendency is that the smaller the maximum relative error $\varepsilon_{max}^{(1)}$ of the initial estimate based on 1-step AES is, the smaller the maximum relative error ratio γ becomes. In this context, the maximum relative error ratio γ takes a rough value such as (1) 10^{-2} for the case of $\varepsilon_{max}^{(1)}$=1.5~5%, (2) 10^{-3} for the case of $\varepsilon_{max}^{(1)}$=0.75~1.1%, (3) 2×10^{-4} for the case of $\varepsilon_{max}^{(1)}$=0.15~0.3%, (4) 10^{-4} for the case of $\varepsilon_{max}^{(1)}$=0.07% and (5) 4×10^{-6} for the case of $\varepsilon_{max}^{(1)}$=0.01%. It can be said again that the smaller the maximum relative error of the initial estimate $\varepsilon_{max}^{(1)}$ is, the greater the degree of improvement in the accuracy of 2-step AES becomes. This reflects the characteristics of Newton's method associated with a quadratic convergence.

Table B-1 Order of maximum relative error, name and range of relative error for 1-step AES and 2-step AES and ratio of maximum relative error.

group	order	formula	relative error $\varepsilon^{(1)}$ (%)	order	formula	relative error $\varepsilon^{(2)}$ (%)	$\varepsilon_{max}^{(2)} / \varepsilon_{max}^{(1)}$
①	1	Eckart[1]	0～**5.24**	1	Fenton[1]	**-0.051**～0.0084	9.7×10^{-3}
	2	Carv14[1]	-2.45～**3.28**	3	YN4[1]	**-0.029**～0.0067	8.8×10^{-3}
	3	Iwagaki[1]	-3.05～**3.14**	2	YN3[1]	**-0.040**～0.012	1.3×10^{-2}
	4	Cham-3	0～**2.81**	5	Cham-3N	**-0.022**～3.1×10^{-4}	7.8×10^{-3}
②	5	FM[1]	-1.39～**1.66**	4	FM-N[1]	-0.0085～**0.023**	1.4×10^{-2}
	6	YN1[1]	-1.52～**1.55**	6	YN5[1]	**-0.0049～0.0049**	3.2×10^{-3}
	7	Carv9[1]	**-1.12**～0	7	YN6[1]	-4×10^{-4}～**1.4×10^{-3}**	1.3×10^{-3}
③	8	Guo[1]	**-0.75～0.75**	8	YN7[1]	**-0.0012～0.0012**	1.6×10^{-3}
	9	YN2[1]	**-0.73～0.73**	9	YN8[1]	**-9×10^{-4}**～8×10^{-4}	1.2×10^{-3}
	10	Cham-2	**-0.74**～0	10	Cham-2N	-1.4×10^{-4}～**6.3×10^{-4}**	8.5×10^{-4}
④	11	Carv5[1]	-0.21～**0.27**	13	YN9[1]	**-1.1×10^{-4}～1.1×10^{-4}**	4.1×10^{-4}
	12	Carv4[1]	-0.12～**0.20**	14	YN10[1]	-7×10^{-6}～**4×10^{-5}**	2.0×10^{-4}
	13	Beji-1	-0.15～**0.19**	12	Beji-1N	**-1.6×10^{-4}**～2.1×10^{-5}	8.4×10^{-4}
	14	Hunt-6[5]	**-0.19**～0	15	Hunt-6N	-2.4×10^{-8}～**3.4×10^{-5}**	1.8×10^{-4}
⑤	15	Cham-4	**-0.16**～0	11	Cham-4N	**-3.1×10^{-4}**～9.6×10^{-6}	1.9×10^{-3}
	16	Hunt-4[5]	**-0.15**～0.14	16	Hunt-4N	-1.5×10^{-5}～**2.1×10^{-5}**	1.4×10^{-4}
	17	Hunt-5[1]	-0.070～**0.078**	18	Hunt-5N	-6.2×10^{-6}～**7.3×10^{-6}**	9.4×10^{-5}
	18	Cham-5	**-0.071**～0	19	Cham-5N	**-6.7×10^{-6}**～1.3×10^{-6}	9.4×10^{-5}
⑥	19	Beji-2	**-0.044**～0.042	17	Beji-2N	**-8.2×10^{-6}**～1.1×10^{-6}	1.9×10^{-4}
	20	Vata-0	-0.016～**0.021**	22	Vata-0N	**-2.8×10^{-7}～2.8×10^{-7}**	1.3×10^{-5}
	21	Vata-1	**-0.019～0.019**	20	Vata-1N	**-1.6×10^{-6}**～3.7×10^{-7}	8.4×10^{-5}
⑦	22	Cham-6	**-0.013**～0	23	Cham-6N	-4.8×10^{-9}～**1.9×10^{-7}**	1.5×10^{-5}
	23	Hunt-9[1]	**-0.0082**～0.0054	24	Hunt-9N	-1.8×10^{-10}～**3.4×10^{-8}**	4.1×10^{-6}
	24	Cham-7	**-0.0035**～0.0013	25	Cham-7N	-8.9×10^{-10}～**1.3×10^{-8}**	3.7×10^{-6}
	25	Vata-2	**-0.0012～0.0012**	21	Vata-2N	**-6.3×10^{-7}**～8.1×10^{-10}	5.3×10^{-4}

Appendix C

Addition of a new AES to the filing list

After the completion of writing our text with two appendices A, B, we discovered that Vatankhah and Aghashariatmadari[13] had published a paper related to a new AES in an international journal by Elsevier(Coastal Engineering). They proposed a Newton's method-based 2-step AES (named Vata-0N here) with the maximum relative error $\varepsilon_{max}=2.8\times10^{-7}$ % which exceeds the accuracy of 2-step AES (Beji-2N) of $\varepsilon_{max}=8.2\times10^{-6}$ % associated with an initial estimate by 1-step AES (Beji-2) and gave the following equation to 1-step AES (Vata-0) to be used in an initial estimation.

$$\beta=\alpha\left[1+\alpha^{0.986}\exp\left\{-\left(1.863+1.198\alpha^{1.366}\right)\right\}\right]\Big/\left(\tanh\alpha\right)^{1/2}, \quad [\text{Vata-0}, \ \varepsilon_{max}=0.021\%] \qquad \text{(C-1)}$$

The equation preceding to Eq.(C-1) named Vata-1 is expressed in Eq.(6) of the text as

$$\beta=\alpha\left[1+\alpha\cdot\exp\left\{-\left(1.835+1.225\alpha^{1.35}\right)\right\}\right]\Big/\left(\tanh\alpha\right)^{1/2}, \quad [\text{Vata-1}, \ \varepsilon_{max}=0.019\%], \quad (6) \qquad \text{(C-2)}$$

Only a small difference in both equations is found in the constant terms and the power index of α term, but a large alteration may occur with the increase of the number of free parameters as the power number of α term preceding the exponential function term is changed from the integer number of 1 to a real number 0.986. Also, Vata-0-based ε_{max} of 0.021% is slightly greater than Vata-1-based ε_{max} of 0.019% and Eq.(C-1) may be regarded as a modified or an extended version of Eq.(6) or Eq.(C-2).

Fig. C-1 shows the relation between the relative error ε and relative water depth h/L_0 for either Vata-0 or Vata-1. Vata-0 yields an oscillating relative error around zero with increase of h/L_0, as does Vata-1. A range of the variation and the corresponding h/L_0 value are as follows.

61) Vata-0 : -0.016% (h/L_0 =0.268) ～0.021% (h/L_0 =0.493) (C-3)

Also, Vata-1-based result in Eq.(11) of the text is rewritten as

25) Vata-1 : -0.019% (h/L_0 =0.064) ～0.019% (h/L_0 =0.011) , (11) (C-4)

Vata-0-based maximum relative error(absolute value) ε_{max} is 0.021 %, which is indeed slightly greater than 0.019% associated with Vata-1, and then a range of relative error ε is somewhat biased to the positive side in comparison with that associated with Eq.(11) in the text, that is Eq.(C-4).

Fig. C-2 illustrates the relation between relative error ε and relative water depth h/L_0 for either Newton's method-based Vata-0N or Vata-1N, in cases where 1-step AES such Vata-0 or Vata-1 is used for an initial estimation respectively. A very small relative error ε associated with Vata-0N varies accompanying a positive or negative peak with increase of h/L_0. A range of the relative error ε including the corresponding h/L_0 is indicated as

62) Vata-0N : -2.8×10^{-7} (h/L_0 =0.053) ～2.8×10^{-7} (h/L_0 =0.271) (C-5)

As indicated in Eq.(40) of the text, Vata-1N-based result is rewritten as

37) Vata-1N : -1.6×10^{-6} % (h/L_0 =0.011) ～3.7×10^{-7} % (h/L_0 =0.277) , (40) (C-6)

A range of relative error related to Vata-0N keeps a balance between the positive and the negative maximum value, while a range with Vata-1N shows some imbalance between the negative and the positive maximum

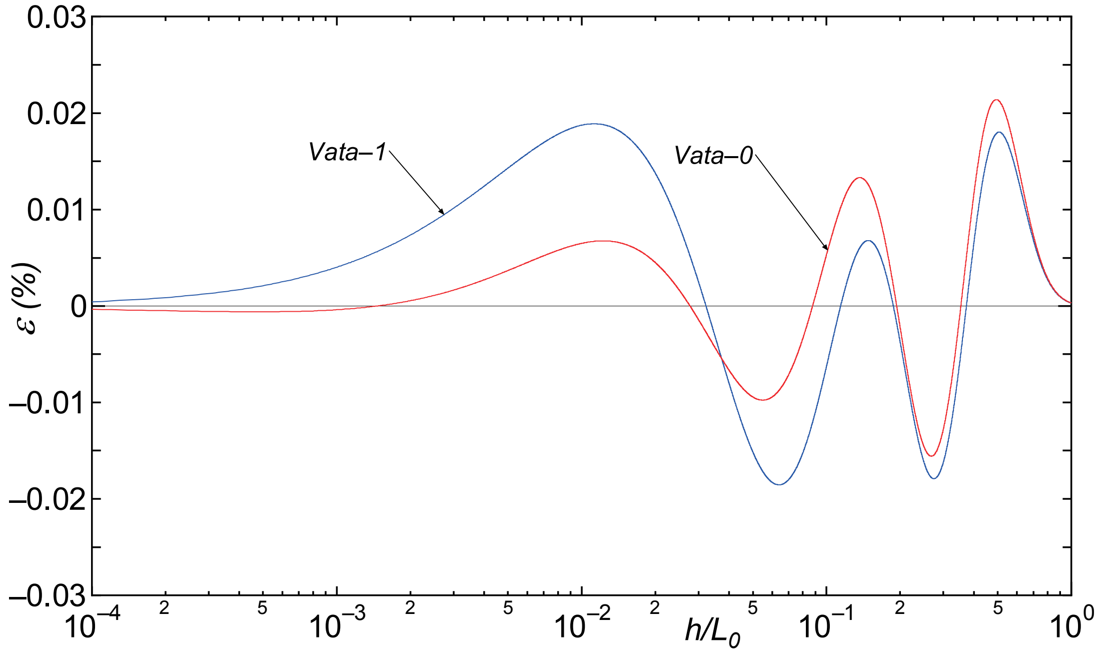

Fig. C-1 Relation between relative error ε and relative water depth h/L_0 for either Vata-0 or Vata-1.

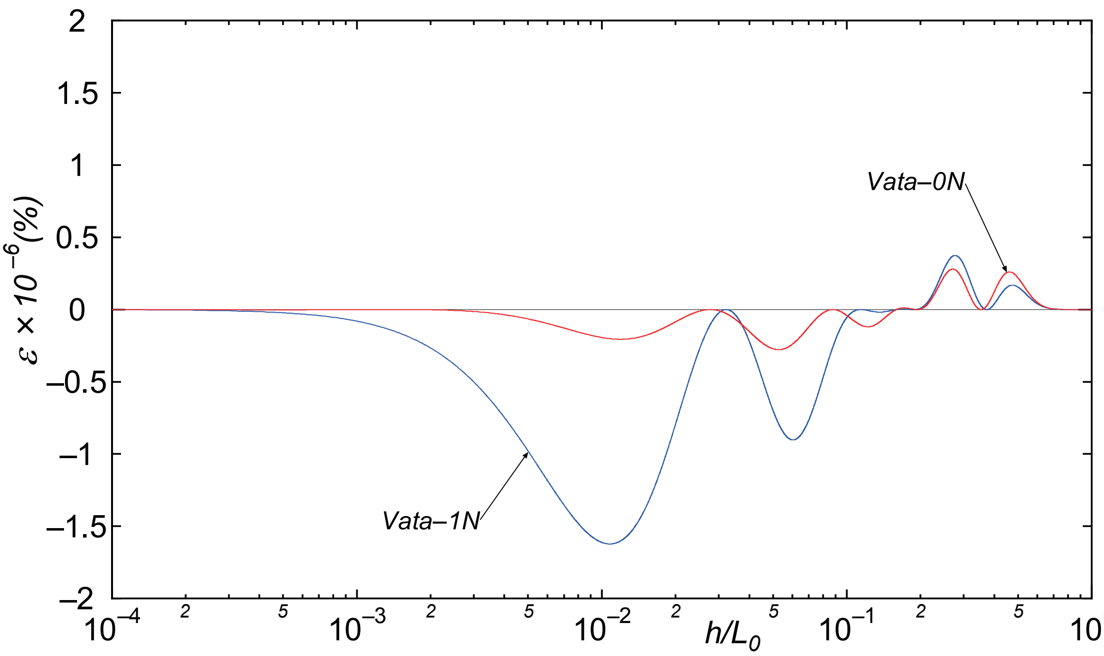

Fig. C-2 Relation between relative error ε and relative water depth h/L_0 for either Vata-0N or Vata-1N.

value and the Vata-0N related maximum relative error of ε_{max}=2.8×10^{-7} % coincides with that given by Vatankhah and Aghashariatmadari[13]. This value signifies nearly one order of magnitude decrease of the error compared to the maximum relative error of (-)1.6×10^{-6}% associated with Vata-1N. We can see a similar relation between FM+FM-N and YN1+YN5 for 1-step AES and 2-step AES in Yamaguchi and Nonaka[1], which realizes a proper balance between the negative and positive maximum relative error associated with 2-step AES and the resulting decrease of the maximum relative error by using 1-step AES with a modified power index for an initial estimation.

Table B-1 reinforced by the results with Vata-0 and Vata-0N tells us that Vata-0N-related ε_{max} of 2.8×10^{-7} % is smaller than Vata-2N-related ε_{max} of (-)6.3×10^{-7} % but greater than any of Cham-6N-related ε_{max} of 1.9×10^{-7} %, Hunt-9N-related ε_{max} of 3.4×10^{-8}% and Cham-7N-related ε_{max} of 1.3×10^{-8}%. In any case, it might be said that Vata-0N associated with a proper balance between the positive and negative maximum relative error of $\varepsilon_{max}=\pm2.8\times10^{-7}$ % yields a very highly accurate estimate.

Next, the long wave approximation(LWA)-based equation(PAES) to nearly $O(\alpha^3)$ for Vata-0 and that for Vata-1 are written in order as

$$\beta_l=\alpha^{1/2}\left\{1+0.1552063\alpha^{0.986}+(1/6)\alpha^2-0.1859372\alpha^{2.352}+0.02586772\alpha^{2.986}+O\left(\alpha^{3.718}\right)\right\},\quad \text{[Vata-0L]} \tag{C-7}$$

$$\beta_l=\alpha^{1/2}\left(1+0.1596135\alpha+(1/6)\alpha^2-0.195526537\alpha^{2.35}+0.02660225\alpha^3\right),\quad \text{[Vata-1L]},\ \text{(A-9)} \tag{C-8}$$

Eq.(C-8) is the same equation as Eq.(A-9) in the text. Also, the residual of Eq.(C-7) from Eq.(A-1)(You-4) in Appendix A to $O(\alpha^2)$ $\Delta\beta$ named Vata-0LR and that corresponding to Eq.(C-8), that is Eq.(A-10) in Appendix A named Vata-1LR are expressed as follows respectively:

$$\Delta\beta=\alpha^{1/2}\left\{0.1552063\alpha^{0.986}-(1/6)\alpha+(49/360)\alpha^2\right\},\quad \underline{\text{[Vata-0LR]}} \tag{C-9}$$

$$\Delta\beta=\alpha^{1/2}\left\{-0.007054\alpha+(49/360)\alpha^2\right\},\quad \underline{\text{[Vata-1LR],}}\ \text{(A-10)} \tag{C-10}$$

Fig. C-3 illustrates the relation between relative difference of wave number δ_L , defined in Section 2 of Appendix A and relative water depth h/L_0 for either Vata-0L/Vata-0 or Vata-1L/Vata-1. Vata-0L/Vata-0-related δ_L falls rapidly from zero toward a negative large value with increase of h/L_0 and the behavior is very similar to that of Vata-1L/Vata-1-related δ_L. Table C-1 reinforces Table A-1 by adding h/L_0 value yielding a relative difference of wave number δ_L of either 0.1% or 1% level for Vata-0L. The h/L_0 value at each level is in close agreement with that for Vata-1L.

In addition, Table C-2 shows a list of the coefficients of the residual terms to $O(\alpha^2)$, percentage of relative residual $\Delta\beta$ to Eq.(A-1)(You-4)-based β; $\Delta\beta/\beta$ for any of $h/L_0 = 10^{-3}$, 5×10^{-4} and 10^{-4} and its order placed from the largest $\Delta\beta/\beta$ in the 13 LWA-based PAESs including Vata-0L. The coefficient of $O(\alpha^2)$ term in Eq.(C-9) coincides with that in Eq.(C-10). The sum of the 1st and 2nd terms in Eq.(C-9) is

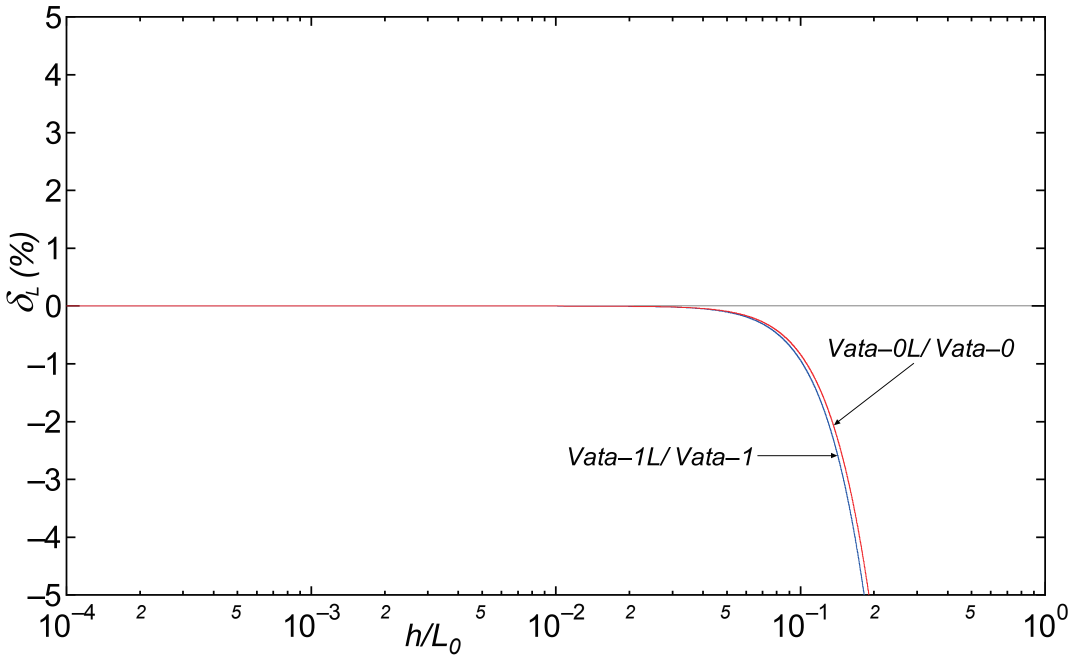

Fig. C-3 Relation between relative difference of wave number δ_L and relative water depth h/L_0 for either Vata-0L/Vata-0 or Vata-1L/Vata-1.

Table C-1 List of critical h/L_0 yielding relative wave number difference of 0.1 % or 1 % level by long wave approximation reinforced by addition of Vata-0-based results.

App. eq.	h/L_0 (δ_L %)		App. eq.	h/L_0 (δ_L %)		App. eq.	h/L_0 (δ_L %)	
	0.1%	1%		0.1%	1%		0.1%	1%
Beji-1	0.010	0.028	Hunt-4	0.042	0.081	Cham-2	0.149	0.232
Beji-2	0.054	0.098	Hunt-5	0.086	0.135	Cham-3	0.070	0.125
Vata-0	**0.051**	**0.102**	Hunt-6	0.276	0.449	Cham-4	0.127	0.248
Vata-1	0.050	0.102	Hunt-9	0.279	0.367	Cham-5	0.084	0.145
Vata-2	0.030	0.096				Cham-6	0.198	0.455
						Cham-7	0.159	0.249

Table C-2 Coefficients of residual terms and relative residual with its order reinforced by addition of Vata-0LR-based results.

Name of approx.	Coeff. of residual terms			Percentage of residual to original eq. $\Delta\beta/\beta(\text{You - 4})$% and order			
	α	α^2	α^a	$h/L_0 = 10^{-3}$	$h/L_0 = 5\times10^{-4}$	$h/L_0 = 10^{-4}$	order
Beji-1LR	-1/6	49/360	$0.332871\,\alpha^{1.3}$	-0.058	-0.034	-0.0082	2
Beji-2LR	-1/6	49/360	$0.212248\,\alpha^{1.09}$	-0.020	-0.013	-0.0036	3
Vata-0LR	**-1/6**	**49/360**	$\mathbf{0.155206\,\alpha^{0.986}}$	**0.00051**	**0.00063**	**0.00035**	**7**
Vata-1LR	-0.007054	49/360		-0.0039 *5	-0.0021 *5	-0.00044	6
Vata-2LR	-0.125905		sum of 6 terms	-0.0011 *6	-0.00099 *6	-0.00064	4
Hunt-4LR		-0.044522		-1.8×10^{-4}	-4.5×10^{-5}	-1.8×10^{-6}	9
Hunt-5LR	0.00723	-0.064094		0.0043 *4	0.0022 *4	0.00045	5
Hunt-6LR	0	0					
Hunt-9LR		0.000030		1.2×10^{-7}	3.0×10^{-8}	1.2×10^{-9}	12
Cham-2LR		1/90		4.4×10^{-5}	1.1×10^{-5}	4.4×10^{-7}	10
Cham-3LR	-1/6	109/360		-0.10	-0.052	-0.010	1
Cham-4LR		1/90		4.4×10^{-5}	1.1×10^{-5}	4.4×10^{-7}	10
Cham-5LR		-1/20		-2.0×10^{-4}	-4.9×10^{-5}	-2.0×10^{-6}	8

approximated as

$$\left\{0.155206\alpha^{0.986} - (1/6)\alpha\right\} \approx (0.0157 \sim 0.0000025)\alpha \quad \text{for} \quad \alpha = 10^{-5} \sim 5\times10^{-3} \qquad \text{(C-11)}$$

because $\alpha^{0.986} \approx \alpha$ such as ① $\alpha^{0.986} = 1.1749\times10^{-5}$ for $\alpha = 10^{-5}$, ② $\alpha^{0.986} = 1.1376\times10^{-4}$ for $\alpha = 10^{-4}$, ③ $\alpha^{0.986} = 1.1015\times10^{-3}$ for $\alpha = 10^{-3}$ and ④ $\alpha^{0.986} = 5.3850\times10^{-3}$ for $\alpha = 5\times10^{-3}$. Vata-0LR-based $\Delta\beta$ takes a positive value for α less than around 5×10^{-3} (to be accurate, around 2×10^{-3} as seen in Fig. C-4) and Vata-1LR-based $\Delta\beta$ gives a negative value for the same range of α at least. Both $\Delta\beta$ s take opposite sign, as indicated by $\Delta\beta/\beta$ for any of $h/L_0 = 10^{-3}$, 5×10^{-4} and 10^{-4} in Table C-2.

Fig. C-4 illustrates the relation between the relative residual of wave number $\Delta\beta/\beta$ and relative water depth h/L_0 for either Vata-0LR or Vata-1LR. With augmentation of h/L_0 Vata-0LR-based residual $\Delta\beta/\beta$ increases from nearly zero, takes a positive and a small negative peak and then rapidly becomes greater. On the other hand, Vata-1LR-based residual $\Delta\beta/\beta$ gives a negative value decreasing with augmentation of h/L_0 and turns into an increasing trend toward a positive value after reaching a negative peak. A smaller absolute value of residual associated with Vata-0LR than that with Vata-1LR shows that Vata-0L is a closer approximate solution to the long wave theory compared to Vata-1LR.

As discussed above, Vata-0-related relative error is similar to Vata-1-related relative error in a rough estimate but availability of Vata-0 for 1-step AES estimator does not exceed that of Vata-1 and is somewhat lower, because a balance between the positive and negative maximum relative error is slightly worse in Vata-0 than in Vata-1. However, accuracy of 2-step AES using Vata-0 for an initial estimation of wave length is nearly one order of magnitude higher than that using Vata-1, in cases where the maximum relative error takes a significantly small value such as 2.8×10^{-7} %. Vata-0-based 2-step AES has a very high accuracy. Accordingly,

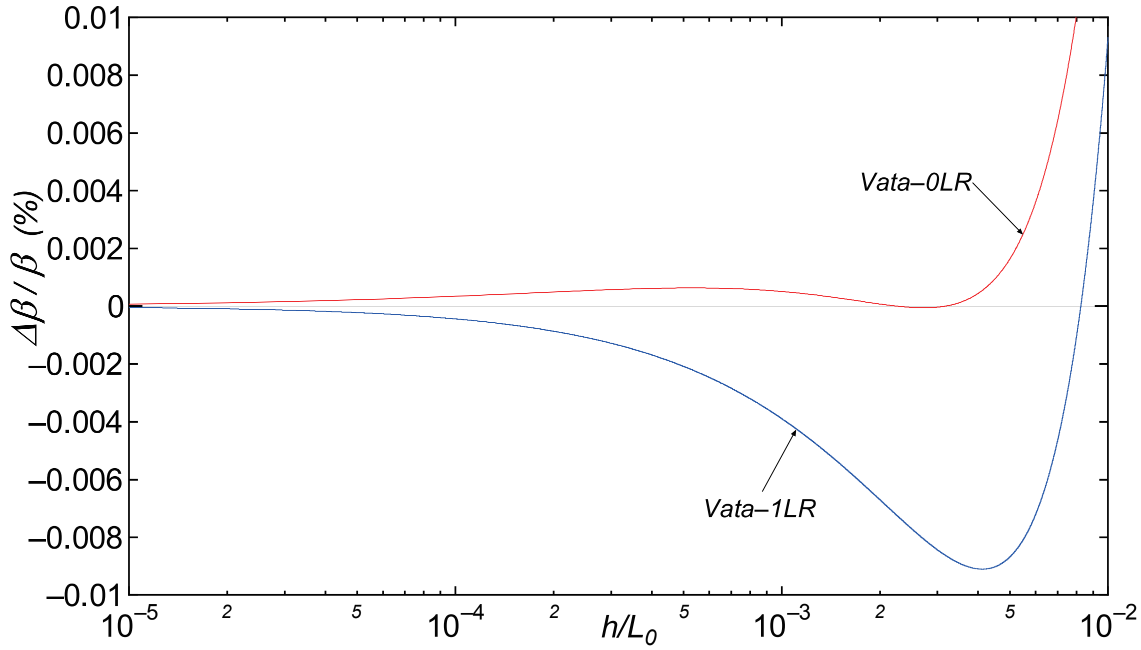

Fig. C-4 Relation between percentage of residual wave number against You-4 PAES-based wave number $\Delta\beta/\beta$ and relative water depth h/L_0 for either Vata-0LR or Vata-1LR.

it may be concluded that Vata-0 is a very efficient 1-step AES estimator for use in a Newton's method-based 2-step AES.