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Multiplicity of $L(k)$ for the Stable Decomposition of $B(\mathbf{Z}/p)^n$

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

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Abstract

The purpose of the paper is to determine the multiplicity of the indecomposable summand $L(k)$ for the classifying spaces of elementary abelian p-groups.

1. Introduction

Let $B(\mathbf{Z}/p)^n$ be the classifying space of an elementary abelian p-group $(\mathbf{Z}/p)^n$. We consider the complete stable wedge decomposition of $B(\mathbf{Z}/p)^n_+$.

In [3], Harris and Kuhn showed that the homotopy classes of the indecomposable summands of such decomposition are in one-to-one correspondence with the isomorphism classes of irreducible $\mathbf{F}_p[M_n(\mathbf{Z}/p)]$-modules, and a given homotopy type appears with the multiplicity equal to the dimension of the corresponding irreducible module.

In [4], we calculated the multiplicity of certain indecomposable summands in the complete stable decomposition of $B(\mathbf{Z}/p)^n_+$.

Let $L(k)$ be the indecomposable summand of $B(\mathbf{Z}/p)^k_+$ associate to the Steinberg module.

The purpose of the paper is to determine the multiplicity of $L(k)$ in the complete stable decomposition of $B(\mathbf{Z}/p)^n_+$. The main theorem is Theorem 3.4.

In section 2, we review general results of stable splitting and multiplicity of stable summands. In section 3, we determine the multiplicity of $L(k)$.

2. Multiplicity of stable summands

Let p be a prime number and $M_n(\mathbf{Z}/p)$ the semigroup of all $n \times n$ matrices with entries in $\mathbf{Z}/p$.

A *p-regular partition*, α, is a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$ of non-negative integers such that $0 \leq \alpha_i - \alpha_{i+1} \leq p-1$ for all $i \geq 1$. The number of posibive entries in a p-regular partition α is called the *length* of α, and we write $l(\alpha)$ for it. Then there are p^n such partitions with length $\leq n$. Put $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{l(\alpha)}$.

Let $W^\alpha(n)$ be the Weyl module ($\mathbf{F}_p[M_n(\mathbf{Z}/p)]$-module) associate to p-regular partition α of length$\leq n$. Let $F^\alpha(n)$ be the top composition factor of $W^\alpha(n)$. Then $\{F^\alpha(n) \mid \alpha$ is a p-regular partition of length$\leq n\}$ is a complete set of irreducible $\mathbf{F}_p[M_n(\mathbf{Z}/p)]$-modules. For detail, see [5] and [3].

Harris and Kuhn [3] showed that the homotopy classes of the indecomposable summands of $B(\mathbf{Z}/p)^n_+$ are in one-to-one correspondence with the isomorphism classes of irreducible $\mathbf{F}_p[M_n(\mathbf{Z}/p)]$-modules.

Let X_α be the indecomposable summand of $B(\mathbf{Z}/p)^n_+$ associate to $F^\alpha(n)$. Let $m(X_\alpha, n)$ be the multiplicity of X_α in the complete stable splitting of $B(\mathbf{Z}/p)^n_+$.

Then by Harris-Kuhn [3], we have

Theorem 2.1. *The multiplicity of X_α in $B(\mathbf{Z}/p)^n_+$ is*

$$m(X_\alpha, n) = \dim F^\alpha(n).$$

Since the Weyl module $W^\alpha(n)$ is not always irreducible, it is difficult to compute the multiplicity $m(X_\alpha, n)$. In a special case we have :

Theorem 2.2. *If the Weyl module $W^\alpha(|\alpha|)$ is irreducible then $W^\alpha(n)$ is irreducible for $n \geq l(\alpha)$.*

To prove the theorem we need the following propositions.

Proposition 2.3. [3, 4] *There are non-negative integers $s(\alpha, i)$ for $i \geq 0$ such that*

(1) $s(\alpha, i) > 0$ *if and only if* $l(\alpha) \leq i \leq |\alpha|$, *and*

(2) $m(X_\alpha, n) = \sum_{i=0}^{n} \binom{n}{i} s(\alpha, i)$ *for* $n \geq 0$.

In a similar way we can easily prove the following :

Proposition 2.4. *There are non-negative integers $s'(\alpha, i)$ for $i \geq 0$ such that*

(1) $s'(\alpha, i)>0$ *if and only if* $l(\alpha)\leq i\leq|\alpha|$,

(2) $0\leq s(\alpha, i)\leq s'(\alpha, i)$ *for* $i\geq 0$, *and*

(3) $\dim W^{\alpha}(n)=\sum_{i=0}^{n}\binom{n}{i}s'(\alpha, i)$ *for* $n\geq 0$.

Proof of Theorem 2.2. Since $W^{\alpha}(|\alpha|)$ is irreducible, we have

$$\dim W^{\alpha}(|\alpha|)=\sum_{i=0}^{|\alpha|}\binom{|\alpha|}{i}s'(\alpha, i)=\sum_{i=0}^{|\alpha|}\binom{|\alpha|}{i}s(\alpha, i).$$

Therefore, by the above propositions, $s'(\alpha, i)=s(\alpha, i)$ for $i\geq 0$. This implies $\dim W^{\alpha}(n)=\dim F^{\alpha}(n)$, hense the theorem holds.

3. Multiplicity of $L(k)$

Let $L(k)$ be the indecomposable summand of $B(\mathbf{Z}/p)_+^k$ associte to the p-regular partition $\alpha(k)=(k(p-1), (k-1)(p-1),\ldots,2(p-1), p-1)$. For detail, see [6] and [3].

Theorem 3.1. [6] *The multiplicity of* $L(k)$ *in* $B(\mathbf{Z}/p)_+^k$ *is*

$$m(L(k), k)=\dim W^{\alpha(k)}(k)=p^{\binom{k}{2}}.$$

In order to compute the multiplicity $m(L(k), n)$ of the indecomposable summand $L(k)$ in $B(\mathbf{Z}/p)_+^n$ for $k\leq n$, we use the following properties of the Weyl module $W^{\alpha(k)}(n)$.

Proposition 3.2. *The dimension of the Weyl module* $W^{\alpha(k)}(n)$ *is*

$$\dim W^{\alpha(k)}(n)=\prod_{i=1}^{k}\frac{p^i i!(n-k-1+ip)!}{(ip)!(n-k-1+i)!}.$$

Proof. It is an immediate consequence from Corollary 8.1.17 of James–Kerber [5].

Proposition 3.3. *The Weyl module* $W^{\alpha(k)}(n)$ *is an irreducible module, that is*

$$W^{\alpha(k)}(n)=F^{\alpha(k)}(n).$$

Proof. Since associated Young diagram $[\alpha(k)]$ of $\alpha(k)$ has no p-hooks, its p-core is $[\alpha(k)]$. Therefore, Theorem 2.19 of Carlisle [1] implies the proposition.

By the above two propositions, we have the following theorem and corollary :

Theorem 3.4. *The multiplicity of* $L(k)$ *in* $B(\mathbf{Z}/p)_+^n$ *is*

$$m(L(k), n)=\prod_{i=1}^{k}\frac{p^i i!(n-k-1+ip)!}{(ip)!(n-k-1+i)!}.$$

Corollary 3.5. *For* $p=2$, *the multiplicity of* $L(k)$ *in* $B(\mathbf{Z}/2)_+^n$ *is*

$$m(L(k), n)=\prod_{i=1}^{k}\frac{i!}{(2i-1)!!}\binom{n-k-1+2i}{i}.$$

References

[1] D.P. Carlisle, The modular representation theory of $GL(n,p)$, and applications to topology, Ph. D. thesis, University of Manchester, 1985.

[2] D. P. Carlisle and N. J. Kuhn, Smash products of summands of $B(\mathbf{Z}/p)^n_+$, *Algebraic Topology*, Contemporary Mathematics, vol. 96, AMS, 1989, 87–102.

[3] J. C. Harris and N. J. Kuhn, Stable decompositions of classifying spaces of finite abelian p-groups, *Math. Proc. Camb. Phil. Soc.*, **103**(1988), 427–449.

[4] K. Hirata, On the multiplicity of indecomposable stable summands for the classifying spaces of elementary abelian p-groups, *Mem. Fac. Educ. Ehime Univ., Nat. Sci.*, **11**(1991), 1–11.

[5] G. James and A. Kerder, *The Representation Theory of the Symmetric Group*, Encyclopedia of Math., vol. 16, Addison-Wesley, 1981.

[6] S. A. Mitchell and S. B. Priddy, Stable splittings derived from the Steinberg module, *Topology*, **22**(1983), 285–298.