〈論文〉

OPTIMAL CONSUMPTION AND EMPLOYMENT POLICIES IN A STOCHASTIC GROWTH MODEL

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Abstract

In this paper, we study dynamic consumption and employment policies in a Cobb-Douglas technology economic growth model under uncertainty. Using the viscosity solutions method, we show that the optimal policies exist in feedback forms. Our results demonstrate that the optimal shortening of working hours arises from uncertain fluctuations in the stochastic economy.

Key words. Economic growth, Employment, Brownian motions, Uncertainty, Viscosity solutions
JEL classifications. D90 ; D81 ; E13 ; O41

1 Introduction

This paper develops a one sector optimal economic growth model which includes random shocks to production and capital depreciation. Possible sources of shocks include uncertainty about environmental changes, natural disasters, morale of workers and technological progress/regress. Our analysis is based on a stochastic version of the Cobb-Douglas economic growth model. The model we propose here differs from the traditional deterministic one sector optimal growth models in two respects : both consumption and employment processes are determined to maximize the discounted sum of utilities of a representative household, and the risk term of the capital accumulation process expands as the each factor of production increases. Under these circumstances, we obtain the optimal consumption and employment policies and demonstrate how the risk elements in the model affect the optimal consumption and working hours of the households.

Following the seminal papers of Koopmans [13] and Cass [2] a number of authors have investigated various properties of neoclassical optimal growth models which are characterized by additive utilities and the Solow-Swan model of capital accumulation. The continuous-time stochastic versions of such models are considered by Merton [14], Chang [3] and Foldes [7]. These studies assume that labor is supplied by each household inelastically and is fully employed. More recently Amilton and Bermin [1] extends the model by allowing the authirities to control the expected growth rate of the labor supply. However, it is also assumed in their model that the labor supply is fully utilized. In a deterministic economy such a full employment assumption makes no serious problem, since the more labor force leads to the more consumption in general (see Section 4 of this paper for details), and the representative household in the economy only cares about its consumption level. However, once uncertainty is introduced, it becomes unclear whether full employment is always good for the society or not. Therefore, in our study, we depart from the assumption of full employment, and focus on the trade-off between the production expansion and the risk increase in the economy.

The main purpose of this paper is to characterize the optimal consumption and employment policy functions in feedback forms, under general assumptions on the representative agent's unility function. We first present the existence and uniqueness of the accumulation process of the capital stock and verify that the control problem is meaningful. Given these existence and uniqueness results, we demonstrate that, at least for some quantity of capital stock, the total labor force in the economy is redundant from the social planner's (not from each individual's) point of view if there exist stochastic fluctuations in the economy. Our findings clearly indicate the importance of the risk factor in the economy, since the full employment policy is certainly optimal in the normal deterministic growth models. In a risky economy, it is possible for the laborers to have more leisure than in a riskless economy to attain optimality. Our analysis of optimal policies is based on the viscosity solutions method developed by Fleming and Soner [6].

The rest of the paper proceeds as follows. Section 2 provides the stochastic economic growth model with Cobb-Douglas production technology and introduces the HJB equation we analyze. In Section 3, we characterize the optimal policies of consumption and employment. In Section 4, we show that, for some quantity of capital stock, the optimal employment plan induces shortening of working hours. Section 5 concludes the paper. All mathematical technicalities are collected in Appendices A, B and C.

2 The model

The Cobb-Douglas production function is of the form

$$L^{1-\alpha}K^{\alpha}, \quad 0<\alpha<1$$

where $L \geq 0$ and $K \geq 0$ respectively denote the employment and the capital stock. We call the economic growth model with such a technology a Cobb-Douglas growth model.

Let us consider the capital accumulation model :

$$dK_t = [L_t^{1-\alpha}K_t^{\alpha} - \delta K_t - c_t K_t]dt + \sigma L_t K_t dW_t, \quad K_0 = k > 0, \quad \sigma \neq 0, \tag{2.1}$$

on a complete probability space $(\Omega, \mathcal{F}, P)$, carrying a standard Brownian motion $\{W_t\}$, endowed with the natural filtration $\mathcal{F}_t$ generated by $\sigma(W_s, s \leq t)$. Here $L_t \geq 0$ and $c_t \geq 0$ respectively denote the employment policy and the total consumption per capital stock at time t, and $\delta > 0$ is the constant expected capital depreciation rate. The uncertain fluctuations in production and capital depreciation are modeled by the risk term $\sigma L_t K_t dW_t$. A pair $(\mathbf{L}, \mathbf{c})$ of $\mathbf{L} = \{L_t\}$ and $\mathbf{c} = \{c_t\}$ is called admissible if each of them is

$\{\mathcal{F}_t\}$-progressively measurable and fulfills

$$\begin{aligned} &\Delta \le L_t \le 1 \quad \text{for all} \quad t \ge 0, \quad a.s., \\ &0 \le c_t \le \overline{c} \quad \text{for all} \quad t \ge 0, \quad a.s., \end{aligned} \tag{2.2}$$

where $\overline{c}$ is some positive constant. We denote by $\mathcal{A}$ the set of all admissible policies $(\mathbf{L}, \mathbf{c})$. Throughout this paper, we assume that the total labor supply in the economy is fixed and normalized as one. The policy maker can assign employment between some fixed minimum level $\Delta > 0$ and unity. We impose the minimum restriction Δ for keeping productivity, skill and morale of laborers.

Our objective is to characterize an optimal policy $(\mathbf{L}^*, \mathbf{c}^*) \in \mathcal{A}$, which maximizes the expected utility function given by

$$J(\mathbf{L}, \mathbf{c}) = E[\int_0^\infty e^{-\beta t} U(c_t K_t) dt], \tag{2.3}$$

where $\beta > 0$ is a discount rate and $U(\cdot)$ is a utility function in $C^2(0, \infty) \cap C[0, \infty)$, which is assumed to have the following properties :

$$\begin{cases} U(x): \text{strictly concave on } [0, \infty), \quad U''(x) < 0, \\ U'(\infty) = U(0) = 0, \quad U'(0+) = U(\infty) = \infty. \end{cases} \tag{2.4}$$

Assuming that the per capita output is a Lipschitz function of the per capita capital stock, Chang and Malliaris [4] proves the existence and uniqueness of the solution for the Solow equation by applying the basic theory of stochastic differential equations. However, in economics, the production functions satisfying Inada conditions are widely used and such functions are clearly non-Lipschitz. The most popular type of such functions is, without doubt, the Cobb-Douglas function which is extensively used in vast literature of economic growth studies. Unfortunately, for non-Lipschitz functions, the uniqueness of the solution for the Solow equation is not satisfied in general. In fact, it is easy to see that the following simple deterministic equation

$$dk_t/dt = k_t^{1/2} - \delta k_t, \text{ (zero consumption)}$$

has two solutions

$$k_t^{(1)} = 0, \quad t \ge 0$$

and

$$k_t^{(2)} = (\frac{1}{\delta} - \frac{1}{\delta} e^{-\frac{\delta t}{2}})^2, \quad t \ge 0$$

for the initial value of per capita capital stock $k_0 = 0$. Therefore, when the technology is represented by such a non-Lipschitz function, there are two paths we can go by, once we used up the whole capital stock in the economy. This example indicates that, to investigate optimal stochastic growth models under Inada conditions, it is important to confirm the existence and uniqueness of the positive solution for each type of production functions respectively. The following proposition provides the existence and uniqueness results for the Cobb-Douglas growth model by showing that, for each admissible policies, 0 is not an accessible boundary of the capital stock. We see that the upper bound constraint for consumption $\overline{c}$ prevents the extinction of the capital stock in the economy.

Proposition 2.1 *For each* $(\mathbf{L}, \mathbf{c}) \in \mathcal{A}$, *there exists a unique positive solution* $\{K_t\}$ *of* (2.1), *which satisfies*

$$E[K_t] \leq \{(1-\alpha)t + k^{1-\alpha}\}^{1/(1-\alpha)}, \tag{2.5}$$

$$E[K_t^2] \leq e^{\sigma^2 t}[2(1-\lambda)t + k^{2(1-\lambda)}]^{1/(1-\lambda)}, \quad \lambda := (1+\alpha)/2. \tag{2.6}$$

Proof. Let $(\mathbf{L}, \mathbf{c}) \in \mathcal{A}$ be arbitrary and set $x_t := K_t^{1-\alpha}$. Then, by Ito's formula, $\{x_t\}$ satisfies

$$dx_t = (1-\alpha)[L_t^{1-\alpha} - (\delta + c_t + \frac{\sigma^2\alpha}{2}L_t^2)x_t]dt + (1-\alpha)\sigma L_t x_t dW_t, \quad x_0 = k^{1-\alpha} > 0. \tag{2.7}$$

By (2.2), there exists a unique solution $\{x_t\}$ of (2.7). By the comparison theorem (Ikeda and Watanabe [11]), we observe that $x_t > 0$ a.s. Thus (2.1) admits a unique positive solution $\{K_t\}$ such that $\sup_{0 \leq t \leq T} E[K_t^4] < \infty$ for each $T > 0$.

Let ζ_t be the right-hand side of (2.5). Then, it is easy to see that

$$d\zeta_t = \zeta_t^\alpha dt, \quad \zeta_0 = k.$$

By (2.1) and Jensen's inequality

$$d(E[K_t]) \leq E[K_t^\alpha - \delta K_t - c_t K_t]dt \leq E[K_t]^\alpha dt.$$

Since $\zeta_0 = E[K_0] = k$, we get $E[K_t] \leq \zeta_t$ and (2.5).

Let η_t be the right-hand side of (2.6). Then $\hat{\eta}_t := e^{-\sigma^2 t}\eta_t$ satisfies

$$d\hat{\eta}_t = 2\hat{\eta}_t^\lambda dt, \quad \hat{\eta}_0 = k^2.$$

Hence

$$d\eta_t = e^{\sigma^2 t}(2\hat{\eta}_t^\lambda + \sigma^2\hat{\eta}_t)dt \geq (2\eta_t^\lambda + \sigma^2\eta_t)dt.$$

By the same line as above, using Ito's formula and Jensen's inequality, we deduce

$$\begin{aligned} d(E[K_t^2]) &= E[2(L_t^{1-\alpha}K_t^{2\lambda} - \delta K_t^2 - c_t K_t^2) + \sigma^2 L_t^2 K_t^2]dt \\ &\leq (2E[K_t^2]^\lambda + \sigma^2 E[K_t^2])dt. \end{aligned}$$

Since $\eta_0^2 = E[K_0^2] = k^2$, we get $E[K_t^2] \leq \eta_t$ and (2.6).

3 The Hamilton-Jacobi-Bellman equation

In this section, we completely characterize the value function defined by

$$V(k) = \sup_{(\mathbf{L}, \mathbf{c}) \in \mathcal{A}} E[\int_0^\infty e^{-\beta t} U(c_t K_t)dt], \tag{3.1}$$

and we derive the optimal policy $(\mathbf{L}^*, \mathbf{c}^*) \in \mathcal{A}$. Using dynamic programming technique, we show that V is a viscosity solution of the associate Hamilton-Jacobi-Bellman (HJB, for short) equation :

$$\beta v = \max_{\Delta \le L \le 1} \{\frac{1}{2}\sigma^2 L^2 k^2 v'' + L^{1-\alpha} k^\alpha v'\} - \delta k v' + \tilde{U}(k, v'), \quad k > 0, \tag{3.2}$$

where

$$\tilde{U}(k, p) := \max_{0 \le c \le \bar{c}} \{U(ck) - ckp), \quad p \in \mathbf{R}. \tag{3.3}$$

The definition of viscosity solutions of (3.2) is given as follows.

Definition 3.1 *Let $v \in C(0, \infty)$. Then v is called a viscosity solution of (3.2) if*

for any $\varphi \in C^2(0, \infty)$ and any local maximum point $y > 0$ of $v - \varphi$,

$$-\beta v + \max_{\Delta \le L \le 1} \{\frac{1}{2}\sigma^2 L^2 k^2 \varphi'' + L^{1-\alpha} k^\alpha \varphi'\} - \delta k \varphi' + \tilde{U}(k, \varphi')\Big|_{k=y} \ge 0, \tag{3.4}$$

and

for any $\varphi \in C^2(0, \infty)$ and any local minimum point $z > 0$ of $v - \varphi$,

$$-\beta v + \max_{\Delta \le L \le 1} \{\frac{1}{2}\sigma^2 L^2 k^2 \varphi'' + L^{1-\alpha} k^\alpha \varphi'\} - \delta k \varphi' + \tilde{U}(k, \varphi')\Big|_{k=z} \le 0. \tag{3.5}$$

Following Morimoto and Zhou [15], we make use of the comparative results on dynamic programming.

Theorem 3.2 *Assume (2.4). Then the value function V of (3.1) is nondecreasing, continuous on 0, ∞) and the dynamic programming principle holds for V, i. e.,*

$$V(k) = \sup_{(\mathbf{L}, \mathbf{c}) \in \mathcal{A}} E[\int_0^\tau e^{-\beta t} U(c_t K_t) dt + e^{-\beta\tau} V(K_\tau)] \tag{3.6}$$

for any $\tau \ge 0$.

Proof. See Appendix A.

Our method is often referred to as the viscosity solutions method which is extensively developed by Fleming and Soner [6]. Since the existence of a viscosity solution substantially follows from the dynamic programming principle for the value function, we can state the following.

Theorem 3.3 *Assume (2.4). Then the value function V is the viscosity solution of the HJB equation (3.2).*

Proof. Let $\varphi \in C^2(0, \infty)$ and $z > 0$ be such that $V(x) - \varphi(x) \ge V(z) - \varphi(z) = 0$ for x near z. Let $\{K_t\}$ be the solution of (2.1) with $K_0 = z$ corresponding to any constant policy $(\mathbf{L}, \mathbf{c}) \in \mathcal{A}$ such that $L_t = L$, $c_t = c$ for all $t \ge 0$. By (3.6), we have

$$\varphi(z) \ge E[\int_0^\tau e^{-\beta t} U(cK_t) dt + e^{-\beta t} \varphi(K_\tau)]$$

for sufficiently small τ. Applying Ito's formula to φ, we have

$$0 \geq E[\int_0^\tau e^{-\beta t}\{-\beta\varphi + \frac{1}{2}\sigma^2 L^2 k^2 \varphi'' + L^{1-\alpha}k^\alpha \varphi' - \delta k \varphi' + U(ck) - ck\varphi'\}\Big|_{k=K_t} dt].$$

By (2.6), we note that

$$E[\sup_{0 \leq t \leq \tau} |K_t - z|^2] \leq CE[\int_0^\tau (1 + |K_t|^2)dt] \to 0 \quad \text{as} \quad \tau \to 0$$

for some constant $C > 0$, independent of τ. Dividing both sides by τ and letting $\tau \to 0$, we deduce (3.5). Similarly, by the same arguments as Fleming and Soner [6], we can obtain (3.4).

Next, we show the existence of a classical solution of the HJB equation.

Theorem 3.4 *Assume (2.4). Then the value function V is the $C^2(0, \infty)$ solution of the HJB equation (3.2).*

Proof. See Appendix B.

Now, let $\Pi(k)$ and $\Phi(k)$ be the maximizers of the HJB equation (3.2) for V respectively. By Theorems 3.2 and 3.3, we observe that $V' \geq 0$ and the maximum of $\frac{1}{2}\sigma^2 L^2 k^2 V'' + L^{1-\alpha}k^\alpha V'$ is attained at $L = 1$ if $V'' \geq 0$. Thus

$$\Pi(k) = \begin{cases} G(-\dfrac{(1-\alpha)k^\alpha V'(k)}{\sigma^2 k^2 V''(k)}) & \text{if} \quad V''(k) < 0, \\ 1 & \text{if} \quad V''(k) \geq 0, \end{cases} \tag{3.7}$$

$$\Phi(k) = I(k, V'(k)), \tag{3.8}$$

where

$$G(x) = \begin{cases} \Delta & \text{if} \quad x^{1/(1+\alpha)} \leq \Delta, \\ x^{1/(1+\alpha)} & \text{if} \quad \Delta < x^{1/(1+\alpha)} < 1, \\ 1 & \text{if} \quad 1 \leq x^{1/(1+\alpha)}, \end{cases} \tag{3.9}$$

and

$$I(k, p) = \begin{cases} (U')^{-1}(p)/k & \text{if} \quad U'(\bar{c}k) < p, \\ \bar{c} & \text{otherwise.} \end{cases} \tag{3.10}$$

By the regularity of the value function, we can derive the optimal policy in a feedback form.

Theorem 3.5 *Assume (2.4). Then the optimal policy $(\mathbf{L}^*, \mathbf{c}^*) \in \mathcal{A}$ for (2.3) is given by*

$$L_t^* = \Pi(K_t^*), \tag{3.11}$$

$$c_t^* = \Phi(K_t^*), \tag{3.12}$$

where $\{K_t^*\}$ is the solution of

$$dK_t^* = [\Pi(K_t^*)^{1-\alpha}(K_t^*)^\alpha - \delta K_t^* - \Phi(K_t^*)K_t^*]dt + \sigma\Pi(K_t^*)K_t^* dW_t, \; K_0^* = k > 0. \tag{3.13}$$

Proof. See Appendix C.

4 Existence of surplus workers

In this section, we illustrate that the economy, facing some quantity of capital stock, has redundant labor force from the viewpoint of social optimality. Finding a quantity of capital stock which yields shortening of working hours is tedious but straightforward.

In order to make the significance of the risk elements clear, let us consider the deterministic version of our Cobb-Douglas growth model (i. e., $\sigma = 0$) first. Then the HJB equation can be rewritten as

$$\begin{aligned} \beta v &= \max_{\Delta \le L \le 1} \{L^{1-\alpha} k^{\alpha} v'\} - \delta k v' + \tilde{U}(k, v') \\ &= k^{\alpha} v' - \delta k v' + \tilde{U}(k, v'), \quad k > 0. \end{aligned}$$

Thus the optimal employment policy leads to full employment $L = 1$ at every moment of time.

In the present case, by (3.7) and Theorem 3.5, the laborers are always fully employed if

$$\Pi(k) = 1 \quad \text{for all} \quad k > 0. \tag{4.1}$$

The optimal employment policy is the same as in the traditional deterministic optimal growth models. Furthermore, V solves

$$\beta V = \frac{1}{2}\sigma^2 k^2 V'' + k^{\alpha} V' - \delta k V' + \tilde{U}(k, V'), \quad k > 0, \tag{4.2}$$

By the same line as Appendix C, we see that the uniqueness of (4.2) holds. Hence we have

$$V(k) = \sup_{(\mathbf{1}, \mathbf{c}) \in \mathcal{A}} E\left[\int_0^{\infty} e^{-\beta t} U(c_t K_t)\, dt\right], \tag{4.3}$$

where $\{K_t\}$ denotes the solution of (2.1) with $\mathbf{1}$ replacing $\mathbf{L}$, i. e.,

$$dK_t = [K_t^{\alpha} - \delta K_t - c_t K_t]\, dt + \sigma K_t\, dW_t, \quad K_0 = k > 0.$$

Furthermore, we can state that if (4.1) is satisfied, then

$$V \text{ is concave on } (0, \infty). \tag{4.4}$$

Indeed, let $k_i > 0$, $i = 1, 2$, and $\varepsilon > 0$ be arbitrary. Then, by (4.3), there exists $(\mathbf{1}, \mathbf{c}^{(i)}) \in \mathcal{A}$ such that

$$V(k_i) - \varepsilon \le E\left[\int_0^{\infty} e^{-\beta t} U(c_t^{(i)} K_t^{(i)})\, dt\right], \tag{4.5}$$

where $\{K_t^{(i)}\}$ is the solution of (2.1) corresponding to $(\mathbf{1}, \mathbf{c}^{(i)}) \in \mathcal{A}$ with $K_0^{(i)} = k_i$. For $0 \le \xi \le 1$, we set

$$\overline{c}_t = \frac{\xi c_t^{(1)} K_t^{(1)} + (1-\xi) c_t^{(2)} K_t^{(2)}}{\xi K_t^{(1)} + (1-\xi) K_t^{(2)}}.$$

Clearly $(\mathbf{1}, \overline{\mathbf{c}})$ belongs to $\mathcal{A}$. Define

$$\begin{aligned} d\overline{K}_t &= [(\overline{K}_t)^{\alpha} - \delta \overline{K}_t - \overline{c}_t \overline{K}_t]\, dt + \sigma \overline{K}_t\, dW_t, \ \overline{K}_0 = \xi k_1 + (1-\xi) k_2, \\ \tilde{K}_t &= \xi K_t^{(1)} + (1-\xi) K_t^{(2)}. \end{aligned}$$

By concavity, we have

$$\tilde{K}_t \leq \xi k_1 + (1-\xi) k_2 + \int_0^t [(\tilde{K}_s)^\alpha - \delta \tilde{K}_s - \overline{c}_t \tilde{K}_s] dt + \sigma \int_0^t \tilde{K}_s dW_s.$$

By the comparison theorem, we get

$$\tilde{K}_t \leq \overline{K}_t \quad \text{for all} \quad t \geq 0, \quad \text{a. s.}$$

It follows from (2.4) and (4.5) that

$$\begin{aligned} V(\xi k_1 + (1-\xi) k_2) &\geq E[\int_0^\infty e^{-\beta t} U(\overline{c}_t \overline{K}_t) dt] \\ &\geq E[\int_0^\infty e^{-\beta t} U(\overline{c}_t \tilde{K}_t) dt] \\ &= E[\int_0^\infty e^{-\beta t} U(\xi c_t^{(1)} K_t^{(1)} + (1-\xi) c_t^{(2)} K_t^{(2)}) dt] \\ &\geq \xi E[\int_0^\infty e^{-\beta t} U(c_t^{(1)} K_t^{(1)})] dt + (1-\xi) E[\int_0^\infty e^{-\beta t} U(c_t^{(2)} K_t^{(2)}) dt] \\ &\geq \xi V(k_1) + (1-\xi) V(k_2) - \varepsilon. \end{aligned}$$

Therefore, letting $\varepsilon \to 0$, we obtain (4.4).

Now, let us examine the issue of surplus workers. By (3.7), our model exhibits surplus workers if

$$\Pi(k) < 1 \quad \text{for some} \quad k > 0. \tag{4.6}$$

The question arises, can the positive liesure existence condition (4.6) actually be satisfied in our model? In case of $\sigma \neq 0$, we shall show that (4.6) generally holds. Suppose that (4.1) is fulfilled. Then, by (4.4), we note that

$$-\sigma^2 k^2 V'' \leq (1-\alpha) k^\alpha V', \tag{4.7}$$

and

$$0 \leq V'(k) k \leq V(k) - V(0+) \leq V(k) \quad \text{for} \quad k > 0. \tag{4.8}$$

By (4.2) and (4.8), we get

$$\begin{aligned} (\beta + \delta + \overline{c}) V &\geq \frac{\alpha+1}{2} k^\alpha V' - \delta k V' + U(\overline{c} k) - \overline{c} k V' + (\delta + \overline{c}) V \\ &\geq U(\overline{c} k) \quad \to \quad \infty \quad \text{as} \quad k \to \infty. \end{aligned}$$

Hence, by (4.4) again

$$V'(k) > 0 \quad \text{for all} \quad k > 0. \tag{4.9}$$

Also, by (4.2)

$$0 \leq \beta V \leq (k^\alpha - \delta k) V' + U(\overline{c} k).$$

Thus, by (2.4)

$$V' \leq \frac{U(\overline{c} k)}{\overline{c} k} \frac{\overline{c} k}{\delta k - k^\alpha} \quad \to \quad U'(\infty) \overline{c} / \delta = 0 \quad \text{as} \quad k \to \infty. \tag{4.10}$$

Furthermore, by (4.7) and (4.10)

$$0 \leq \lim_{k \to \infty} \frac{V''(k)}{(\alpha - 1)k^{\alpha - 2}} \leq \lim_{k \to \infty} \frac{V'(k)}{\sigma^2} = 0.$$

Therefore, by L'Hospital's rule

$$\lim_{k \to \infty} \frac{V'(k)}{k^{\alpha - 1}} = 0. \tag{4.11}$$

We set $H(k) = V'(k)k^{1-\alpha}$. By (4.11), there exists $k_n \geq n$, for any $n \in \mathbf{N}$, such that

$$H'(k_n) = V''(k_n)k_n^{1-\alpha} + (1-\alpha)V'(k_n)k_n^{-\alpha} \leq 0.$$

Hence, by (4.7)

$$(1-\alpha)V'(k_n) \leq -V''(k_n)k_n \leq \frac{1-\alpha}{\sigma^2}V'(k_n)\frac{k_n^\alpha}{k_n^2}k_n.$$

Thus, by (4.9), we deduce

$$1 \leq k_n^{\alpha-1}/\sigma^2 \quad \to \quad 0 \quad \text{as} \quad n \to \infty,$$

which is a contradiction. Therefore we obatin (4.6).

5 Conclusion

This paper developed an optimal stochastic growth model characterized by uncertainty about the capital accumulation process, in which the authorities determines both consumption and employment policies. We apply the viscosity solutions method to obtain the optimal policies in feedback forms under rather mild assumptions on the utility function. Our results are normative, in the sense that they rigorously show the optimal positive paths of capital stock, consumption and employment if the regulation of the households' working hours and consumption is possible.

Furthermore, while it is not obvious whether the optimal policies involve full employment all the time or not, we found out that the optimal unemployment actually occurs for some quantity of capital stock in our stochastic economy. We notice that this result requires no extra specifications or assumptions on the primitives of the model.

The optimal unemployment or leisure in our model should be regarded as the consequence of the risk averse behavior of the representative agent. Our findings obtained here indicate that it is possible that the full employment policy is socially unoptimal in the risky economy, and if so, we must lower the output level less than full employment level to achieve social optimality.

Appendix A: Proof of Theorem 3.2

To prove the theorem, we need the following two lemmas.

Lemma A.1 *For each* $(\mathbf{L}, \mathbf{c}) \in \mathcal{A}$, *let* $\{K_t^{(i)}\}$, $i = 1, 2$, *be two solutions of (2.1) with* $K_0^{(i)} = k_i > 0$. *Then there exits* $C_\varepsilon > 0$, *for any* $\varepsilon > 0$, *such that*

$$E[|K_t^{(1)}-K_t^{(2)}|]\le C_\varepsilon|k_1-k_2|+\varepsilon(1+t^{1/(1-\alpha)}+k_1+k_2). \qquad (A.1)$$

Proof.　Define

$$x_t:=(K_t^{(1)})^{1-\alpha},\quad y_t:=(K_t^{(2)})^{1-\alpha}.$$

By (2.7), we have

$$d(x_t-y_t)=-(1-\alpha)(\delta+c_t+\frac{\sigma^2\alpha}{2}L_t^2)(x_t-y_t)dt+(1-\alpha)\sigma L_t(x_t-y_t)dW_t,$$

which implies

$$x_t-y_t=(x_0-y_0)\exp\{-(1-\alpha)(\int_0^t(\delta+c_s+\frac{\sigma^2}{2}L_s^2)ds)+\int_0^t(1-\alpha)\sigma L_s dW_s\}. \qquad (A.2)$$

By (2.2), we note that

$$E[\exp\{\int_0^t\sigma L_s dW_s-\int_0^t\frac{\sigma^2}{2}L_s^2)ds\}]=1.$$

Then

$$E[|x_t-y_t|^\theta]\le|x_0-y_0|^\theta E[\exp\{\int_0^t\sigma L_s dW_s-\int_0^t\frac{\sigma^2}{2}L_s^2)ds\}]$$
$$=|k_1^{1-\alpha}-k_2^{1-\alpha}|^{1/(1-\alpha)}\le|k_1-k_2|,$$

where $\theta=1/(1-\alpha)>1$.　By Young's inequality, for any $\varepsilon>0$

$$|x^\theta-y^\theta|=|\int_x^y\theta t^{\theta-1}dt$$
$$\le\theta|x-y|(x^{\theta-1}+y^{\theta-1})$$
$$\le\theta[\frac{1}{\theta}(\frac{1}{\varepsilon})^\theta|x-y|^\theta+\frac{\theta-1}{\theta}\{\varepsilon(x^{\theta-1}+y^{\theta-1})\}^{\theta/(\theta-1)}],\quad x,\ y\ge 0.$$

Hence, there exists $C_\varepsilon>0$ such that

$$|x^\theta-y^\theta|\le C_\varepsilon|x-y|^\theta+\varepsilon(1+x^\theta+y^\theta),\quad x,\ y\ge 0.$$

Thus, by (2.5), we deduce

$$E[|K_t^{(1)}-K_t^{(2)}|]=E[|x_t^\theta-y_t^\theta|]$$
$$\le C_\varepsilon E[|x_t-y_t|^\theta]+\varepsilon E[1+x_t^\theta+y_t^\theta]$$
$$\le C_\varepsilon|k_1-k_2|+\varepsilon E[1+K_t^{(1)}+K_t^{(2)}]$$
$$\le C_\varepsilon|k_1-k_2|+\varepsilon\{1+2^\theta(t^\theta+k_1)+2^\theta(t^\theta+k_2)\},$$

which implies (A.1).

Lemma A.2　*Under (2.4), there exists* $\Theta_0>0$ *such that*

$$0\le V(k)\le k+\Theta_0,\quad k>0, \qquad (A.3)$$

and there exists $C_\rho > 0$, *for any* $\rho > 0$, *such that*

$$|V(k_1) - V(k_2)| \leq C_\rho |k_1 - k_2| + \rho(1 + k_1 + k_2), \quad k_1, k_2 > 0. \tag{A.4}$$

Proof. We take $\Xi_1, \Xi_2 > 0$ such that

$$\beta + \delta > \Xi_2, \quad k^\alpha \leq \Xi_1 + \Xi_2 k, \quad \forall k \geq 0.$$

Then we see that $\Theta(k) := k + \Theta_0$ satisfies

$$-\beta\Theta + \frac{1}{2}\sigma^2 k^2 \Theta'' + (k^\alpha - \delta k)\Theta' + \tilde{U}(k, \Theta') \leq -\beta\Theta_0 + \Xi_1 + U \circ (U')^{-1}(1) < 0$$

for a suitable choice of $\Theta_0 > 0$. By Ito's formula, we have

$$\begin{aligned}
0 &\leq e^{-\beta t}\Theta(K_t) \\
&= \Theta(k) + \int_0^t e^{-\beta s}\{-\beta\Theta(K_s) + [L_s^{1-\alpha}K_s^\alpha - \delta K_s - c_s K_s]\Theta'(K_s) \\
&\qquad + \frac{1}{2}\sigma^2 L_s^2 K_s^2 \Theta''(K_s)\} ds + \int_0^t e^{-\beta s}\sigma\Theta'(K_s) L_s K_s dW_s \\
&\leq \Theta(k) - \int_0^t e^{-\beta s} U(c_s K_s) ds + \int_0^t e^{-\beta s}\sigma\Theta'(K_s) L_s K_s dW_s.
\end{aligned}$$

Taking the expectation of the both sides, by (2.6), we obtain (A.3).

Under (2.4), for any $\rho > 0$, there exist $x_\rho > 0$ such that $U(x) < \rho$ for all $x \leq x_\rho$. Taking $C_\rho = U'(x_\rho) > 0$, we have

$$|U(x) - U(y)| \leq C_\rho |x - y| + \rho, \quad x, y \geq 0. \tag{A.5}$$

Therefore, by (A.1), we deduce that

$$\begin{aligned}
|V(k_1) - V(k_2)| &\leq \sup_{(\mathbf{L}, \mathbf{c}) \in \mathcal{A}} E\left[\int_0^\infty e^{-\beta t} |U(c_t K_t^{(1)}) - U(c_t K_t^{(2)})| dt\right] \\
&\leq \sup_{(\mathbf{L}, \mathbf{c}) \in \mathcal{A}} C_\rho E\left[\int_0^\infty e^{-\beta t} |K_t^{(1)} - K_t^{(2)}| dt\right] + \rho/\beta \\
&\leq C_\rho E\left[\int_0^\infty e^{-\beta t}\{C_\varepsilon |k_1 - k_2| + \varepsilon(1 + t^{1/(1-\alpha)} + k_1 + k_2)\} dt\right] + \rho/\beta,
\end{aligned}$$

which implies (A.4).

Proof of Theorem 3.2

By (A.2) and (A.4), it is clear that V is nondecreasing and continuous. To prove the theorem, let $\tilde{V}(k)$ denote the right-hand side of (3.6). Define

$$\begin{aligned}
\tilde{K}_t &= K_{t+\tau}, \quad \tilde{W}_t = W_{t+\tau} - W_\tau, \\
\tilde{c}_t &= c_{t+\tau}, \quad \tilde{L}_t = L_{t+\tau}.
\end{aligned}$$

Then $\{\tilde{W}_t\}$ is a Brownian motion and $\{\tilde{K}_t\}$ satisfies

$$d\tilde{K}_t = [\tilde{L}_t^{1-\alpha}\tilde{K}_t^{\alpha} - \delta\tilde{K}_t - \tilde{c}_t\tilde{K}_t]dt + \sigma\tilde{L}_t\tilde{K}_t d\tilde{W}_t, \quad \tilde{K}_0 = K_\tau.$$

Using the conditional expectation $E[\cdot|\mathcal{F}_t]$, we have

$$e^{\beta t}E[\int_\tau^\infty e^{-\beta t}U(c_tK_t)dt|\mathcal{F}_\tau] = E[\int_0^\infty e^{-\beta t}U(\tilde{c}_t\tilde{K}_t)dt|\mathcal{F}_\tau] = J_{K_\tau}(\tilde{\mathbf{L}}, \tilde{\mathbf{c}}), \quad a.s.$$

Thus

$$\begin{aligned} J_k(\mathbf{L}, \mathbf{c}) &= E[\int_0^\tau e^{-\beta t}U(c_tK_t)dt + \int_\tau^\infty e^{-\beta t}U(c_tK_t)dt] \\ &\leq E[\int_0^\tau e^{-\beta t}U(c_tK_t)dt + e^{-\beta\tau}V(K_\tau)]. \end{aligned}$$

Taking the supremum, we obtain $V(k) \leq \tilde{V}(k)$.

To prove the reverse inequality, we fix any $\varepsilon > 0$. Select a sequence $\{S_j : j = 1, \ldots, n+1\}$ of disjoint subsets of $(0, \infty)$ such that

$$diam(S_j) < \nu, \ \bigcup_{j=1}^{n} S_j = (0, R) \quad \text{and} \quad S_{n+1} = [R, \infty)$$

for ν, $R > 0$ chosen later. We pick $k^{(j)} \in S_j$ and $(\mathbf{L}^{(j)}, \mathbf{c}^{(j)}) \in \mathcal{A}$ such that

$$V(k^{(j)}) - \varepsilon \leq J_{k^{(j)}}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)}), \quad j = 1, \ldots, n+1. \tag{A.6}$$

By the same argument as (A. 4), we can show that

$$|J_{k_1}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)}) - J_{k_2}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)})| + |V(k_1) - V(k_2)| \leq C_\varepsilon|k_1 - k_2| + \frac{\varepsilon}{4}(1 + k_1 + k_2), \quad k_1, k_2 > 0$$

for some constant $C_\varepsilon > 0$. Taking $0 < \nu < 1$ such that $C_\varepsilon\nu < \varepsilon/2$, we have

$$\begin{aligned} &|J_{k_1}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)}) - J_{k_2}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)})| + |V(k_1) - V(k_2)| \leq \varepsilon(1 + k_1), \\ &\qquad k_1, k_2 \in S_j, \quad j = 1, 2, \ldots, n \end{aligned} \tag{A.7}$$

If $K_\tau \in S_j$, we see by (A. 6) and (A. 7) that

$$\begin{aligned} J_{K_\tau}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)}) &= J_{K_\tau}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)}) - J_{k^{(j)}}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)}) + J_{k^{(j)}}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)}) \\ &\geq -\varepsilon(1 + K_\tau) + V(k^{(j)}) - \varepsilon \\ &\geq -2\varepsilon(1 + K_\tau) + V(K_\tau) - \varepsilon \\ &\geq -3\varepsilon(1 + K_\tau) + V(K_\tau), \quad j = 1, 2, \ldots, n. \end{aligned} \tag{A.8}$$

By definition, there exists $(\mathbf{L}, \mathbf{c}) \in \mathcal{A}$ such that

$$\tilde{V}(k) - \varepsilon \leq E[\int_0^\tau e^{-\beta t}U(c_tK_t)dt + e^{-\beta\tau}V(K_\tau)].$$

Define

$$\begin{aligned} &L_t^\tau = L_t 1_{\{t<\tau\}} + L_{t-\tau}^{(j)} 1_{\{\tau\leq t\}} \quad \text{if} \quad K_\tau \in S_j, \ j = 1, \ \ldots, \ n+1, \\ &c_t^\tau = c_t 1_{\{t<\tau\}} + c_{t-\tau}^{(j)} 1_{\{\tau\leq t\}} \quad \text{if} \quad K_\tau \in S_j, \ j = 1, \ \ldots, \ n+1, \\ &\qquad \mathbf{L}^\tau := \{L_t^\tau\}, \ \mathbf{c}^\tau := \{c_t^\tau\} \end{aligned}$$

Then, $(\mathbf{L}^\tau, \mathbf{c}^\tau)$ belongs to $\mathcal{A}$. Let $\{K_t^\tau\}$ be the solution of

$$dK_t^\tau = [(L_t^\tau)^{1-\alpha}(K_t^\tau)^\alpha - \delta K_t^\tau - c_t^\tau K_t^\tau]dt + \sigma L_t^\tau K_t^\tau dW_t, \quad K_0^\tau = k > 0.$$

Clearly, $K_t^\tau = K_t$ a. s. if $t < \tau$. Moreover, for each $j = 1, \ldots, n$, we have on $\{K_\tau \in S_j\}$

$$\begin{aligned} K_{t+\tau}^\tau &= K_\tau + \int_\tau^{t+\tau} [(L_s^\tau)^{1-\alpha}(K_s^\tau)^\alpha - \delta K_s^\tau - c_s^\tau K_s^\tau]ds + \int_\tau^{t+\tau} \sigma L_s^\tau K_s^\tau dW_s \\ &= K_\tau + \int_0^t [(L_s^{(j)})^{1-\alpha}(K_{s+\tau}^\tau)^\alpha - \delta K_{s+\tau}^\tau - c_s^{(j)} K_{s+\tau}^\tau]ds + \int_0^t \sigma L_s^{(j)} K_{s+\tau}^\tau d\tilde{W}_s, \quad a.\ s. \end{aligned}$$

Hence, $K_{t+\tau}^\tau$ coincides with the solution $K_t^{(j)}$ of (2. 1) with $K_0^{(j)} = K_\tau$ corresponding to $(\mathbf{L}^{(j)}, \mathbf{c}^{(j)})$ and $\{\tilde{W}_t\}$ on $\{K_\tau \in S_j\}$. Thus

$$\begin{aligned} J_{K_\tau}(\tilde{\mathbf{L}}^\tau, \tilde{\mathbf{c}}^\tau) &= E[\int_0^\infty e^{-\beta t} U(c_{t+\tau}^\tau K_{t+\tau}^\tau)dt | \mathcal{F}_\tau] \\ &= E[\int_0^\infty e^{-\beta t} U(c_t^{(j)} K_t^{(j)})dt | \mathcal{F}_\tau] \\ &= J_{K_\tau}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)}), \quad a.\ s. \quad \text{on} \quad \{K_\tau \in S_j\}, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{A. 9}$$

By Proposition 2. 1 and (A. 3), for each $k > 0$, we choose $R > 0$ such that

$$\begin{aligned} \sup_{(\mathbf{L}, \mathbf{c}) \in \mathcal{A}} E[V(K_\tau) 1_{\{K_\tau \geq R\}}] &\leq \sup_{(\mathbf{L}, \mathbf{c}) \in \mathcal{A}} E[\Theta(K_\tau) 1_{\{K_\tau \geq R\}}] \\ &\leq \sup_{(\mathbf{L}, \mathbf{c}) \in \mathcal{A}} \frac{1}{R} E[K_\tau^2 + \Theta_0 K_\tau] \\ &\leq \frac{C_0}{R}(1 + k + k^2) < \varepsilon, \end{aligned} \tag{A. 10}$$

where the constant $C_0 > 0$ depends only on τ and Θ_0. By (A. 6) - (A. 10), we have

$$\begin{aligned} E[\int_\tau^\infty e^{-\beta t} U(c_t^\tau K_t^\tau)dt] &= E[E[\int_0^\infty e^{-\beta t} U(c_t^\tau K_t^\tau)dt | \mathcal{F}_\tau]] \\ &= E[e^{-\beta\tau} J_{K_\tau}(\tilde{\mathbf{L}}^\tau, \tilde{\mathbf{c}}^\tau)] \\ &= E[\sum_{j=1}^{n+1} e^{-\beta\tau} J_{K_\tau}(\mathbf{L}^{(j)}, \mathbf{c}^{(j)}) 1_{\{K_\tau \in S_j\}}] \\ &\geq E[\sum_{j=1}^{n+1} e^{-\beta\tau} \{V(K_\tau) - 3\varepsilon(1 + K_\tau)\} 1_{\{K_\tau \in S_j\}}] \\ &\geq E[e^{-\beta\tau}\{V(K_\tau) - V(K_\tau) 1_{\{K_\tau \geq R\}}\}] - 3\varepsilon E[1 + K_\tau] \\ &\geq E[e^{-\beta\tau} V(K_\tau)] - \varepsilon - 3\varepsilon C(1 + k) \end{aligned}$$

for some constant $C > 0$, independent of ε. Thus

$$\begin{aligned} V(k) &\geq E[\int_0^\tau e^{-\beta t} U(c_t^\tau K_t^\tau)dt + \int_\tau^\infty e^{-\beta t} U(c_t^\tau K_t^\tau)dt] \\ &\geq E[\int_0^\tau e^{-\beta t} U(c_t K_t)dt + e^{-\beta t} V(K_\tau)] - \varepsilon - 3\varepsilon C(1 + k) \\ &\geq \tilde{V}(k) - 2\varepsilon - 3\varepsilon C(1 + k). \end{aligned}$$

Therefore, letting $\varepsilon \to 0$, we obtain $\tilde{V}(k) \leq V(k)$. The proof is complete.

Appendix B : Proof of Theorem 3.4

To prove the theorem, we consider the boundary value problem :

$$\begin{cases}\beta w = \max_{\Delta \le L \le 1} \{\frac{1}{2}\sigma^2 L^2 k^2 w'' + L^{1-\alpha} k^\alpha w'\} - \delta k w' + \tilde{U}(k, w') \quad \text{in} \quad (a, b) \\ w(a) = V(a), \quad w(b) = V(b), \end{cases} \tag{B.1}$$

for any fixed $0 < a < b$.

Lemma B.1 *Assume (2.4). Let* $w_i \in C(a, b)$, $i = 1, 2$, *be two viscosity solutions of (B.1). Then we have* $w_1 = w_2$.

Proof. Suppose that there exists $x_0 \in (a, b)$ such that

$$w_1(x_0) > w_2(x_0).$$

Then we can find $\overline{x} \in (a, b)$ such that

$$\varpi := \sup_{x \in [a, b]} \{w_1(x) - w_2(x)\} = w_1(\overline{x}) - w_2(\overline{x}) > 0.$$

Define

$$\Psi_n(x, y) := w_1(x) - w_2(y) - \frac{n}{2}|x-y|^2, \quad n \in \mathbf{N}$$

Then, for each $n \in \mathbf{N}$, there exists $(x_n, y_n) \in [a, b]^2$ such that

$$\Psi_n(x_n, y_n) = \sup_{(x, y) \in [a, b]^2} \Psi_n(x, y) \ge \Psi_n(\overline{x}, \overline{x}) = \varpi \tag{B.2}$$

Hence

$$\frac{n}{2}|x_n - y_n|^2 < w_1(x_n) - w_2(y_n),$$

and

$$|x_n - y_n| \to 0 \quad as \quad n \to \infty. \tag{B.3}$$

Since $\Psi_n(x_n, y_n) \ge \Psi_n(x_n, x_n)$ and w_2 is uniformly continuous, we have

$$\frac{n}{2}|x_n - y_n|^2 \le w_2(x_n) - w_2(y_n) \to 0 \quad as \quad n \to \infty. \tag{B.4}$$

By (B.2), (B.3) and (B.4), extracting a subsequence, we deduce that

$$(x_n, y_n) \to (\hat{x}, \hat{x}) \in (a, b)^2 \quad as \quad n \to \infty. \tag{B.5}$$

Now, by (B.5), we may consider $(x_n, y_n) \in (a, b)^2$. By Ishii's lemma (Crandall, Ishii and Lions [5]), there exist $X, Y \in R$ such that

$$
\begin{aligned}
&(n(x_n-y_n),\ X)\in \overline{J}^{2,+}w_1(x_n),\\
&(n(x_n-y_n),\ Y)\in \overline{J}^{2,-}w_2(y_n), \qquad\qquad (\mathrm{B}.6)\\
&\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\le 3n\begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix}
\end{aligned}
$$

where $\overline{J}^{2,+}$, $\overline{J}^{2,-}$ respectively denote the second-order superjets and subjets. From the equivalent definition of viscosity solutions, it follows that

$$
\begin{aligned}
\beta w_1(x_n)&\le \max_{\Delta\le L\le 1}\{\frac{\sigma^2}{2}L^2x_n^2X+L^{1-\alpha}x_n^\alpha n(x_n-y_n)\}-\delta x_n n(x_n-y_n)+\tilde{U}(x_n,\ n(x_n-y_n)),\\
\beta w_2(y_n)&\ge \max_{\Delta\le L\le 1}\{\frac{\sigma^2}{2}L^2y_n^2X+L^{1-\alpha}y_n^\alpha n(x_n-y_n)\}-\delta y_n n(x_n-y_n)+\tilde{U}(y_n,\ n(x_n-y_n)).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\beta\{w_1(x_n)-w_2(y_n)\}&\le \max_{\Delta\le L\le 1}\{\frac{\sigma^2}{2}L^2(x_n^2X-y_n^2Y)+L^{1-\alpha}(x_n^\alpha-y_n^\alpha)n(x_n-y_n)\}\\
&\quad -\delta n(x_n-y_n)^2+\tilde{U}(x_n,\ n(x_n-y_n))-\tilde{U}(y_n,\ n(x_n-y_n))\\
&\le \frac{\sigma^2}{2}(x_n^2X-y_n^2Y)^+ +n(x_n^\alpha-y_n^\alpha)(x_n-y_n)\\
&\quad +\tilde{U}(x_n,\ n(x_n-y_n))-\tilde{U}(y_n,\ n(x_n-y_n)).
\end{aligned}
$$

By (B.4) and (B.6), we get

$$
\begin{aligned}
&\frac{\sigma^2}{2}(x_n^2X-y_n^2Y)^+\le \frac{3\sigma^2 n}{2}(x_n-y_n)^2\to 0,\\
&n(x_n^\alpha-y_n^\alpha)(x_n-y_n)\le n\alpha a^{\alpha-1}(x_n-y_n)^2\to 0 \quad as \quad n\to\infty.
\end{aligned}
$$

Moreover, by (3.3), (A.5), (B.4) and (B.5)

$$
\begin{aligned}
\tilde{U}(x_n,\ n(x_n-y_n))-\tilde{U}(y_n,\ n(x_n-y_n))&\le \max_{0\le c\le \overline{c}}|U(cx_n)-U(cy_n)|+n|x_n-y_n|^2\\
&\le C_\rho|x_n-y_n|+\rho+n|x_n-y_n|^2\to 0\\
&\qquad as \quad n\to\infty \quad \text{then} \quad \rho\to 0.
\end{aligned}
$$

Therefore, by (B.2) and (B.5), we deduce that

$$
0<\varpi\le w_1(\hat{x})-w_2(\hat{x})\le 0,
$$

which is a contradiction. Interchanging w_1 and w_2, we obtain $w_1=w_2$.

Lemma B.2 *Assume (2.4). Then there exists a unique solution* $w\in C^2(a,\ b)\cap C[a,\ b]$ *of (B.1).*

Proof. For each L, $c\in[0,\ 1]\cap\mathbf{Q}$, denote $\varrho=(L,\ c)$. We define

$$
\begin{aligned}
&F^\varrho(k,z,p,r)=-\beta z+\{\frac{1}{2}L^2\sigma^2k^2r+L^{1-\alpha}k^\alpha p\}-\delta kp+\{U(ck)-cxp\},\\
&\qquad (k,\ z,\ p,\ r)\in(a,\ b)\times\mathbf{R}^3.
\end{aligned}
$$

Then F^{ϱ} has the following properties :

F. 1 : $(z, p, r) \mapsto F^{\varrho}(k, z, p, r)$: convex,

F. 2 : $z \mapsto F^{\varrho}(k, z, p, r)$: nonincreasing,

F. 3 : $0 < (\Delta\sigma a)^2/2 \leq F_r^{\varrho} \leq (\sigma b)^2/2$

F. 4 : $|F_p^{\varrho}|, |F_z^{\varrho}|, |F_{rk}^{\varrho}|, |F_{pk}^{\varrho}|, |F_{zk}^{\varrho}| \leq C$,

F. 5 : $|F_k^{\varrho}|, |F_{kk}^{\varrho}| \leq C(1+|p|+|r|)$,

where the constant $C>0$ is independent of $\varrho(k, z, p, r)$.

According to Gilbarg and Trudinger [10, Thm. 17. 18], there exists a unique solution $w \in C^2(a, b) \cap C[a, b]$ of

$$\sup_{\varrho} F^{\varrho}(k, w, w', w'') = 0 \quad \text{in} \quad (a, b),$$
$$w(a) = V(a), \quad w(b) = V(b),$$

which is equivalent to (B. 1).

Proof of Theorem 3. 4

By Lemma B. 2, $w \in C^2(a, b) \cap C[a, b]$ is a viscosity solution of (B. 1). By Theorem 3. 3, V is a viscosity solution of (B. 1). By Lemma B. 1, we have $w = V$ on $[a, b]$. Therefore, we deduce $V \in C^2(0, \infty)$. The proof is complete.

Appendix C : Proof of Theorem 3. 5

Before going into the proof of the theorem, we prepare the following lemma. Let $v \in C^2(0, \infty)$ be the nonnegative nondecreasing solution of (3. 2) and π, ϕ denote Π, Φ of (3. 7) and (3. 8) with v replacing V.

Lemma C. 1 *We make the assumption of Theorem 3. 5. Then $\pi(k)$ is locally Lipschitz on $(0, \infty)$.*

Proof. Define the pairwise disjoint open sets O_i, $i = 1, 2, \ldots, 7$, by

$$O_1 = \{k>0 : \Delta^{1+\alpha} < g(k) < 1,\ v''(k) < 0\},$$
$$O_2 = \{k>0 : g(k) < \Delta^{1+\alpha},\ v''(k) < 0\},$$
$$O_3 = \{k>0 : g(k) > 1,\ v''(k) < 0\},$$
$$O_4 = int\{k>0 : g(k) = \Delta^{1+\alpha},\ v''(k) < 0\},$$
$$O_5 = int\{k>0 : g(k) = 1,\ v''(k) < 0\},$$
$$O_6 = int\{k>0 : v''(k) = 0\},$$
$$O_7 = \{k>0 : v''(k) > 0\},$$

where $g(k) := (1-\alpha)k^{\alpha}v'/\sigma^2 k^2(-v'')$ and int denotes the interior of $\{\cdot\}$. Each O_i can be expressed as

$$O_i = \bigcup_{I \in \mathcal{J}_i} I,\ i = 1, 2, \ldots, 7,$$

where $\mathcal{J}_i$ is a countable collection of non-overlapping closed intervals I. We split the proof into several steps.

(1) Let $0 < a < b$ be arbitrary. For any $I \in \mathcal{J}_1$, we have

$$\Delta^{1+\alpha} < g < 1,\ v'' < 0 \quad \text{on} \quad I_1 := I \cap [a,\ b]. \tag{C.1}$$

Substituting (3.7) and (3.8) into (3.2), we get

$$\beta v = \frac{1}{2}\sigma^2 \pi(k)^2 k^2 v'' + \pi(k)^{1-\alpha} k^\alpha v' - \delta k v' + U(\phi(k)k) - \phi(k) k v'. \tag{C.2}$$

Hence

$$\begin{aligned} Q &:= \beta v + \delta k v' - U(\phi(k)k) + \phi(k) k v' \\ &= \frac{1}{2}\sigma^2 \{\frac{(1-\alpha)k^\alpha v'}{\sigma^2 k^2(-v'')}\}^{2/(1+\alpha)} k^2 v'' + \{\frac{(1-\alpha)k^\alpha v'}{\sigma^2 k^2(-v'')}\}^{(1-\alpha)/(1+\alpha)} k^\alpha v'. \end{aligned} \tag{C.3}$$

Also, by (C.1) and (C.2)

$$Q \geq \frac{1}{2}\sigma^2 \Delta^2 k^2 v'' + \Delta^{1-\alpha} k^\alpha v' \geq \{\frac{1}{2}\Delta^2 \frac{\alpha-1}{\Delta^{1+\alpha}} + \Delta^{1-\alpha}\} k^\alpha v' \geq \frac{\alpha+1}{2}\Delta^{1-\alpha} \inf_{k \in I_1} \{k^\alpha v'\} > 0.$$

Multiplying both sides of (C.3) by $(-v'')^{(1-\alpha)/(1+\alpha)}/Q$, we get

$$(-v'')^{(1-\alpha)/(1+\alpha)} = [-\frac{1}{2}\sigma^2 \{\frac{1-\alpha}{\sigma^2} k^{\alpha-2} v'\}^{2/(1+\alpha)} k^2 + \{\frac{1-\alpha}{\sigma^2} k^{\alpha-2} v'\}^{(1-\alpha)/(1+\alpha)} k^\alpha v']/Q. \tag{C.4}$$

By (3.8), we note that

$$\phi(k)k = \begin{cases} (U')^{-1}(v'(k)) & \text{if} \ \ U'(\bar{c}k) < v'(k), \\ \bar{c}k & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} &\frac{d}{dk}(U(\phi(k)k) - \phi(k) k v'(k)) \\ &\quad = \begin{cases} -(U')^{-1}(v'(k)) v''(k) & \text{if} \ \ U'(\bar{c}k) < v'(k), \\ \bar{c}U'(\bar{c}k) - \bar{c}v'(k) - \bar{c}k v''(k) & \text{otherwise.} \end{cases} \end{aligned}$$

By (C.3), this yields that

$$\sup_{k \in I_1} |Q'(k)| < \infty.$$

Differentiating (C.4), we have

$$\sup_{k \in I_1} |v'''(k)| < \infty,$$

and thus

$$\sup_{k \in I_1} |g'(k)| < \infty.$$

Therefore, by (3.7), π is Lipschitz on I_1.

(2) Let $I_i = I \cap [a,\ b]$ for $I \in \mathcal{J}_i$, $i = 2,\ 4$. By (3.7) and (3.9), $\pi(k) = \Delta$ on I_i. Thus π is Lipschitz

on I_i.

(3)　Let $I_i = I \cap [a, b]$ for $I \in \mathcal{J}_i$, $i = 3, 5, 6, 7$. By (3.7) and (3.9), $\pi(k) = 1$ on I_i. Thus π is Lipschitz on I_i.

(4)　Let $0 < a \le k_1 \le k_2 \le b$. We take a sequence of $J_j = [a_j, b_j] \in \mathcal{J}_i$ for some $i \in \{1, 2, \dots, 6\}$ such that

$$[k_1, k_2] = \text{the closure of } \bigcup_{j=1}^{\infty} J_j,$$

where $a_j = k_1$ if $k_1 \in J_j$ and $b_j = k_2$ if $k_2 \in J_j$. By (1)-(3), we have

$$\begin{aligned}|\pi(k_1) - \pi(k_2)| &= |\sum_j \{\pi(a_j) - \pi(b_j)\}| \le \sum_j |\pi(a_j) - \pi(b_j)| \\ &\le \sum_j C|a_j - b_j| = C|k_1 - k_2|\end{aligned}$$

for some constant $C > 0$. Therefore π is Lipschitz on $[a, b]$, and then locally Lipschitz on $(0, \infty)$.

Proof of Theorem 3.5

We set $A(k) = \Pi(k)^{1-\alpha} k^{\alpha} - \delta k - \Phi(k)k$ and $B(k) = \sigma \Pi(k) k$. By (3.7), (3.8) and Lemma C.1, we observe that $A(k)$ and $B(k)$ are locally Lipschitz on $(0, \infty)$. Taking into account $A(0) = B(0) = 0$, we can obtain their locally Lipschitz extensions on $\mathbf{R}$, also denoted by $A(k)$ and $B(k)$. Furthermore,

$$\begin{aligned}|A(k)| + |B(k)| &\le k^{\alpha} + \delta k + \bar{c}k + |\sigma|k \\ &\le 1 + (1 + \delta + \bar{c}^2 + |\sigma|)k \quad \text{if} \quad k \ge 0.\end{aligned}$$

Thus, by a standard result on SDEs (Friedman [9]), there exists a unique solution $\{K_t^*\}$ of (3.13). By the same calculation as (2.7), we have $K_t^* > 0$ a.s.

Now, in view of (A.3), let the nonnegative, nondecreasing solution v of (3.2) satisfy

$$0 \le v(k) \le k + \Theta_1, \quad k \ge 0 \tag{C.5}$$

for some constant $\Theta_1 > 0$. Applying Ito's formula to (3.2), by (3.11) and (3.12), we have

$$\begin{aligned}e^{-\beta t} v(K_t^*) &= v(k) + \int_0^t e^{-\beta s} \{-\beta v + (\pi(k)^{1-\alpha} k^{\alpha} - \delta k - \phi(k)k) v' \\ &\quad + \frac{1}{2}\sigma^2 \pi(k)^2 k^2 v''\}\Big|_{k = K_s^*} ds + \int_0^t e^{-\beta s} v'(K_s^*) \sigma \pi(K_s^*) K_s^* dW_s \\ &= v(k) - \int_0^t e^{-\beta s} U(\phi(K_s^*) K_s^*) ds + M_t,\end{aligned}$$

where $M_t = \int_0^t e^{-\beta s} v'(K_s^*) \sigma \pi(K_s^*) K_s^* dW_s$. Hence

$$E[e^{-\beta(t \wedge \tau_n)} v(K_{t \wedge \tau_n}^*)] = v(k) - E[\int_0^{t \wedge \tau_n} e^{-\beta s} U(c_s^* K_s^*) ds] \tag{C.6}$$

for a localizing sequence $\{\tau_n\}$ of stopping times of $\{M_t\}$ with $\tau_n \uparrow \infty$. By (3.13) and (2.6), we see that

$$\sup_n E[(K_{t \wedge \tau_n}^*)^2] < \infty.$$

Thus, by (C.5), $\{e^{-\beta(t \wedge \tau_n)} v(K_{t \wedge \tau_n}^*)\}$ is uniformly integrable in n. Letting $n \to \infty$ in (C.6), we get

$$E[e^{-\beta t} v(K_t^*)] = v(k) - E[\int_0^t e^{-\beta s} U(c_s^* K_s^*) ds].$$

By (C. 5) and (2. 5)

$$E[e^{-\beta t} v(K_t^*)] \leq E[e^{-\beta t}(K_s^* + \Theta_1)] \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore we deduce

$$v(k) = E[\int_0^\infty e^{-\beta s} U(c_s^* K_s^*) ds].$$

By the same calculation as above, we have

$$v(k) \geq E[\int_0^\infty e^{-\beta s} U(c_s K_s) ds]$$

for all $(\mathbf{L}, \mathbf{c}) \in \mathcal{A}$. This yields the optimality of $(\mathbf{L}^*, \mathbf{c}^*)$. We remark that under (C. 4), we have $v = V$ and the uniqueness of (3. 2) holds. The proof is complete.

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