〈論文〉

Optimal Investment and Exit Decisions under Financial Constraints

Kazuhito KAWAGUCHI* and Hiroaki MORIMOTO**

*Department of Comprehensive Policy Making,
Faculty of Law and Letters,
Ehime University, Matsuyama 790-0826, Japan
kawa@ll.ehime-u.ac.jp
**Department of Mathematics,
Faculty of Science,
Ehime University, Matsuyama 790-0826, Japan
morimoto@mserve.sci.ehime-u.ac.jp

Abstract

We study the interaction between financial constraints and exit decisions of a firm under uncertainty. We consider a nonlinear variational inequality associated with the firm's value maximizing problem. The optimal policies are shown to exist from the optimality conditions in the inequality. The model developed herein can be solved explicitly when the exit payoff of the firm is a first order function of its replacement cost. We examine the properties of the explicit solution to explain how the irreversible investment and the exit time are distorted by the binding financial constraint. The relationship between Tobin's q theory of investment and exit strategies of the firm is discussed.

Key words : Irreversible investment ; Exit ; Financial constraint ; Variational inequality ; Penalty method
JEL classifications. D21, D84, G33, C61

1. Introduction

This paper studies the determination of the exit time and irreversibe investment rates for a private business in a stochastic environment. To analyze the relationship between the exit option and financial status, we impose a financing constraint to the firm and examine how it affects the optimal policies. We treat the model with uncertainty for the current price of an investment good to change according to a Ito process. The firm maximizes the expected present value of all future net cash flows (i. e., expected fundamental value) and has an option to sell the business once for all at any time.

Recent studies of exit decisions investigate mainly "now-or-never" investments and an option to shutdown (Alvarez[2], Dixit[7], McDonald and Siegel[19]). However, the relationship between investment and exit decisions can have dynamically interactive features. In fact, the gradual investment policy may reduce the adjustment costs of investment and lead to the higher fundamental value of the firm, which affects the firm's exit policy. This paper examines the simultaneous determination of these policies to explore the interaction between investment and exit decisions. Moreover, the integration of the exit option theory and the irreversible investment theory gives us results which have not been explicitly obtained by standard neoclassical investment theory (Jorgenson [13]) or Tobin's q theory of investment (Abel[1], Tobin[22]).

The main goal of this paper is to find the optimal investment and exit policies which maximize the expected firm's fundamental value including its scrap value or exit payoff. These optimal policies depend critically on the shape of the exit payoff function and the availability of funds. In addition, we confirm that our optimality conditions are consistent with Tobin's q theory of investment.

The remainder of the paper is organized as follows. In the next section, we present the investment and exit model of the firm and introduce the variational inequality which gives the conditions for optimality. In Section 3, we obtain optimal policies by using the penalty method (Morimoto [20], Koike and Morimoto [17]). In Section 4, we illustrate our results explicitly by considering the first order resale value function. Finally, Section 5 concludes. The investment-exit problem described here is one of optimal stopping problems in the stochastic dynamic programming (Bensoussan [4], Bensoussan and Lions [5]). All mathematical technicalities are collected in Appendix A, B and C.

2. The model

We consider a competitive firm that sells its whole business at any time τ. The firm has to integrate capital goods in the production process before they become productive and this "integration" is costly. Therefore the selling price of the firm itself is not necessarily the same as the current market value of capital goods owned by it. We assume that the current market price of the capital good P_t evolves according to the stochastic differential equation,

$$dP_t = -\nu P_t dt + \sigma P_t dW_t, \quad P_0 = p > 0, \tag{2.1}$$

where $\nu \in \mathbf{R}$ is the expected decline rate of the price, $\sigma > 0$ the standard deviation of the decline rate, and W_t denotes a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with the natural filtration $\mathcal{F}_t$ generated by $\sigma(W_s, s \leq t)$.

The firm's capital stock decreases as a result of depreciation at a constant rate $d > 0$. Hence the total change in the capital stock K_t is

$$\dot{K}_t = (z_t - d)K_t, \quad K_0 > 0 : \text{given} \tag{2.2}$$

where z_t denotes the gross investment-capital ratio. Since the investment is irreversible, z_t only takes nonnegative values.

We introduce a Penrose-Uzawa type installation or adjustment cost function in the investment decision process (see Uzawa [23] and Yoshikawa [24]).
Let $\phi : [0, \infty) \to [0, \infty)$ be the Penrose-Uzawa function satisfying

$$\phi(z) \geq 0, \quad \phi'(z) > 0, \quad \phi''(z) \geq 0$$
$$\phi(0) = 0, \quad \phi'(0) = 1, \quad \phi'(\infty) = \infty,$$

and $\phi(z_t)P_tK_t$ denotes the total cost of investing at time t. We impose an exogenous financial restriction on the firm's investing behaviour such that

$$\phi(z_t)P_tK_t \leq \overline{\phi}P_tK_t, \quad \overline{\phi} > 0.$$

This inequality implies that the maximum investing cost which the firm can afford is proportional to its replacement cost, $P_tK_t > 0$. We call the control $z = \{z_t\}$ admissible if it satisfies $0 \leq z_t \leq \phi^{-1}(\overline{\phi}) := \overline{z}$. We denote the set of all admissible controls by $\mathcal{A}$.

The objective of the firm is to maximize the expected present value $\tilde{J}(z, \tau)$ of future net cash flow up to time τ defined by

$$\tilde{J}(z, \tau) = E\left[\int_0^\tau (r - \phi(z_t))P_tK_te^{-\alpha t}dt + g(P_\tau K_\tau)e^{-\alpha\tau}\right], \quad (z, \tau) \in \mathcal{A} \times S, \tag{2.3}$$

where S denotes all stopping times with respect to the filtration $\mathcal{F}_t$. The firm has stationary expectations concerning the future profit rate $r > 0$ and the discount rate $\alpha > 0$. The terminal value function $g : [0, \infty) \to \mathbf{R}$ is called the scrap value or the exit payoff of the firm.

Changing the variable by $x_t = P_tK_t > 0$, we see that (2.3) is reduced to the one-dimensional maximization problem

$$J(z, \tau) := E\left[\int_0^\tau (r - \phi(z_t))x_te^{-\alpha t}dt + g(x_\tau)e^{-\alpha\tau}\right]$$

subject to

$$dx_t = (z_t - \mu)x_tdt + \sigma x_tdW_t, \quad x_0 = x := P_0K_0 > 0, \tag{2.4}$$

where $\mu := \nu + d$.

We make the following assumptions on α and g :

$$\alpha_0 := \overline{z} - \mu < \alpha, \tag{2.5}$$
$$g \in \mathcal{B}, \quad g \geq 0, \quad g(0) = 0, \tag{2.6}$$

where $\mathcal{B}$ denotes the Banach space of all continuous functions h on $[0, \infty)$ such that

$$\begin{cases} \|h\| := \sup_{x \geq 0}|h(x)|/(1 + x) < \infty, \\ \forall\rho > 0, \exists C_\rho > 0 ; |h(x) - h(y)| \leq C_\rho|x - y| + \rho(1 + x + y), \quad x, y \geq 0. \end{cases} \tag{2.7}$$

In the following, we solve the optimization problem of the firm and obtain the optimal investment-exit policies. To this end, we cosider the nonlinear variational inequality :

$$\begin{cases} H(x, v, v', v'') := -\alpha v + \frac{1}{2}\sigma^2x^2v'' - \mu xv' + rx + \Lambda(v')x \leq 0, \\ v(x) \geq g(x) \quad \text{for} \quad x \geq 0, \\ H(x, v, v', v'')(v - g)^+ = 0, \\ v(0) = 0, \end{cases} \tag{2.8}$$

where $\Lambda(y) := \sup_{0 \leq z \leq \overline{z}}(zy - \phi(z))$ and the supremum is attained by

$$\lambda(y)=\begin{cases} \bar{z} & \text{if} \quad y \geq \phi'(\bar{z}), \\ \phi'^{-1}(y) & \text{if} \quad 1 < y < \phi'(\bar{z}), \\ 0 & \text{if} \quad y \leq 1. \end{cases} \tag{2.9}$$

3. Optimal Policies

In this section, we present a synthesis of optimal policies for the investment-exit problem (2.3). We find a smooth solution of the nonlinear variational inequality (2.8) by using the penalty method then give our main results for the optimization problem.

Define the nonlinear penalty equation of the form :

$$H_\varepsilon(x,u,u',u''):=-\alpha u+\frac{1}{2}\sigma^2x^2u''-\mu xu'+rx+\Lambda(u')x+\frac{1}{\varepsilon}(u-g)^-=0, \quad x>0, \tag{3.1}$$

for $\varepsilon > 0$, with boundary condition

$$u(0)=0.$$

The proofs of existence and uniqueness results are rather long, so we put them all into Appendices. Theorems A. 4 and A. 6 show that a unique smooth solution $u = u_\varepsilon$ of (3.1) exists, by using the viscosity solutions method (Crandall, Ishii and Lions [6]).

We suppose that g belongs to $C^2(0,\infty)\cap C[0,\infty)$ and satisfies

$$-\alpha g+\frac{1}{2}\sigma^2x^2g''+\alpha_0xg'+\alpha h=0 \tag{3.2}$$

for some $h \in \mathcal{B}$. The simplest function with these characteristics is the first order type : $g(x)=Ax$, $A>0$.

Theorem 3. 1 *We assume (2.5), (2.6) and (3.2). Then the solutions* $\{u_{\varepsilon_n}\}$ *of (3.1) converge to* v *in* $\mathcal{B}$ *as* $\varepsilon_n := 2^{-n} \to 0$, *and* v *is a viscosity solution of (2.8).*

Theorem 3. 2 *We make the assumptions of Theorem 3. 1 Then we have*

$$v \in C^{1,1}_{loc}(0,\infty). \tag{3.3}$$

Theorem 3. 3 *We make the assumptions of Theorem 3. 1. Then the optimal policy* $(z^*,\tau^*)\in\mathcal{A}\times S$ *is given by*

$$z_t^*=\lambda\circ v'(P_tK_t^*), \tag{3.4}$$

$$\tau^*=\inf\{t\geq 0: v(P_tK_t^*)=g(P_tK_t^*)\}, \tag{3.5}$$

where

$$\dot{K}_t^*=(\lambda\circ v'(P_tK_t^*)-d)K_t^*, \quad K_0=k>0. \tag{3.6}$$

4. First order exit payoff

The special economic interest of Theorm 3.3 arises from its application to the financially constrained firm with the first order resale value : $g(x)=Ax,\ A>0$. The acquirers' bid for the target firm is greater or less than its replacement cost according to $A>1$ or $A<1$. In the following, we discuss the relationship between Tobin's q theory of investment and the firm's optimal exit decision. Throughout this section we suppose that the firm's profit rate is greater than the real interest rate : $r>\alpha+\mu$.

For simplicity of exposition we first consider a specific case where the scrap value equals the replacement cost : $g(x)=x$. Let us consider a function F : $[0,\infty)\rightarrow\mathbf{R}$ defined by

$$F(z)=(r-\phi(z))-\phi'(z)(\alpha+\mu-z)$$

which has following properties :

$$F(0)=r-(\alpha+\mu)>0,$$
$$F'(z)=-\phi''(z)(\alpha+\mu-z)<0, \qquad \text{for} \quad z<\alpha+\mu.$$

We set

$$\zeta(x):=\begin{cases}\dfrac{r-\phi(\hat{z})}{\alpha+\mu-\hat{z}}x, & \text{if} \quad F(\bar{z})<0,\\ \dfrac{r-\phi(\bar{z})}{\alpha+\mu-\bar{z}}x, & \text{if} \quad F(\bar{z})\geq 0,\end{cases} \tag{4.1}$$

where $\hat{z}(\leq\bar{z})$ is a unique root of $F=0$, if it exists. It is easily verified that ζ solves both (2.8) and (3.1) for any $\varepsilon>0$. Therefore, as we have already shown in Section 3, ζ gives the maximum fundamental value of the firm.

We observe that the more severe is the financing restriction, the lower the firm's fundamental value, since

$$\frac{\partial\zeta(x;\bar{z})}{\partial\bar{z}}=\frac{(r-\phi(\bar{z}))-\phi'(\bar{z})(\alpha+\mu-\bar{z})}{(\alpha+\mu-\bar{z})^2}>0, \quad \text{if} \quad F(\bar{z})>0.$$

Note that $F(\bar{z})>0$ means that the firm never attain its maximum value owing to the financial constraint. Furthermore, in view of the optimality condition (3.5), the exercise of the exit option may be optimal for the stakeholders of the firm if there exists a well-functioning market for corporate control.

Intuitively speaking, the shareholders of the firm will not accept the offer from the acquirers unless the price equals or exceeds the discounted present value of its net cash flows. In this first order case, this requirement can be shown as follows. By (4.1), we can take $A>0$ such that

$$A:=\begin{cases}\dfrac{r-\phi(\hat{z})}{\alpha+\mu-\hat{z}}, & \text{if} \quad F(\bar{z})<0,\\ \dfrac{r-\phi(\bar{z})}{\alpha+\mu-\bar{z}}, & \text{if} \quad F(\bar{z})\geq 0,\end{cases} \tag{4.2}$$

and Ax is a solution of (2.8) and (3.1). Hence, by (3.5), there exists some trigger value of A such that the firm should be sold at once. Consequently our example shows that the optimal exit behavior of a firm may occur owing to the severe borrowing constraint.

5. Conclusion

This paper has developed a model of investment and exit decisions of the firm and obtained the optimal policies which depend on Tobin's q and the exit payoff. Our principal results are as follows :

1) We establish the existence of a solution to the nonlinear variational inequality associated with the firm's investment-exit problem, by using the penalty method, and find the optimal policies.
2) Our optimality conditions entail Tobin's q theory of investment. Furthermore, we obtain an exact condition for determining the optimal exit time.
3) In the special case of the first order exit payoff function, we provide an explicit solution to the problem. We analyze this solution and discuss the effect of the borrowing constraints on the takeover activity. The financing constraint lowers the value of the firm and, given the severe restriction to its financing behavior, the shareholders of the firm may choose to sell the business at the bid equals to its constrained maximum value.

Theorem 3.3 shows that the optimal investment rate is positive iff the value of the firm is strictly larger than its resale value and the marginal value of its replacement cost is greater than unity. We remark that the marginal value condition is equivalent to Tobin's q theory in our stochastic context. It should be noted that the solution given by (4.2) in Section 4 is independent of the volatility of the price of the capital good. However, this result heavily depends upon the first order assumption, since the solution of (2.8) is nonlinear and $v'' \neq 0$ if g is a general nonlinear function.

There are several interesting extensions of the model analyzed here. Our analysis has focused on the competitive firm which has static expectations of the profit rates. An obvious extension of our model would consider the more general situation where the firm has various kinds of expectations, each of which has different implications for the optimal investment rates.

Appendix

A The Penalized problem

Appendix A proves the existence of a unique smooth solution of the penalty equation (3.1). Here, we introduce the notion of viscosity solutions (Crandall, Ishii and Lions [6]).

Definition A.1 *Let $u \in C[0, \infty)$ and $u(0)=0$. Then u is a viscosity solution of (3.1) if*

(i) *u is a viscosity subsolution of (3.1), that is, for any $\varphi \in C^2(0, \infty)$ and any local minimum point $x > 0$ of $u-\varphi$,*

$$H_\varepsilon(x, u, \varphi', \varphi'') \leq 0,$$

and

(ii) *u is a viscosity supersolution of (3.1), that is, for any $\varphi \in C^2(0, \infty)$ and any local maximum point $x > 0$ of $u-\varphi$,*

$$H_\varepsilon(x, u, \varphi', \varphi'') \geq 0,$$

Next, we rewrite (3.1) as

$$-(\alpha+\frac{1}{\varepsilon})u+\frac{1}{2}\sigma^2x^2u''-\mu xu'+rx+\Lambda(u')x+\frac{1}{\varepsilon}u\vee g=0,$$

and consider the equation

$$u(x)=\sup_{z\in A} E\left[\int_0^\infty e^{-(\alpha+\frac{1}{\varepsilon})t}\{(r-\phi(z_t))x_t+\frac{1}{\varepsilon}(u\vee g)(x_t)\}dt\right]. \tag{A.1}$$

Lemma A. 2 *We assume (2.5). Then there exist* $\gamma>1$ and $\nu>0$ such that

$$E[e^{-\alpha\tau}x_\tau^\gamma]+\nu E\left[\int_0^\tau e^{-\alpha t}x_t^\gamma dt\right]\leq x^\gamma \tag{A.2}$$

for any $(z, \tau)\in\mathcal{A}\times S$.

Proof. We choose $\gamma>1$ such that

$$-\alpha+\gamma\alpha_0+\frac{1}{2}\sigma^2\gamma(\gamma-1)<0,$$

and set

$$\nu=\alpha-\gamma\alpha_0-\frac{1}{2}\sigma^2\gamma(\gamma-1).$$

By Ito's formula, we have

$$e^{-\alpha t}x_t^\gamma=x^\gamma+\int_0^t e^{-\alpha t}(-\alpha+\gamma(z_t-\mu)+\frac{1}{2}\sigma^2\gamma(\gamma-1))x_t^\gamma ds+\int_0^t e^{-\alpha t}\gamma\sigma x_t^\gamma dW_t\leq x^\gamma-\int_0^t e^{-\alpha t}\nu x_t^\gamma ds$$
$$+\int_0^t e^{-\alpha t}\gamma\sigma x_t^\gamma dW_t.$$

Hence we obtain

$$E[e^{-\alpha\tau\wedge\tau_n}x_{\tau\wedge\tau_n}^\gamma]+E\left[\int_0^{\tau\wedge\tau_n}e^{-\alpha t}\nu x_t^\gamma ds\right]\leq x^\gamma,$$

where $\{\tau_n\}$ is a sequence of the localizing stopping times of the local martingale. Since $\tau_n\rightarrow\infty$ as $n\rightarrow\infty$, this proves (A.2).

Theorem A. 3 *We assume (2.5) and (2.6). Then the equation (A.1) admits a unique solution* $u=u_\varepsilon\in\mathcal{B}$ *such that* $u\geq 0$ *and* $u(0)=0$.

Proof. We define

$$Th(x)=\sup_{z\in\mathcal{A}}T_zh(x),$$
$$T_zh(x)=E\left[\int_0^\infty e^{-(\alpha+\frac{1}{\varepsilon})t}\left\{(r-\phi(z_t))x_t+\frac{1}{\varepsilon}h\vee g(x_t)\right\}dt\right] \tag{A.3}$$

for any $h\in\mathcal{B}$ and show that $T:\mathcal{B}\rightarrow\mathcal{B}$ is a contraction.

From (2.5) and the comparison theorem [16], it follows that

$$E[x_t] \leq xE[\exp(a_0 t+\sigma W_t-\frac{1}{2}\sigma^2 t)] = xe^{a_0 t}. \tag{A.4}$$

Hence, taking into account $\phi(z) \geq 0$, we have

$$0 \leq Th(x) \leq \int_0^\infty e^{-(\alpha+\frac{1}{\varepsilon})t}\left\{rxe^{a_0 t}+\frac{\|h\|+\|g\|}{\varepsilon}(1+xe^{a_0 t})\right\}dt \leq C(1+x), \tag{A.5}$$

for some constant $C>0$. This implies $\|Th\|<\infty$. Furthermore,

$$E[|x_t-y_t|] \leq |x-y|e^{a_0 t}, \qquad x, y \geq 0$$

where y_t is the solution of (2.4) with $y_0=y$. Then, by (2.7) and $(A.4)$, we find $C'_\rho>0$, for any $\rho>0$, such that

$$\begin{aligned}|Th(x)-Th(y)| &\leq \sup_{z\in A} E\left[\int_0^\infty e^{-(\alpha+\frac{1}{\varepsilon})t}\{(r+\phi(\bar{z}))|x_t-y_t|+\frac{1}{\varepsilon}(|h(x_t)-h(y_t)|+|g(x_t)-g(y_t)|)\}dt\right]\\ &\leq \sup_{z\in A} E\left[\int_0^\infty e^{-(\alpha+\frac{1}{\varepsilon})t}\{(r+\phi(\bar{z}))|x_t-y_t|+\frac{2}{\varepsilon}(C_\rho|x_t-y_t|+p(1+x_t+y_t))\}dt\right]\\ &\leq C'_\rho|x-y|+\rho(1+x+y), \qquad x, y \geq 0,\end{aligned} \tag{A.6}$$

which implies $Th \in \mathcal{B}$.

Next, we have by (2.7) and $(A.4)$

$$\begin{aligned}|Th_1(x)-Th_2(x)| &\leq \sup_{z\in A} E\left[\int_0^\infty e^{-(\alpha+\frac{1}{\varepsilon})t}\{\frac{1}{\varepsilon}(|h_1\vee g(x_t)-h_2\vee g(x_t)|)\}dt\right]\\ &\leq \sup_{z\in A} E\left[\int_0^\infty e^{-(\alpha+\frac{1}{\varepsilon})t}\{\frac{1}{\varepsilon}(1+x_t)\}dt\right]\|h_1-h_2\|\\ &\leq \frac{1+x}{(\alpha-a_0)\varepsilon+1}\|h_1-h_2\|, \quad h_1, h_2 \in \mathcal{B}.\end{aligned} \tag{A.7}$$

Thus T is a contraction mapping, which has a fixed point $u \in \mathcal{B}$. Therefore u is a unique solution to $(A.1)$. Clearly, by $\phi(0)=0$, we have $u \geq 0$. Further, by $(A.1)$

$$u(0) \leq \frac{1}{\alpha\varepsilon+1}u \vee g(0),$$

which implies $u(0)=0$.

Theorem A. 4 *We assume* (2.5), (2.6). *Then u of* $(A.1)$ *is a viscosity solution of* (3.1).

Proof. We can apply a standard result on the theory of viscosity solutions in [8] since every function in B is uniformly continuous on each compact interval. So, for the viscosity property of u, it is sufficient to show that the dynamic programming principle holds, i.e.,

$$u(x) = \sup_{z\in A} E\left[\int_0^\tau e^{-(\alpha+\frac{1}{\varepsilon})t}\left\{(r-\phi(z_t))x_t+\frac{1}{\varepsilon}u\vee g(x_t)\right\}dt+e^{-(\alpha+\frac{1}{\varepsilon})\tau}u(x_\tau)\right] \tag{A.8}$$

for any $\tau \in S$, which may depend on z. We denote the right-hand side of $(A.8)$ by $\overline{u}(x)$.

By means of Markov property [8], we have

$$T_z u(x) = E\left[\int_0^\tau e^{-(\alpha+\frac{1}{\varepsilon})t}\{(r-\phi(z_t))x_t + \frac{1}{\varepsilon}(u\vee g)(x_t)\}dt\right.$$
$$+ e^{-(\alpha+\frac{1}{\varepsilon})\tau} E\left[\int_0^\infty e^{-(\alpha+\frac{1}{\varepsilon})t}\{(r-\phi(z_{t+\tau}))x_{t+\tau} + \frac{1}{\varepsilon}(u\vee g)(x_{t+\tau})\}dt \,|\, \mathcal{F}_\tau\right]\Bigg]$$
$$\le E\left[\int_0^\tau e^{-(\alpha+\frac{1}{\varepsilon})t}\{(r-\phi(z_t))x_t + \frac{1}{\varepsilon}(u\vee g)(x_t)\}dt + e^{-(\alpha+\frac{1}{\varepsilon})\tau}u(x_\tau)\right],$$

where T_z is as in $(A.3)$. By taking the supremum over $\mathcal{A}$, we deduce $u(x) \le \overline{u}(x)$.

To prove the reverse inequality, let $\rho > 0$ be arbitrary. By the same calculation as $(A.6)$, we find $C_\rho > 0$ such that

$$|u(x)-u(y)| \le \sup_{z\in A}|T_z u(x) - T_z u(y)| \le C_\rho |x-y| + \rho(1+x+y), \quad x,y \ge 0. \tag{A.9}$$

Taking $\delta \in (0, 1)$ such that $C_\rho \delta < \rho$, we have for $|x-y| < \delta$,

$$|u(x)-u(y)| \le \rho(2+x+y) \le 3\rho(1+x). \tag{A.10}$$

Let $\{S_i\}$ be a sequence of disjoint subsets of $[0,\infty)$ such that

$$diam(S_i) < \delta \quad and \quad \cup_i S_i = [0,\infty).$$

For any i, we take $x^{(i)} \in S_i$ and $z^{(i)} \in \mathcal{A}$ such that

$$T_{z^{(i)}} u(x^{(i)}) \ge u(x^{(i)}) - \rho.$$

Then, by (A.10)

$$\begin{aligned} T_{z^{(i)}} u(x_\tau) &= \{T_{z^{(i)}} u(x_\tau) - T_{z^{(i)}} u(x_i)\} + T_{z^{(i)}} u(x_i) \\ &\ge -3\rho(1+x_\tau) + T_{z^{(i)}} u(x_i) \\ &\ge -3\rho(1+x_\tau) + u(x_i) - \rho \\ &\ge -7\rho(1+x_\tau) + u(x_\tau) \qquad \text{if} \quad x_\tau \in S_i. \end{aligned}$$

By definition, there exists $z \in \mathcal{A}$ such that

$$\overline{u}(x) - \rho \le E\left[\int_0^\tau e^{-(\alpha+\frac{1}{\varepsilon})t}\left\{(r-\phi(z_t))x_t + \frac{1}{\varepsilon}u\vee g(x_t)\right\}dt + e^{-(\alpha+\frac{1}{\varepsilon})\tau}u(x_\tau)\right].$$

Define $z^\tau \in \mathcal{A}$ by

$$z_t^\tau = z_t 1_{\{t<\tau\}} + z_{t-\tau}^{(i)} 1_{\{t\ge\tau\}} \quad \text{if} \quad x_\tau \in S_i.$$

Using Lemma A. 2 and $0 \le x \le (1+x^\gamma)$ for $\gamma > 1$, we deduce

$$\overline{u}(x) - \rho \le \sum_i E\left[\int_0^\tau e^{-(\alpha+\frac{1}{\varepsilon})t}\left\{(r-\phi(z_t))x_t + \frac{1}{\varepsilon}u\vee g(x_t)\right\}dt + e^{-(\alpha+\frac{1}{\varepsilon})\tau}\left\{T_{z^{(i)}}u(x_\tau) + 7\rho(1+x_\tau)\right\} : x_\tau \in S_i\right]$$
$$\le T_{z^\tau} u(x) + E\left[e^{-(\alpha+\frac{1}{\varepsilon})\tau}\{7\rho(1+x_\tau)\}\right]$$
$$\le u(x) + 7\rho(2+x^\gamma).$$

Thus, letting $\rho \to 0$, we deduce $\overline{u}(x) \le u(x)$. The proof is complete.

Theorem A. 5 *We assume* (2.5), (2.6). Let u_1 *and* u_2 *be two viscosity solutions of* (3.1) *in* $\mathcal{B}$. *Then we have* $u_1 = u_2$.

Proof. We shall show that

$$u_1 \leq u_2.$$

Suppose that $u_1(x_0) - u_2(x_0) > 0$ for some $x_0 > 0$. We take $\gamma > 1$ in Lemma A. 2 and find a constant $\eta > 0$ such that

$$\sup_{x \geq 0} \{u_1(x) - u_2(x) - 2\eta\phi(x)\} > 0,$$

where $\phi(x) := x^\gamma$. Since $u_1(x) - u_2(x) - 2\eta\phi(x) \to -\infty$ as $x \to \infty$, there exists $\tilde{x} > 0$ such that

$$\sup_{x \geq 0} \{u_1(x) - u_2(x) - 2\eta\phi(x)\} = u_1(\tilde{x}) - u_2(\tilde{x}) - 2\eta\phi(\tilde{x}) > 0.$$

Define

$$\Phi(x, y) := u_1(x) - u_2(y) - \frac{n}{2}|x - y|^2 - \eta(\phi(x) + \phi(y))$$

for each positive integer n. It is clear that

$$\Phi(x, y) \leq (\| u_1 \| + \| u_2 \|)(2 + x + y) - \eta(\phi(x) + \phi(y)) \to -\infty \quad \text{as} \quad x + y \to \infty.$$

Hence we find $(x_n, y_n) \in [0, \infty)^2$ such that

$$\begin{aligned} \sup_{x,y \geq 0} \Phi(x, y) &= \Phi(x_n, y_n) \\ &= u_1(x_n) - u_2(y_n) - \frac{n}{2}|x_n - y_n|^2 - \eta(\phi(x_n) + \phi(y_n)) \\ &\geq u_1(\tilde{x}) - u_2(\tilde{x}) - 2\eta\phi(\tilde{x}) > 0, \end{aligned} \tag{A.11}$$

from which

$$\frac{n}{2}|x_n - y_n|^2 < u_1(x_n) - u_2(y_n) - \eta(\phi(x_n) + \phi(y_n)).$$

Thus the sequences $\{x_n\}$, $\{y_n\}$ and $\{n|x_n - y_n|^2\}$ are bounded by some constant $C > 0$. Therefore, extracting a subsequence, we deduce

$$|x_n - y_n| \leq C/\sqrt{n} \to 0,$$
$$x_n, y_n \to \overline{x} \in [0, \infty) \quad \text{as} \quad n \to \infty.$$

Furthermore, we recall $\Phi(x_n, y_n) \geq \Phi(x_n, x_n)$ to obtain

$$\frac{n}{2}|x_n - y_n|^2 < u_2(x_n) - u_2(y_n) - \eta(\phi(x_n) - \phi(y_n)) \to 0.$$

Passing to the limit in (A.11), we get

$$u_1(\overline{x}) - u_2(\overline{x}) - 2\eta\phi(\overline{x}) > 0, \qquad \overline{x} \neq 0. \tag{A.12}$$

Now we apply Ishii's lemma to

$$\Phi(x, y) = U_1(x) - U_2(y) - \frac{n}{2}|x - y|^2,$$

where $U_1(x) := u_1(x) - \eta\phi(x)$ and $U_2(y) := u_2(y) + \eta\phi(y)$. Then we obtain $X_1, X_2 \in \mathbf{R}$ such that

$$
\begin{aligned}
&(n(x_n - y_n), X_1) \in \overline{J}^{2,+} U_1(x_n),\\
&(n(x_n - y_n)\ X_2) \in \overline{J}^{2,-} U_2(y_n),\\
&\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \le 3n \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},
\end{aligned}
$$

where $J^{2,+}$ and $J^{2,-}$ are the second-order superjets and subjets [6], and

$$
\overline{J}^{2,\pm} U_i(x) = \left\{ (p, X) : \begin{array}{c} \exists x_r \to x, \exists (p_r, X_r) \in J^{2,\pm} U_i(x) \\ (U_i(x_r), p_r, X_r) \to (U_i(x), p, X), \end{array} \right\}, \quad i = 1, 2.
$$

It is easy to see that

$$
\begin{aligned}
J^{2,+} u_1(x) &= \{(p + \eta\phi'(x), X + \eta\phi''(x) : (p, X) \in J^{2,+} U_1(x)\},\\
J^{2,-} u_2(x) &= \{(p - \eta\phi'(x), X - \eta\phi''(x) : (p, X) \in J^{2,-} U_2(x)\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&(p_1, \overline{X}_1) := (n(x_n - y_n) + \eta\phi'(x_n), X_1 + \eta\phi''(x_n)) \in \overline{J}^{2,+} u_1(x_n)\\
&(p_2, \overline{X}_2) := (n(x_n - y_n) - \eta\phi'(y_n), X_2 - \eta\phi''(y_n)) \in \overline{J}^{2,-} u_2(y_n),\\
&x_n^2 X_1 - y_n^2 X_2 \le 3n(x_n - y_n)^2 \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

By the equivalent definition of viscosity solutions, we have

$$
\begin{aligned}
&-\alpha u_1(x) + \frac{1}{2}\sigma^2 x^2 \overline{X}_1 - \mu x p_1 + rx + \Lambda(p_1)x + \frac{1}{\varepsilon}(u_1 - g)^-|_{x = x_n} \ge 0,\\
&-\alpha u_2(y) + \frac{1}{2}\sigma^2 y^2 \overline{X}_2 - \mu y p_2 + ry + \Lambda(p_2)y + \frac{1}{\varepsilon}(u_2 - g)^-|_{y = y_n} \le 0,
\end{aligned}
$$

Putting these inequalities together, we have

$$
\begin{aligned}
\alpha(u_1(x_n) - u_2(y_n)) \le\ & \frac{1}{2}\sigma^2(x_n^2 \overline{X}_1 - y_n^2 \overline{X}_2)\\
& - \mu(p_1 x_n - p_2 y_n) + r(x_n - y_n)\\
& + \Lambda(p_1)x_n - \Lambda(p_2)y_n\\
& + \frac{1}{\varepsilon}\{(u_1 - g)^-(x_n) - (u_2 - g)^-(y_n)\}\\
=:\ & I_1 + I_2 + I_3 + I_4, \quad \text{say.}
\end{aligned}
$$

We note that

$$
\begin{aligned}
I_1 &= \frac{\sigma^2}{2}(x_n^2 X_1 - y_n^2 X_2 + x_n^2 \eta\phi''(x_n) + y_n^2 \eta\phi''(y_n)) \to \sigma^2 \overline{x}^2 \eta\phi''(\overline{x}),\\
I_2 &= -\mu n(x_n - y_n)^2 - \mu x_n \eta\phi'(x_n) - \mu y_n \phi'(y_n) + r(x_n - y_n) \to -2\mu\overline{x}\eta\phi'(\overline{x}),\\
I_3 &\le \overline{z}\,|p_1 x_n - p_2 y_n| + \phi(\overline{z})\,|x_n - y_n| \to 2\overline{z}\,\overline{x}\eta\phi'(\overline{x}),\\
I_4 &\to \frac{1}{\varepsilon}\{(u_1 - g)^-(\overline{x}) - (u_2 - g)^-(\overline{x})\} \le 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Thus we deduce

$$\alpha(u_1(\bar{x})-u_2(\bar{x}))\leq 2\eta\{\frac{1}{2}\sigma^2\bar{x}^2\psi''(\bar{x})+\alpha_0\bar{x}\psi'(\bar{x})\}$$
$$=2\eta\psi(\bar{x})\{\frac{1}{2}\sigma^2\gamma(\gamma-1)+\gamma\alpha_0\}$$
$$<2\alpha\eta\psi(\bar{x}),$$

which is contrary with (A.12). The proof is complete.

Theorem A. 6 *We assume* (2.5), (2.6). *Then a unique solution* $u=u_\varepsilon$ *of* (3.1) *belongs to* $C^2(0,\infty)$.

Proof. Let $0<a<b$ be arbitrary, and we consider the boundary value problem :

$$-(\alpha+\frac{1}{\varepsilon})\xi+\frac{1}{2}\sigma^2x^2\xi''-\mu x\xi'+rx+\Lambda(\xi')x+\frac{1}{\varepsilon}\xi\vee g=0 \quad in \quad (a,\ b),$$
$$\xi(a)=u(a),\quad \xi(b)=u(b). \tag{A.13}$$

By the uniform ellipticity and the Lipschitz continuity of Λ, it is known in [11] that (A.13) has a smooth solution ξ. By replacing η in the proof of Theorem A. 5 by zero, we observe that the viscosity solution of (A.13) is unique. Therefore we conclude that $u=\xi\in C^2(a,b)$, and thus $u\in C^2(0,\infty)$.

B　Nonlinear variational inequality

Now we present another representation of u, which will be needed in the proof of Theorem 3. 1.

Lemma B. 1 *Under* (2.5) *and* (2.6), *we have*

$$u(x)=\sup_{(z,\tau)\in\mathcal{A}\times S}E\left[\int_0^\tau e^{-\alpha t}(r-\phi(z_t))x_tdt+e^{-\alpha\tau}\{g-(u-g)^-\}(x_\tau)\right]. \tag{B. 1}$$

Proof. By (A. 1), we have

$$u(x)\geq E\left[\int_0^\tau e^{-\alpha t}\{(r-\phi(z_t))x_t+\frac{1}{\varepsilon}(u-g)^-(x_t)\}dt+e^{-\alpha\tau}u(x_\tau)\right]$$
$$\geq E\left[\int_0^\tau e^{-\alpha t}\{(r-\phi(z_t))x_tdt+e^{-\alpha\tau}u\wedge g(x_\tau)\right]$$

for any $\tau\in S$. Since $\lambda\circ u'(x)$ is locally Lipschitz and

$$|(\lambda\circ u'(x)-\mu)x|\leq C(1+|x|)$$

for some constant $C>0$, there exists a unique solution $\{\hat{x}_t\}$ to the equation

$$d\hat{x}_t=(\lambda\circ u'(\hat{x}_t)-\mu)\hat{x}_tdt+\sigma\hat{x}_tdW_t,\quad \hat{x}_0=x>0.$$

Define

$$\hat{\tau}=\inf\{t\geq 0:u(\hat{x}_t)\leq g(\hat{x}_t)\}.$$

Note $u\wedge g=g-(u-g)^-$. Then, using Ito's formula and Lemma A. 2, we deduce

$$u(x)=E\left[\int_0^{\hat{\tau}} e^{-\alpha t}\{(r-\phi(\lambda\circ u'(\hat{x}_t))\hat{x}_t+\frac{1}{\varepsilon}(u-g)^-(\hat{x}_t)\}dt+e^{-\alpha\hat{\tau}}u(\hat{x}_{\hat{\tau}})\right]$$
$$=E\left[\int_0^{\hat{\tau}} e^{-\alpha t}\{(r-\phi(\lambda\circ u'(\hat{x}_t))\hat{x}_t dt+e^{-\alpha\hat{\tau}}\{g-(u-g)^-\}(\hat{x}_{\hat{\tau}})\right],$$

which completes the proof.

We obtain a viscosity solution of the variatioal inequality (2.8) given in Section 2. The viscosity solution of (2.8) is defined in the following.

Definition B.2 *Let $v\in C[0,\infty)$ and $v(0)=0$. Then v is a viscosity solution of (2.8) if*

(i) *For any $\varphi\in C^2(0,\infty)$ and any local minimum point $x>0$ of $v-\varphi$,*

$$H(x,v,\varphi',\varphi'')\leq 0,$$

(ii) $v(x)\geq g(x)$ *for all $x\in[0,\infty)$,*

and

(iii) *For any $\varphi\in C^2(0,\infty)$ and any local maximum point $x>0$ of $v-\varphi$,*

$$H(x,v,\varphi',\varphi'')(v-g)^+\geq 0.$$

Proof of Theorem 3.1

We use (2.5), (2.6) and (A.4) to obtain

$$0\leq E[e^{-(\alpha+\frac{1}{\varepsilon})t}g(x_t)]\leq\|g\|e^{-(\alpha+\frac{1}{\varepsilon})t}(1+xe^{\alpha_0 t})\to 0 \qquad as \quad t\to\infty.$$

Then, by (3.2), g can be rewritten as

$$g(x)=E\left[\int_0^{\infty} e^{-(\alpha+\frac{1}{\varepsilon})t}(\alpha h+\frac{1}{\varepsilon}g)(\overline{x}_t)dt\right],$$

where $\{\overline{x}_t\}$ is the response to the constant control $\overline{z}$. By Theorem A. 3, we have

$$u_\varepsilon(x)\geq E\left[\int_0^{\infty} e^{-(\alpha+\frac{1}{\varepsilon})t}\{(r-\phi(\overline{z}))\overline{x}_t+\frac{1}{\varepsilon}u_\varepsilon\vee g(\overline{x}_t)\}dt\right].$$

Hence, by (A.4)

$$(u_\varepsilon-g)(x)\geq E\left[\int_0^{\infty} e^{-(\alpha+\frac{1}{\varepsilon})t}\{(r-\phi(\overline{z}))\overline{x}_t-\alpha h(\overline{x}_t)+\frac{1}{\varepsilon}(u_\varepsilon\vee g-g)(\overline{x}_t)\}dt\right]$$
$$\geq -\frac{|r-\phi(\overline{z})|\varepsilon x}{(\alpha-\alpha_0)\varepsilon+1}-\frac{\alpha\|h\|\varepsilon(1+x)}{(\alpha-\alpha_0)\varepsilon+1}$$
$$\geq\frac{-\varepsilon C(1+x)}{(\alpha-\alpha_0)\varepsilon+1}$$

for some constant $C>0$, independent of ε. Thus, we get

$$(u_\varepsilon-g)^-(x)\leq C(1+x)\varepsilon. \qquad (B.2)$$

By Lemma B. 1 we have

$$|u_{\varepsilon_{n+1}}(x)-u_{\varepsilon_n}(x)|\leq \sup_{z\in A\tau\in S} E[e^{-\alpha\tau}|(u_{\varepsilon_{n+1}}-g)^- -(u_{\varepsilon_n}-g)^-|(x_\tau)]$$
$$\leq(\varepsilon_{n+1}+\varepsilon_n)C(1+x).$$

Therefore

$$\sum_{n=1}^{\infty}\| u_{\varepsilon_{n+1}}-u_{\varepsilon_n}\|\leq C\sum_{n=1}^{\infty}(\varepsilon_{n+1}+\varepsilon_n)<\infty.$$

This yields that $\{u_{\varepsilon_n}\}$ is a Cauchy sequence and u_{ε_n} converges to v in $\mathcal{B}$.

Now we show the viscosity property of v in the sense of Definition B. 2. Let $\varphi\in C^2(0,\infty)$ and $x>0$ be the minimizer of $v-\varphi$. We may consider that

$$v(x)-\varphi(x)<v(y)-\varphi(y),\quad \forall y\in\overline{B}(x,\delta),\quad y\neq x,$$

where $\overline{B}(x,\delta)$ denotes the closed ball centered at x with radius δ. By the above argument, we see that

$$u_{\varepsilon_n}\to v \text{ uniformly on } \overline{B}(x,\delta).$$

Let $x_n\in\overline{B}(x,\delta)$ be a minimum point of $u_{\varepsilon_n}-\varphi$. Extracting a subsequence if necessary, we have

$$x_n\to x,\quad u_{\varepsilon_n}(x)-\varphi(x)\leq u_{\varepsilon_n}(x_n)-\varphi(x_n).$$

Hence, by (3. 1)

$$-\alpha u_{\varepsilon_n}+\frac{1}{2}\sigma^2x^2\varphi''-\mu x\varphi'+rx+\Lambda(\varphi')x\Big|_{x=x_n}\leq 0.$$

Passing to the limit, we see that (i) is verified. (ii) is immediate from (B. 2).
Finally, let $\overline{x}$ be the maximizer of $v-\varphi$, and $\overline{x}_n$ be the maximum point of $u_{\varepsilon_n}-\varphi$ such that $\overline{x}_n\to\overline{x}$. Then, by (3. 1)

$$(-\alpha u_{\varepsilon_n}+\frac{1}{2}\sigma^2x^2\varphi''-\mu x\varphi'+rx+\Lambda(\varphi')x)(u_{\varepsilon_n}-g)^+\Big|_{x=\overline{x}_n}\geq 0.$$

Passing to the limit, we obtain (iii).

Proof of Theorem 3. 2

By Theorem 3. 1 and the Ascoli-Arzelà theorem, we can obtain (3. 3) if

$$\sup_{0<\varepsilon<1}\| u_\varepsilon''\|_{L^\infty(c,d)}<\infty \quad \text{for any} \quad 0<c<d. \tag{B. 3}$$

Changing the variable by $y=\log x$ in (3. 1) and setting $f_\varepsilon(y)=u_\varepsilon(e^y)$, $\tilde{g}(y)=g(e^y)$, we get

$$-\alpha f_\varepsilon+\frac{1}{2}\sigma^2 f_\varepsilon''-(\frac{1}{2}\sigma^2+\mu)f_\varepsilon'+re^y+\Lambda(f_\varepsilon'e^{-y})e^y+\frac{1}{\varepsilon}(f_\varepsilon-\tilde{g})^-=0 \quad \text{in } \mathbf{R}. \tag{B. 4}$$

To prove (B. 3), it suffices to show stepwise that

$$\sup_{0<\varepsilon<1}\|(f_\varepsilon')^2\|_{L^1(B)}<\infty, \tag{B. 5}$$

$$\sup_{0<\varepsilon<1}\|(f_\varepsilon')^2\|_{L^k(B)}<\infty,\quad k\geq 1, \tag{B. 6}$$

$$\sup_{0<\varepsilon<1}\| f_\varepsilon'\|_{L^\infty(B)}<\infty, \tag{B. 7}$$

where $B := (a,b) \subset \mathbf{R}$. We set $B_1 := (a-1, b+1)$. Then, by (B. 4)

$$f'' = G(y, f, f') \quad \text{in} \quad B_1, \tag{B. 8}$$

where we suppress ε of f_ε and

$$G(y,f,f') = \frac{2}{\sigma^2}\{\alpha f + (\frac{1}{2}\sigma^2 + \mu)f' - re^y - \Lambda(f'e^{-y})e^y - \frac{1}{\varepsilon}(f - \tilde{g})^-\}.$$

By (B.2), we observe that

$$|G(y,f,f')| \le C(1 + |f'|), \quad y \in B_1, \tag{B. 9}$$

for some constant $C > 0$, independent of ε. Let $\eta \in C_0^\infty(B_1)$ be such that $0 \le \eta \le 1$ on B_1 and $\eta = 1$ on B. Multiplying (B. 8) by $-\eta^2 f$ and using integration by parts, we get

$$-\int f''\eta^2 f dy = \int f'(f'\eta^2 + 2f\eta\eta')dy \le C(1 + \int |f'|\eta^2 f dy).$$

By Theorem 3. 1, we have

$$\| u_{\varepsilon_n} - v \| \le C,$$

so that

$$0 \le f_{\varepsilon_n}(y) = u_{\varepsilon_n}(e^y) \le |v(e^y)| + |u_{\varepsilon_n}(e^y) - v(e^y)| \le C(1 + e^y). \tag{B.10}$$

Using a simple relation $|xy| \le (\theta x^2 + \theta^{-1}y^2)$ for $\theta > 0$, we have

$$\int (f')^2(\eta)^2 dy \le C(1 + \int f^2(\eta')^2 dy), \tag{B.11}$$

which implies (B. 5). Substituting $f'\eta^2$ into η of (B.11) we get

$$\int (f')^4\eta^4 dy \le C.$$

By induction, we obtain

$$\int (f')^{2^k}\eta^{2^k} dy \le C,$$

which yields (B. 6).

To see (B. 7), we set $\phi = (f')^2$ and $H(y) = 2f'G(y,f,f')$. By (B. 8), ϕ solves

$$\phi' = H(y).$$

Hence, by (B. 9)

$$\phi'' = H'(y), \quad |H(y)| \le C(1 + |f'|^2),$$

and, by (B. 6) with B_1 replacing B

$$H \in L^2(B_1).$$

By virtue of the Hölder estimate for the solution ϕ [11, Thm 8. 24, p.202], we obtain

$$\| \phi \|_{L^\infty(B)} \le C(\| \phi \|_{L^2(B_1)} + \| H \|_{L^2(B_1)}),$$

which yields (B. 7). The proof is complete.

C Optimal policies

In Appendix C, we prove Theorem 3. 3 and obtain optimal exit and investing policies.

Proof of Theorem 3. 3

First, we take some constant $C' > 0$ such that

$$(1+x)^\gamma \le C'(1+x^\gamma)$$

for all $x \ge 0$. Then, by Lemma A. 2 and (B.10), we have

$$\begin{aligned}\sup_{\tau \in S} E[|e^{-\alpha\tau} u_{\varepsilon_n}(x_\tau)|^\gamma] &\le \sup_{\tau \in S} E[e^{-\alpha\tau} C^\gamma (1+x_\tau)^\gamma] \\ &\le C^\gamma C'(1+\sup_{\tau \in S} E[e^{-\alpha\tau} x_\tau^\gamma]) < \infty.\end{aligned}$$

Hence, by uniform integrability, Ito's formula gives

$$\begin{aligned}E[e^{-\alpha\tau} u_{\varepsilon_n}(x_\tau)] &= u_{\varepsilon_n}(x) + E\left[\int_0^\tau e^{-\alpha t}\{-\alpha u_{\varepsilon_n} + \frac{1}{2}\sigma^2 x^2 u''_{\varepsilon_n} + (z_t - \mu) u'_{\varepsilon_n} x\}|_{x=x_t} dt\right] \\ &\le u_{\varepsilon_n}(x) - E\left[\int_0^\tau e^{-\alpha t}(r - \phi(z_t)) x_t dt\right].\end{aligned}$$

Letting $\varepsilon_n \to 0$, by Theorem 3. 1, we deduce $v(x) \ge J(z, \tau)$ for all $(z, \tau) \in \mathcal{A} \times S$.

Now we consider

$$dx_t^* = (\lambda \circ v'(x_t^*) - u) x_t^* dt + \sigma x_t^* dW_t, \quad x_0^* = x. \tag{C. 1}$$

Since $\lambda \circ v'$ is bounded and locally Lipschitz, there exists a unique solution $\{x_t^*\}$ to (C. 1). Then, it is clear that $K_t^* = x_t^*/P_t$ solves (3. 6). Let

$$\tau_\nu = \inf\{t \ge 0 : v(x_t^*) - \frac{1}{\nu}(1 + x_t^*) \le g(x_t^*)\}, \quad \nu = 1, 2, \ldots.$$

By Theorem 3. 1, we deduce

$$\begin{aligned}E\left[\int_0^{\tau_\nu} e^{-\alpha t}(u_{\varepsilon_n} - g)^-(x_t^*) dt\right] &\le E\left[\int_0^{\tau_\nu} e^{-\alpha t}(u_{\varepsilon_n} - (v - \frac{1}{\nu}(1+x)))^-|_{x=x_t^*} dt\right] \\ &\le E\left[\int_0^{\tau_\nu} e^{-\alpha t}(-\| u_{\varepsilon_n} - v \| + \frac{1}{\nu})^-(1 + x_t^*) dt\right] \\ &= 0 \qquad \text{for sufficiently large } n.\end{aligned}$$

We set $\tau_R = \inf\{t \ge 0 : x_t^* > R \text{ or } x_t^* < 1/R\}$ for $R > 1$ and $\hat{\tau} = \tau_R \wedge \tau_\nu$.
Applying Ito's formula to $e^{-\alpha t} u_{\varepsilon_n}(x_t^*)$, we have

$$
\begin{aligned}
E[e^{-\alpha\hat{t}}u_{\varepsilon_n}(x^*_{\hat{t}})] &= u_{\varepsilon_n}(x)+E\left[\int_0^{\hat{t}} e^{-\alpha t}\{-\alpha u_{\varepsilon_n}+\frac{1}{2}\sigma^2x^2u''_{\varepsilon_n}+(\lambda\circ v'-\mu)xu'_{\varepsilon_n}\}|_{x=x^*_t}dt\right]\\
&= u_{\varepsilon_n}(x)-E\left[\int_0^{\hat{t}} e^{-\alpha t}\{(r-\phi(\lambda\circ u'_{\varepsilon_n}))x+\frac{1}{\varepsilon_n}(u_{\varepsilon_n}-g)^-\}|_{x=x^*_t}dt\right]\\
&\quad +E\left[\int_0^{\hat{t}} e^{-\alpha t}\{\lambda\circ v'-\lambda\circ u'_{\varepsilon_n}\}xu'_{\varepsilon_n}|_{x=x^*_t}dt\right]\\
&= u_{\varepsilon_n}(x)-E\left[\int_0^{\hat{t}} e^{-\alpha t}(r-\phi(z^*_t))x^*_tdt\right]+E\left[\int_0^{\hat{t}} e^{-\alpha t}\{\lambda\circ v'-\lambda\circ u'_{\varepsilon_n}\}xu'_{\varepsilon_n}|_{x=x^*_t}dt\right]\\
&\quad -E\left[\int_0^{\hat{t}} e^{-\alpha t}\{\phi(\lambda\circ v'(x^*_t))-\phi(\lambda\circ u'_{\varepsilon_n}(x^*_t))\}dt\right]
\end{aligned}
$$

for sufficiently large n. Taking into account (B. 3), we apply the Ascoli-Arzelà theorem to obtain

$u'_{\varepsilon_n}\to v'$ uniformly on $[1/R, R]$.

Letting $n\to\infty$, $R\to\infty$, and then $\nu\to\infty$, we deduce that $\tau_\nu\uparrow\tau^*$, and

$$E[e^{-\alpha\tau^*}v(x^*_{\tau^*})]=v(x)-E[\int_0^{\tau^*} e^{-\alpha t}(r-\phi(z^*_t))x^*_tdt)].$$

This yields $J(z^*,\tau^*)=v(x)$, completing the proof.

REFERENCES

[1] A. B. Abel, Optimal investment under uncertainty, A. E. R., 73 (1983), 228–233.

[2] L. H. R. Alvarez, Exit strategies and price uncertainty : a Greenian approach, J. Math. Econ., 29 (1998), 43-56.

[3] M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhäuser, Boston, 1997.

[4] A. Bensoussan, Stochastic Control by Functional Analysis Methods, North-Holland, Amsterdam, 1982.

[5] A. Bensoussan and J. L. Lions, Applications des Inéquations Variationnelles en Contrôle Stochastique, Dunod, Paris, 1978.

[6] M. G. Crandall, H. Ishii, and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992), 1-67.

[7] A. K. Dixit, Entry and exit decisions under uncertainty, J. Polit. Econ., 97 (1989), 620-638.

[8] W. H. Fleming and H. M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, New York, 1993.

[9] A. Friedman, Stochastic Differential Equations and Applications, vol. 2, Academic Press, New York, 1976.

[10] M. Gertler, Financial capacity and output fluctuation in an economy with multiperiod financial relationship, R. E. S., 59 (1992), 455-472.

[11] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, New York, 1998.

[12] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, Amsterdam, 1981.

[13] D. W. Jorgenson, Capital theory and investment behavior, A. E. R. 53 (1963), 47-56.

[14] I. Karatzas and H. Wang, Utility maximization with discretionary stopping, SIAM J. Control Optim. 39 (2000), 306-329.

[15] I. Karatzas and D. Ocone, A leavable bounded-velocity stochastic control problem, Stochastic Process. Appl., 99 (2002), 31- 51.

[16] I. Karatzas and S. E. Shreve, Brownian Motion and Stochastic Calculus, Springer-Verlag, New York, 1991.

[17] S. Koike and H. Morimoto, On variational inequalities for leavable bounded-velocity control, Appl. Math. Optim., 48 (2003), 1-20.

[18] O. A. Ladyzhenskaya and N. N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.

[19] R. L. McDonald and D. R. Siegel, Investment and the valuation of firms when there is an option to shutdown, I. E. R., 26 (1985), 331-348.

[20] H. Morimoto, Variational inequalities for combined control with stopping, SIAM J. Control Optim., 42 (2003), 686-708.

[21] J. E. Stiglitz and A. Weiss, Credit rationing in markets with imperfect information, A. E. R. 71 (1981), 393-410.

[22] J. Tobin, A General equilibrium approach to monerary theory, Journal of Money, Credit and Banking., 1 (1969), 15-29.

[23] H. Uzawa, Time preference and the Penrose effect in a two-class model of economic growth, J. Polit. Econ., 77(Part 2)(1969), 628-652, 15-29.

[24] H. Yoshikawa, On the "q" theory of investment, A. E. R., 70 (1980), 739-743.