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Alternate Proofs of Art Gallery Theorems

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

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Abstract

In this paper we introduce alternate proofs of Chvátal's Art Gallery Theorem and O'Rourke's Mobile Guards Theorem. The main theorems are refined versions of the above theorems.

Keywords: art gallery; vertex guards; mobile guards.

1 Introduction

In 1975 Chvátal [1] proved the following art gallery theorem. In 1978 Fisk [2] showed a simpler proof of the theorem.

Theorem 1 *Any polygon of n edges can be covered by $\lfloor \frac{n}{3} \rfloor$ vertex guards.*

In 1983 O'Rourke [3] proved the following mobile guards theorem.

Theorem 2 *Any polygon of $n \geq 4$ edges can be covered by $\lfloor \frac{n}{4} \rfloor$ diagonal guards.*

In the paper we will state and prove refined versions of the above theorems. The main results are Theorem 3 and Theorem 4. For simplicity, we discuss only combinatorial guards.

A *triangulation graph* G of a polygon P with n vertices is a graph obtained by triangulating P with internal diagonals between vertices. The nodes of G correspond to the n vertices of P, and the arcs correspond to the n edges and $n-3$ diagonals. G has $n-2$ triangular faces.

Define a *guard* in a triangulation graph G of a polygon P to be a subset of the nodes of G. Then a *vertex guard* in G is a single node of G, and a *diagonal guard* in G is a pair of nodes adjacent across any arc of G. Finally, a collection of guards $C=\{g_1,\ldots,g_k\}$ is said to *dominate* G if every triangular face of G has at least one of its three nodes in some $g_i \in C$.

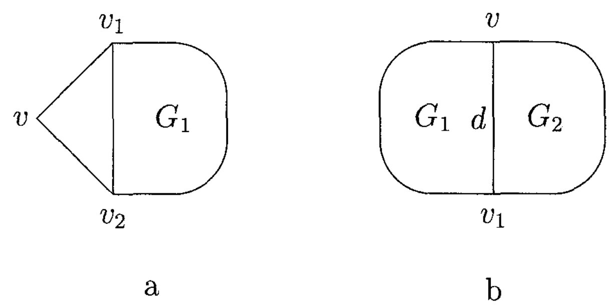

Figure 1: (1) $n=3k$.

2 Chvátal's art gallery theorem

Theorem 3 *Let P be a polygon of $n \geq 3$ vertices and*

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G a triangulation graph of P.

(1) *If $n=3k$, for any given vertex v of G, G can be dominated by k vertices such that one of them coincides with v.*

(2) *If $n=3k+1$, for any given boundary edge e of G, G can be dominated by k vertices such that one of them coincides with an end point of e.*

(3) *If $n=3k+2$, G can be dominated by k vertices.*

Proof. The theorem is true for $n=3$, so assume that $n>3$, and that the theorem holds for all $n'<n$.

To prove (1), let v be any given vertex of G and assume $n=3k$. We will consider possible cases in turn.

Case 1a. Suppose there are no internal diagonals with one end at v. Let vv_1v_2 be the triangle of G whose one vertex at v, and G_1 the remainder of G (see Figure 1a). Since G_1 has $n-1=3(k-1)+2$ vertices, it can be dominated by $k-1$ vertices by the induction hypothesis. Together with single vertex guard v of triangle vv_1v_2, all of G is dominated by k vertices such that one of them coincides with the given vertex v.

Case 1b. Suppose there is at least one internal diagonal with one end at v. Let $d=vv_1$ be such an internal diagonal, which partitions G into two graphs G_1 and G_2 (see Figure 1b). Let $n_1\geq3$ (resp. $n_2\geq3$) be the number of vertices of G_1 (resp. G_2), and put $n_1=3k_1+r_1$ and $n_2=3k_2+r_2$ $(0\leq r_1,r_2<3)$. Since $n=n_1+n_2-2$, we have $3k=3(k_1+k_2)+r_1+r_2-2$. Thus (r_1,r_2) must be one of (0,2), (1,1) and (2,0). Each will be considered in turn.

Case 1b.1 $((r_1,r_2)=(0,2))$. Since G_1 has $n_1=3k_1$ vertices, by induction hypothesis it is dominated by k_1 vertices such that one of them coincides with v. Similarly, G_2 has $n_2=3k_2+2$ vertices, so it is dominated by k_2 vertices. This yields a domination of G by $k_1+k_2=k$ vertices which contain the given vertex v.

Case 1b.2 $((r_1,r_2)=(1,1))$. Since $n_1=3k_1+1$ and $n_2=3k_2+1$, by induction hypothesis there is a domination of G_1 (resp. G_2) by k_1 (resp. k_2) vertices which contain one of v and v_1. If one of the dominations contain vertex v, then all of G is dominated by $k_1+k_2=k$ vertices which contain the given vertex v. If both dominations contain vertex v_1 simultaneously, then replace one of v_1's with v. This yields a domination of G by $k_1+k_2=k$ vertices which contain the given vertex v.

Case 1b.3 $((r_1,r_2)=(2,0))$. This is the mirror image of Case 1b.1.

To prove (2), let $e=v_1v_2$ be given boundary edge of G and assume $n=3k+1$. Let T be the triangle of G supported by $e=v_1v_2$, with its apex at v_3. We will consider possible cases in turn.

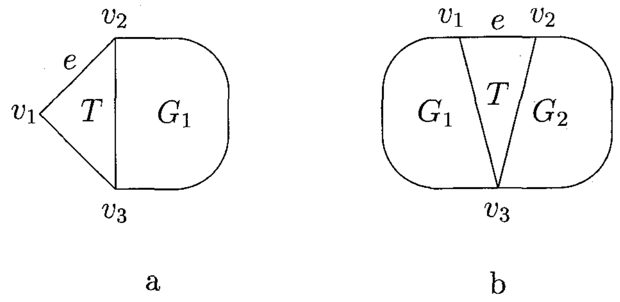

Figure 2: (2) $n=3k+1$.

Case 2a. Suppose one of v_1v_3 and v_2v_3, say v_1v_3, is boundary edge of G. Let G_1 be the remainder of G sharing v_2v_3 (see Figure 2a). Since G_1 has $n-1=3k$ vertices, by the induction hypothesis it can be dominated by k vertices which contain v_2. Since triangle T is covered by single vertex v_2, all of G is dominated by k vertices such that one of them coincides with an endpoint v_2 of e.

Case 2b. Suppose both v_1v_3 and v_2v_3 are internal diagonals, and they partition G into three graph G_1, G_2 and T as in Figure 2b. Let $n_1\geq3$ (resp. $n_2\geq3$) be the number of vertices of G_1 (resp. G_2), and put $n_1=3k_1+r_1$ and $n_2=3k_2+r_2$ $(0\leq r_1,r_2<3)$. Since $n=n_1+n_2-1$, we have $3k=3(k_1+k_2)+r_1+r_2-2$. Thus (r_1,r_2) must be one of (0,2), (1,1) and (2,0).

Case 2b.1 $((r_1,r_2)=(0,2))$. Since G_1 has $n_1=3k_1$ vertices, by induction hypothesis it is dominated by k_1 vertices such that one of them coincides with v_1. Since G_2 has $n_2=3k_2+2$ vertices, it is dominated by k_2 vertices by induction hypothesis. Triangle T is covered by single vertex v_1, thus all of G is dominated by $k_1+k_2=k$ vertices such that one of them coincides with an endpoint v_1 of e.

Case 2b.2 $((r_1,r_2)=(1,1))$. Since $n_1=3k_1+1$ and $n_2=3k_2+1$, by induction hypothesis there is a domination of G_1 by k_1 vertices which contain one of v_1 and v_3. Similarly, there is a domination of G_2 by k_2 vertices which contain one of v_2 and v_3. If one of the dominations contains either v_1 or v_2, then all of G is dominated by $k_1+k_2=k$ vertices which contain an endpoint of given edge e. If both dominations contain vertex v_3 simultaneously, then replace one of v_3's with v_1. This yields a domination of G by $k_1+k_2=k$ vertices which contain an endpoint of given edge e.

Case 2b.3 ($(r_1,r_2)=(2,0)$). This is the mirror image of Case 2b.1.

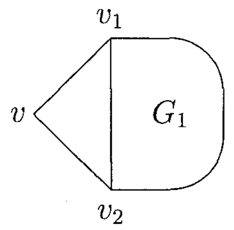

Figure 3: (3) $n=3k+2$.

To prove (3), assume $n=3k+2$. By Two Ears Theorem, G has at least two ears. Let vv_1v_2 be an ear of G such that v_1v_2 is an internal diagonal, and G_1 the remainder of G sharing v_1v_2 (Figure 3). Since G_1 has $n-1=3k+1$ vertices, by induction hypothesis it is dominated by k vertices such that one of them coincides with v_1 or v_2. Since triangle vv_1v_2 is covered by v_1 or v_2, all of G is dominated by k vertices.

This completes the proof of the theorem.

3 O'Rourke's mobile guards theorem

Theorem 4 *Let P be a polygon of $n\geq 4$ vertices and G a triangulation graph of P.*

(1) *If $n=4k$, for any given boundary edge e of G, G can be dominated by k diagonals such that one of them coincides with e.*

(2) *If $n=4k+1$, for any given vertex v of G, G can be dominated by k diagonals such that one of their endpoints coincides with v.*

(3) *If $n=4k+2$, for any given boundary edge e of G, G can be dominated by k diagonals such that one of their endpoints coincides with an end point of e.*

(4) *If $n=4k+3$, G can be dominated by k diagonals.*

Proof. The theorem is true for $n=4$, so assume that $n>4$, and that the theorem holds for all $n'<n$.

To prove (1), let $e=v_1v_2$ be given boundary edge of G and assume $n=4k\geq 8$. Let T be the triangle of G supported by $e=v_1v_2$, with its apex at v_3. We will consider possible cases in turn.

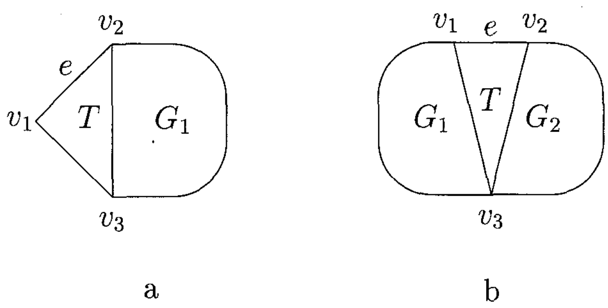

Figure 4: (1) $n=4k$.

Case 1a. Suppose one of v_1v_3 and v_2v_3, say v_1v_3, is boundary edge of G. Let G_1 be the remainder of G sharing v_2v_3 (see Figure 4a). Since G_1 has $n-1=4(k-1)+3$ vertices, by the induction hypothesis it can be dominated by $k-1$ diagonals. Together with single diagonal guard e of triangle T, all of G is dominated by k diagonals including given edge e.

Case 1b. Suppose both v_1v_3 and v_2v_3 are internal diagonals, and they partition G into three graph G_1, G_2 and T as in Figure 4b. Let $n_1\geq 3$ (resp. $n_2\geq 3$) be the number of vertices of G_1 (resp. G_2).

Case 1b.1 ($n_1=3$). Assume one of n_1 and n_2, say n_1, is equal to 3, and G_1 is a triangle $v_1v_3v_4$. Since $n_2=4(k-1)+2\geq 6$, by induction hypothesis G_2 is dominated by $k-1$ diagonals. Together with single diagonal guard e of quadrilateral $v_1v_2v_3v_4$, all of G is dominated by k diagonals including given edge e.

Now assume $n_1,n_2\geq 4$, and put $n_1=4k_1+r_1$ and $n_2=4k_2+r_2$ $(0\leq r_1,r_2<4)$. Since $n=n_1+n_2-1$, we have $4k=4(k_1+k_2)+r_1+r_2-1$. Thus (r_1,r_2) must be one of (0,1), (1,0), (2,3) and (3,2). Only two of these cases are distinct.

Case 1b.2 ($(r_1,r_2)=(0,1)$). Since G_1 has $n_1=4k_1$ vertices, by induction hypothesis it is dominated by k_1 diagonals such that one of them coincides with v_1v_3. Since G_2 has $n_2=4k_2+1$ vertices, by induction hypothesis it is dominated by k_2 diagonals such that one of their endpoints coincides with v_3. Replace the diagonal guard v_1v_3 with $e=v_1v_2$, then all of G is dominated by $k_1+k_2=k$ diagonals including given edge e.

Case 1b.3 ($(r_1,r_2)=(2,3)$). Since G_1 has $n_1=4k_1+2$ vertices, by induction hypothesis it is dominated by k_1 diagonals. Similarly, G_2 is dominated by k_2 diagonals. Together with single diagonal guard e of

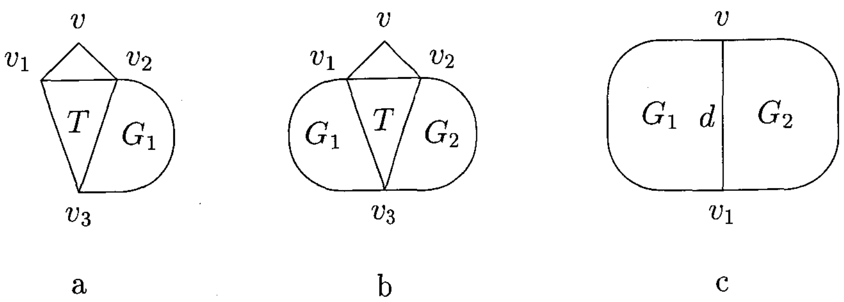

Figure 5: (2) $n=4k+1$.

T, all of G is dominated by $k_1+k_2+1=k$ diagonals including given edge e.

To prove (2), let v be given vertex of G and assume $n=4k+1\geq 5$. We will consider possible cases in turn.

First, suppose there are no internal diagonals with one end at v. Let vv_1v_2 be the triangle of G whose one vertex at v, and G' the remainder of G. Let T be the triangle of G' supported by v_1v_2, with its apex at v_3.

Case 2a. Suppose one of v_1v_3 and v_2v_3, say v_1v_3, is boundary edge of G'. Let G_1 be the remainder of G' sharing v_2v_3 (see Figure 5a).

Case 2a.1. Suppose $n=5$. Then G is a pentagon and dominated by single diagonal guard vv_2.

Case 2a.2. Suppose $n>5$. Since G_1 has $n-2=4(k-1)+3\geq 4$ vertices, by the induction hypothesis it can be dominated by $k-1$ diagonals. Together with single diagonal guard vv_2 of quadrilateral $vv_1v_3v_2$, all of G is dominated by k diagonals such that one of their endpoints coincides with given vertex v.

Case 2b. Suppose both v_1v_3 and v_2v_3 are internal diagonals, and they partition G' into three graph G_1, G_2 and T as in Figure 5b. Let $n_1\geq 3$ (resp. $n_2\geq 3$) be the number of vertices of G_1 (resp. G_2). Since $n=4k+1\geq 6$, we have $n\geq 9$.

Case 2b.1 ($n_1=3$). Assume one of n_1 and n_2, say n_1, is equal to 3, and G_1 is a triangle $v_1v_3v_4$. Since $n_2=4(k-1)+2\geq 6$, by induction hypothesis G_2 is dominated by $k-1$ diagonals. Together with single diagonal guard vv_1 of pentagon $vv_1v_4v_3v_2$, all of G is dominated by k diagonals such that one of their endpoints coincides with given vertex v.

Now assume $n_1,n_2\geq 4$, and put $n_1=4k_1+r_1$ and $n_2=4k_2+r_2$ $(0\leq r_1,r_2<4)$. Since $n=n_1+n_2$, we have $4k=4(k_1+k_2)+r_1+r_2-1$. Thus (r_1,r_2) must be one of (0,1), (1,0), (2,3) and (3,2). Only two of these cases are distinct.

Case 2b.2 ($(r_1,r_2)=(0,1)$). Since G_1 has $n_1=4k_1$ vertices, by the induction hypothesis it can be dominated by k_1 diagonals such that one of them coincides with v_1v_3. Since G_2 has $n_2=4k_2+1$ vertices, by the induction hypothesis it can be dominated by k_2 diagonals such that one of their endpoints coincides with v_3. Replace v_1v_3 with vv_1, then all of G is dominated by $k_1+k_2=k$ diagonals such that one of their endpoints coincides with given vertex v.

Case 2b.3 ($(r_1,r_2)=(2,3)$). Since G_1 has $n_1=4k_1+2$ vertices, by the induction hypothesis it can be dominated by k_1 diagonals. Similarly, G_2 can be dominated by k_2 diagonals. Together with single diagonal guard vv_1 of quadrilateral $vv_1v_3v_2$, all of G is dominated by $k_1+k_2+1=k$ diagonals such that one of their endpoints coincides with given vertex v.

Case 2c. Now suppose there is at least one internal diagonal with one end at v. Let $d=vv_1$ be such an internal diagonal, which partitions G into two graphs G_1 and G_2 (see Figure 5c). Let $n_1\geq 3$ (resp. $n_2\geq 3$) be the number of vertices of G_1 (resp. G_2).

Case 2c.1 ($n_1=3$). Assume one of n_1 and n_2, say n_1, is equal to 3, and G_1 is a triangle vv_1v_2. Since $n_2=4k\geq 4$, by induction hypothesis G_2 is dominated by k diagonals such that one of them coincides with $d=vv_1$. Since the triangle G_1 is covered by d, all of G is dominated by k diagonals such that one of their endpoints coincides with given vertex v.

Now assume $n_1,n_2\geq 4$, and put $n_1=4k_1+r_1$ and $n_2=4k_2+r_2$ $(0\leq r_1,r_2<4)$. Since $n=n_1+n_2-2$, we have $4k=4(k_1+k_2)+r_1+r_2-3$. Thus (r_1,r_2) must be one of (0,3), (1,2), (2,1) and (3,0). Only two of these cases are distinct.

Case 2c.2 ($(r_1,r_2)=(0,3)$). Since G_1 has $n_1=4k_1$ vertices, by induction hypothesis it is dominated by

k_1 diagonals such that one of them coincides with $d=vv_1$. Similarly, G_2 is dominated by k_2 diagonals. Thus all of G is dominated by $k_1+k_2=k$ diagonals such that one of their endpoints coincides with given vertex v.

Case 2c.3 ($(r_1,r_2)=(1,2)$). Since G_1 has $n_1=4k_1+1$ vertices, by induction hypothesis it is dominated by k_1 diagonals such that one of their endpoints coincides with v. Similarly, G_2 is dominated by k_2 diagonals. Thus all of G is dominated by $k_1+k_2=k$ diagonals such that one of their endpoints coincides with given vertex v.

To prove (3), let $e=v_1v_2$ be given boundary edge of G and assume $n=4k+2\geq 6$. Let T be the triangle of G supported by $e=v_1v_2$, with its apex at v_3. We will consider possible cases in turn.

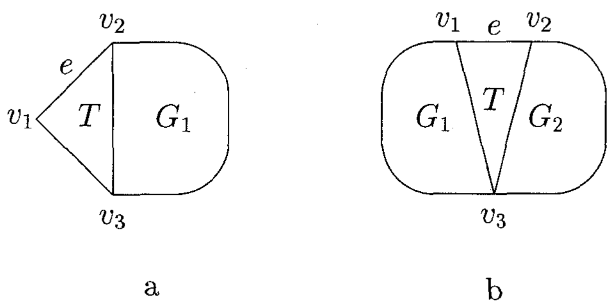

Figure 6: (3) $n=4k+2$.

Case 3a. Suppose one of v_1v_3 and v_2v_3, say v_1v_3, is boundary edge of G. Let G_1 be the remainder of G sharing v_2v_3 (see Figure 6a). Since G_1 has $n-1=4k+1\geq 5$ vertices, by the induction hypothesis it can be dominated by k diagonals such that one of their endpoints coincides with v_2. Since the triangle T is dominated by the vertex v_2, all of G is dominated by k diagonals such that one of their endpoints coincides with an endpoint of given edge e.

Case 3b. Suppose both v_1v_3 and v_2v_3 are internal diagonals, and they partition G into three graph G_1, G_2 and T as in Figure 6b. Let $n_1\geq 3$ (resp. $n_2\geq 3$) be the number of vertices of G_1 (resp. G_2).

Case 3b.1 ($n_1=3$). Assume one of n_1 and n_2, say n_1, is equal to 3, and G_1 is a triangle $v_1v_3v_4$. Since $n_2=4k\geq 4$, by induction hypothesis G_2 is dominated by k diagonals such that one of them coincides with v_2v_3. Since the quadrilateral $v_1v_2v_3v_4$ is covered by v_2v_3, all of G is dominated by k diagonals such that one of their endpoints coincides with an endpoint of given edge e.

Now assume $n_1,n_2\geq 4$, and put $n_1=4k_1+r_1$ and $n_2=4k_2+r_2$ $(0\leq r_1,r_2<4)$. Since $n=n_1+n_2-1$, we have $4k=4(k_1+k_2)+r_1+r_2-3$. Thus (r_1,r_2) must be one of (0,3), (1,2), (2,1) and (3,0). Only two of these cases are distinct.

Case 3b.2 ($(r_1,r_2)=(0,3)$). Since G_1 has $n_1=4k_1$ vertices, by induction hypothesis it is dominated by k_1 diagonals such that one of them coincides with v_1v_3. Since G_2 has $n_2=4k_2+3$ vertices, by induction hypothesis it is dominated by k_2 diagonals. Since the triangle T is covered by v_1v_3, all of G is dominated by $k_1+k_2=k$ diagonals such that one of their endpoints coincides with an endpoint of given edge e.

Case 1b.3 ($(r_1,r_2)=(1,2)$). Since G_1 has $n_1=4k_1+1$ vertices, by induction hypothesis it is dominated by k_1 diagonals such that one of their endpoints coincides with v_1. Similarly, G_2 is dominated by k_2 diagonals. Since the triangle T is covered by v_1, all of G is dominated by $k_1+k_2=k$ diagonals such that one of their endpoints coincides with an endpoint of given edge e.

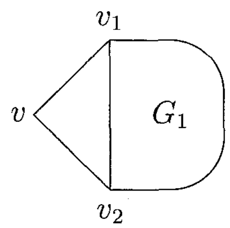

Figure 7: (4) $n=4k+3$.

To prove (4), assume $n=4k+3\geq 7$. By Two Ears Theorem, G has at least two ears. Let vv_1v_2 be an ear of G such that v_1v_2 is internal diagonal, and G_1 the remainder of G sharing v_1v_2 (Figure 7). Since G_1 has $n-1=4k+2$ vertices, by induction hypothesis it is dominated by k diagonals such that one of their endpoints coincides with one of v_1 and v_2. Since triangle vv_1v_2 is covered by v_1 or v_2, all of G is dominated by k diagonals.

This completes the proof of the theorem.

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