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Characterization of Eight Points in the Plane Containing no Convex Pentagons

Dedicated to Professor Yasutoshi Nomura on his 60th birthday

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1. Introduction

It is well know that if five points in the plane are in general position (no three are collinear), then they contain a convex quadrilateral. This is the case $n = 4$ of the conjecture [3] that in the plane any set of at least $2^{n-2} + 1$ points in general position contains a convex n-gons. For the case $n = 5$, this has been proved by [1] and [4], but the conjecture is still open for $n > 5$.

In [4], Kalbfleisch, Kalbfleisch and Stanton showed examples of eight points containing no convex pentagons. The purpose of this paper is to characterize eight points in the plane which contains no convex pentagons. The main results are Theorem 3.1, 3.2 and 3.3.

Throughout the paper we assume every set of finite points in the plane must be in general position.

2. Preliminaries

Let S be a set of finite points in the plane, we write conv$S$ for the set of vertices of the convex hull of S. First we recall the following definition of Bonnice [1].

Definition 2.1. Let S be a set of finite points in the plane, then we call S is of type $(k_1, k_2, \dots, k_r)$ if there is a sequence $S = S_1, S_2, \dots, S_r$ such that

(1) convS_i consists of k_i points, for $1 \leq i \leq r$,
(2) $S_{i+1} = S -$ convS_i for $1 \leq i < r$, and
(3) $S_r =$ convS_r.

From the definition above, any set of eight points in the plane which contains no convex pentagons must be one of the following type : (4, 4), (4, 3, 1), (3, 4, 1), or (3, 3, 2). The following lemma is a result of Bonnice [1].

Lemma 2.2. *If a set S of eight points is of type* (4, 3, 1) *or* (3, 3, 2)*, then S contains a convex pentagon.*

To characterize a set of eight points which contains no convex pentagons, we introduce the following definition.

Definition 2.3. We call a set of eight points in the plane has a *convex quadrilateral cell decomposition* if it has a quadrilateral cell decomposition AA′B′B, BB′C′C, CC′D′D, DD′A′A and ABCD such that all the quadrilaterals are convex (Figure 1 and 2). We denote it by ABCD–A′B′C′D′.

3. Main results

Theorem 3.1. *If a set of eight points in the plane contains no convex pentagons, then it has a convex quadrilateral cell decomposition.*

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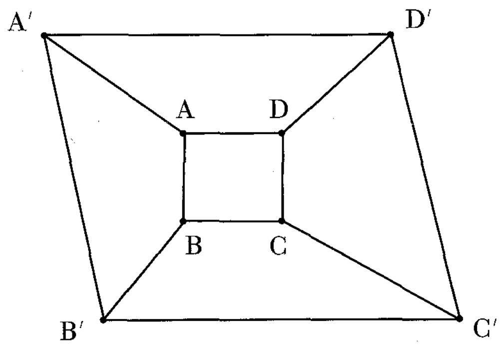

Figure 1

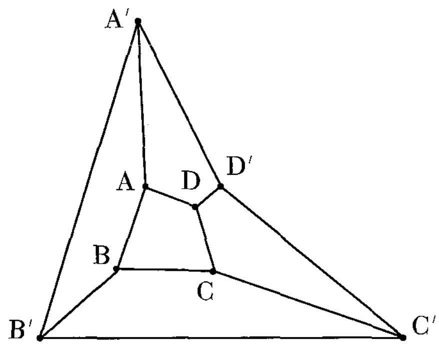

Figure 2

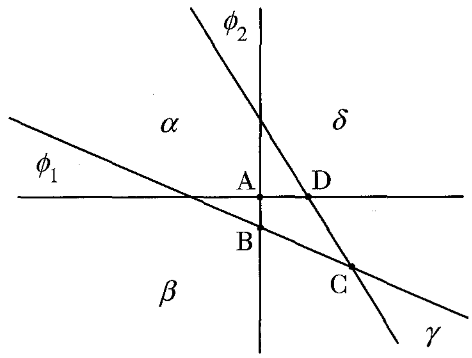

Figure 3

Proof. Let S be a set of eight points in the plane which contains no convex pentagons. By Lemma 2.2, the set S is of type (4, 4) or (3, 4, 1).

First, we assume S is of type (4, 4). We may assume the interior quadrilateral ABCD is in the situation illustrated in Figure 3. Since one of the exterior four points and quadrilateral ABCD do not form a convex pentagon, the four exterior points are contained in $\alpha \cup \beta \cup \gamma \cup \delta \cup \phi_1 \cup \phi_2$. If α contains two exterior points, then the two exterior points and the triangle BCD form a convex pentagon. Hence α, similarly β, γ and δ contains at most one of exterior points. If ϕ_1 contains one of exterior points, then either $\alpha \cup \delta \cup \phi_1 \cup \phi_2$ or $\beta \cup \gamma \cup \phi_1$ contains at least three exterior points. If one of them, say $\alpha \cup \delta \cup \phi_1 \cup \phi_2$, contains three exterior points, then the tree exterior points and AD form a convex pentagon. Hence ϕ_1, similarly ϕ_2 contains no exterior points. Let A′, B′, C′ and D′ be the exterior point in the area α, β, γ and δ, respectively. Then S has a convex quadrilateral cell decomposition ABCD–A′B′C′D′.

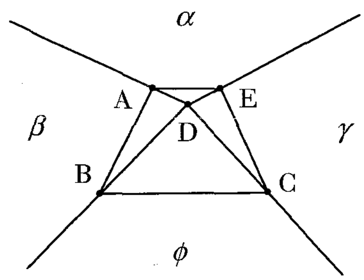

Figure 4

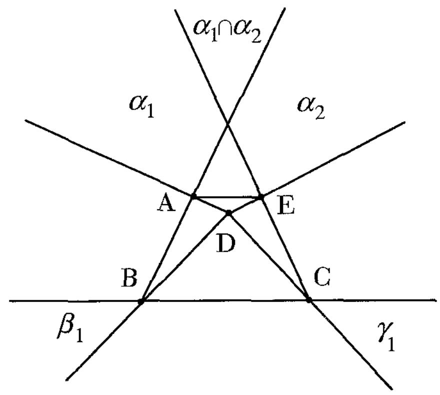

Figure 5

Next, we assume S is of type (3, 4, 1). We may assume the interior five points A, B, C, D and E are in the situation illustrated in Figure 4, where ABCD, ABCE and BCED are convex quadrilaterals. If α contains two exterior points, then the two exterior points and the triangle ADE form a convex pentagon. Hence α, similarly β, γ and ϕ contains at most one of the exterior points. If ϕ contains one exterior point, then either $\beta \cup \phi$ or $\gamma \cup \phi$ contains two exterior points. If one of them, say $\beta \cup \phi$, contains two exterior points, then the two exterior points and the triangle ACD form a convex pentagon. Thus ϕ contains no exterior points. Let A′, B′ and C′ be the exterior point contained in α, β, and γ, respectively. Next, consider

Figure 5. Since one of the exterior three points and one of the interior quadrilaterals ABCD, ABCE and BCED do not form a convex pentagon, the three exterior points A′, B′, and C′ must be contained in $\alpha_1 \cup \alpha_2$, β_1 and γ_1, respectively. If A′ is contained in α_1 then ABCD–A′B′C′E is a convex quadrilateral cell decomposition. If A′ is contained in α_2 then DBCE–AB′C′A′ is a convex quadrilateral cell decomposition. This completes the proof of the theorem.

Now, we consider the converse of Theorem 3.1.

Theorem 3.2. *If a set of eight points in the plane of type* (4, 4) *having a convex quadrilateral cell decomposition* ABCD–A′B′C′D′ *satisfies*

(1) *in Figure* 3, *none of the vertices* A′, B′, C′ *and* D′ *are contained in* $\phi_1 \cup \phi_2$,

(2) A′ *and* C′ *are on the other side of line* AC, *and*

(3) B′ *and* D′ *are on the other side of line* BD,

then the set contains no convex pentagons.

Proof. Let S be a set of eight points of type (4, 4) having a convex quadrilateral cell decomposition ABCD–A′B′C′D′ with the conditions (1), (2) and (3). If S contains a convex pentagon, then the vertices of the pentagon must be one of the followings :

(a) one vertex is from {A, B, C, D} and four vertices are from {A′, B′, C′, D′},

(b) two vertices are from {A, B, C, D} and three vertices are from {A′, B′, C′, D′},

(c) three vertices are from {A, B, C, D} and two vertices are from {A′, B′, C′, D′}, or

(d) four vertices are from {A, B, C, D} and one vertex is from {A′, B′, C′, D′}.

From the fact that S is of type (4, 4), the case (a) is not possible. Consider Figure 3. Since S has a convex quadrilateral cell decomposition and satisfies the condition (1), the vertices A′, B′, C′ and D′ must be contained in α, β, γ and δ, respectively. Therefore the cases (c) and (d) are not possible. From the conditions (2) and (3), the case (b) is not possible. Hence the theorem holds.

Theorem 3.3. *If a set of eight points in the plane of type* (3,4,1) *having a convex quadrilateral cell decomposition* ABCD–A′B′C′D′ *satisfies*

(1) B′ *and* D′ *are on the other side of line* BD, *then the set contains no convex pentagons, where triangle* A′B′C′ *is the convex hull.*

Proof. Let S be a set of eight points of type (3, 4, 1) having a convex quadrilateral cell decomposition ABCD–A′B′C′D′ with the property (1), and assume the convex hull of S is triangle A′B′C′. If S contains a convex pentagon, then the vertices of the pentagon must be one of the followings :

(a) two vertices are from {A, B, C, D, D′} and three vertices are from {A′, B′, C′},

(b) three vertices are from {A, B, C, D, D′} and two vertices are from {A′, B′, C′},

(c) four vertices are from {A, B, C, D, D′} and one vertex is from {A′, B′, C′}, or

(d) five vertices are from {A, B, C, D, D′}.

From the fact that S is of type (3, 4, 1), the cases (a) and (d) are not possible. We may assume the interior five points A, B, C, D and D′ is in the situation illustrated in Figure 5 with E=D′, where ABCD, ABCD′ and BCD′D are convex quadrilateral. Since S has a convex quadrilateral cell decomposition and satisfies the condition (1), the vertices A′, B′ and C′ must be contained in α_1, β_1, and γ_1, respectively. Therefore the cases (b) and (c) are not possible. This completes the proof of the theorem.

4. Application

For an application of Theorem 3.1, we show a simple proof of the following theorem, which has been proved before by Kalbfleisch, Kalbfleisch and Stanton [4], and Bonnice [1].

Theorem 4.1. *Every set of nine points in the plane contains a convex pentagon.*

Proof. Let S be a set of nine points in the plane. Let P be a point of the convex hull, and $T = S -$ {P}. Suppose T has no convex pentagons, otherwise T, hence S has a convex pentagon. By Theorem 3.1, the set T of eight points has a convex quadrilateral cell decomposition. Therefore the point P and one of the four outer quadrilaterals of T form a convex pentagon.

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