EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS

田 中　敏

1 9 9 9

①

EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS

Satoshi Tanaka

Graduate School of Science and Engineering,
Doctor Course, Ehime University

1999

Acknowledgment. The author thanks Professor Manabu Naito for reading the manuscript and making a number of helpful suggestions.

Contents

1. Introduction

In this thesis we are concerned with the existence and asymptotic behavior of solutions of neutral differential equations of the form

$$\frac{d^n}{dt^n}[x(t)+h(t)x(\tau(t))]+f(t,x(g(t)))=q(t), \tag{1.1}$$

where the following conditions are assumed:

(1.2) $n \in \mathbb{N}$;

(1.3) $t_0 > 0$;

(1.4) $h \in C[t_0, \infty)$;

(1.5) $\tau \in C[t_0,\infty)$ is strictly increasing, $\lim_{t\to\infty}\tau(t) = \infty$ and $\tau(t) < t$ for $t \geq t_0$;

(1.6) $g \in C[t_0,\infty)$ and $\lim_{t\to\infty} g(t) = \infty$;

(1.7) $f \in C([t_0,\infty)\times\mathbb{R})$;

(1.8) $q \in C[t_0, \infty)$.

By a solution of (1.1), we mean a function $x(t)$ which is continuous and satisfies (1.1) on $[t_x, \infty)$ for some $t_x \geq t_0$. Therefore, if $x(t)$ is a solution of (1.1), then $x(t) + h(t)x(\tau(t))$ is n-times continuously differentiable on $[t_x, \infty)$. Note that, in general, $x(t)$ itself is not n-times continuously differentiable.

A solution of (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. This means that a solution $x(t)$ is oscillatory if and only if there is a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to \infty$ as $i \to \infty$ and $x(t_i) = 0$ $(i = 1, 2, \dots)$, and a solution $x(t)$ is nonoscillatory if and only if $x(t) \neq 0$ for all large t.

In recent years there has been an increasing interest in oscillation theory for neutral differential equations, and a number of results have been obtained. For typical results we refer to the papers [1, 3–7, 9–23, 26–39, 42–50, 52–69] and the monographs [8] and [24]. In oscillation theory the problem of the existence and asymptotic behavior of solutions is quite important, and in the present paper this problem is discussed in detail for the neutral differential equation (1.1). Although the problem of finding oscillation criteria (that is, sufficient conditions for all solutions to be oscillatory) is also important, it is not considered in the present paper. In this paper we focus our attention to the problem of the existence and asymptotic behavior of solutions of (1.1). For oscillation criteria, see the papers [1, 5–22, 26–28, 30–32, 34, 37–43, 52, 53, 59–62, 64–66, 69] and the references cited therein.

It is possible to discuss more general neutral differential equations of the form

(1.9)
$$\frac{d^n}{dt^n}\left[x(t) + \sum_{i=1}^{k} h_i(t)x(\tau_i(t))\right] + f(t, x(g_1(t)), \cdots, x(g_m(t))) = q(t).$$

But, for simplicity, we restrict our attention to neutral differential equations of the form (1.1).

The neutral differential equations (1.1) and (1.9) may theoretically be regarded as an extended form of the differential equation with a deviating argument

$$x^{(n)}(t) + f(t, x(g(t))) = q(t), \tag{1.10}$$

for which the asymptotic and oscillatory behavior of solutions has been intensively studied in the last three decades.

Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. See, for example, [2], [25], [40], [41] and [51]. Let us give an example of this type. We consider the network in Figure 1.

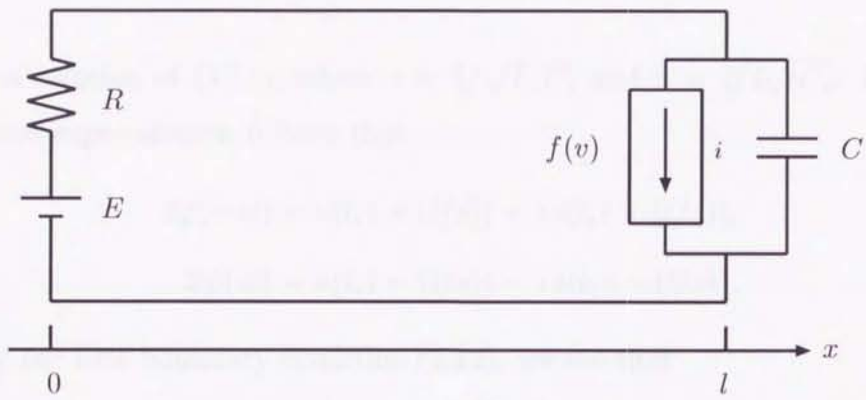


Figure 1.

In this circuit the section between 0 and l is a lossless transmission line with specific inductance L_s, specific capacitance C_s, one end of which is switched on a power source E with resistance R, while the other end is switched on an oscillating circuit formed by a condenser C and a nonlinear element, the volt-ampere characteristic of which is $i = f(v)$. The voltage v across this line and the electric current i flowing through it are functions of x and t and obey the following partial differential equations:

$$L_s \frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}, \quad C_s \frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}, \qquad (1.11)$$

$$0 < x < l, \quad t > 0$$

with the boundary conditions

$$\begin{aligned} 0 &= E - v(0,t) - R\,i(0,t), \\ -C\frac{d}{dt}v(l,t) &= -i(l,t) + f(v(l,t)). \end{aligned} \qquad (1.12)$$

Let $\varphi, \psi \in C^1(\mathbb{R})$ be arbitrary. Then we easily see that

$$\begin{aligned} v(x,t) &= [\varphi(x - st) + \psi(x + st)], \\ i(x,t) &= \frac{1}{z}[\varphi(x - st) - \psi(x + st)], \end{aligned}$$

is a solution of (1.11), where $s = 1/\sqrt{L_sC_s}$ and $z = \sqrt{L_s/C_s}$. From these expressions it follows that

$$2\varphi(-st) = v(l, t + (l/s)) + z\,i(l, t + (l/s)),$$

$$2\psi(st) = v(l, t - (l/s)) - z\,i(l, t - (l/s)).$$

By the first boundary condition (1.12), we see that

$$\begin{aligned} E &= \varphi(-st) + \psi(st) + \frac{R}{z}[\varphi(-st) - \psi(st)] \\ &= \frac{z+R}{z}\varphi(-st) + \frac{z-R}{z}\psi(st) \\ &= \frac{z+R}{2z}v(l, t + (l/s)) + \frac{z+R}{2}i(l, t + (l/s)) \\ &\quad + \frac{z-R}{2z}v(l, t - (l/s)) - \frac{z-R}{2}i(l, t - (l/s)). \end{aligned} \tag{1.13}$$

Multiplying (1.13) by $2/(z + R)$ and substituting $t - (l/s)$ for t, we have

$$i(l, t) - Ki(l, t - \tau) = \alpha - \frac{1}{z}v(l, t) - \frac{K}{z}v(l, t - \tau),$$

where

$$K = \frac{z-R}{z+R}, \quad \alpha = \frac{2E}{z+R} \quad \text{and} \quad \tau = \frac{2l}{s}.$$

Using the second boundary condition (1.12) and putting $y(t) = v(l, t)$, we obtain

$$\begin{aligned} &C\frac{d}{dt}[y(t) - Ky(t - \tau)] \\ &\qquad = \alpha - \frac{1}{z}y(t) - \frac{K}{z}y(t - \tau) - f(y(t)) + Kf(y(t - \tau)). \end{aligned}$$

This is a special form of (1.9).

Now let $\omega(t)$ be a solution of the unperturbed equation

$$\frac{d^n}{dt^n}[\omega(t) + h(t)\omega(\tau(t))] = q(t).$$

It is natural to expect that, if f is small enough in some sense, equation (1.1) has a solution $x(t)$ which behaves like the function $\omega(t)$ as $t \to \infty$. In Section 2 we study the existence and asymptotic behavior of solutions $\omega(t)$. In Section 4 we shall be concerned with the existence of solutions $x(t)$ of (1.1) with the asymptotic properties

$$x(t) = \omega(t) + o(t^k) \quad (t \to \infty), \quad k = 0, 1, 2, \dots, n-1.$$

For this end, in Section 3 we introduce the operator $\Phi : C[T, \infty) \longrightarrow C[T, \infty)$ such that

$$\Phi[u](t) + h(t)\Phi[u](\tau(t)) = u(t), \quad u \in C[T, \infty).$$

This operator Φ is useful to discuss the existence of solutions of the neutral differential equation (1.1). In fact, if the integral equation

$$u(t) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, \Omega_0[u](g(s)))ds$$

or

$$u(t) = \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, \Omega_k[u](g(r)))drds$$
$$\text{for some } k \in \{1, 2, \dots, n-1\}$$

has a solution $u(t)$, then $x(t) = \Omega_k[u](t)$ is a solution of (1.1), where

$$\Omega_k[u](t) = \omega(t) + (-1)^{n-k-1}\Phi[u](t), \quad k = 0, 1, 2, \dots, n-1.$$

Here and hereafter, $C[T, \infty)$ is regarded as the Fréchet space of all continuous functions on $[T, \infty)$ with the topology of uniform convergence on every compact subinterval of $[T, \infty)$.

In Section 5 we derive sufficient conditions and necessary and sufficient conditions for the unforced neutral differential equation

$$\text{(1.14)} \qquad \frac{d^n}{dt^n}[x(t) + h(t)x(\tau(t))] + f(t, x(g(t))) = 0$$

to have certain nonoscillatory solutions. If $x(t)$ is a positive solution of (1.14), then $y(t) = -x(t)$ is a negative solution of

$$\frac{d^n}{dt^n}[y(t) + h(t)y(\tau(t))] + \tilde{f}(t, y(g(t))) = 0,$$

where $\tilde{f}(t, u) = -f(t, -u)$. Thus we will state our results for the existence of positive solutions only.

In Section 6 we establish sufficient conditions for (1.14) to have a positive solution for the case $h(t) \leq 0$. In Sections 7–9 we consider the equation

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + f(t, x(g(t))) = q(t),$$

where $\lambda \in \mathbb{R}$ and $\tau > 0$. In Section 10, we consider the equation (1.14) for the case where $h(t) = h(\tau^N(t)) + o(1)$ as $t \to \infty$ for some integer $N \geq 1$. (The notation of $\tau^N(t)$ is explained just below.) The more precise case $h(t) = h(\tau(t))$ is discussed in Section 11, and its particular case $\tau(t) = t - \tau$ is further discussed in Section 12.

Throughout this paper we use the notation:

$$\tau^0(t) = t; \quad \tau^i(t) = \tau(\tau^{i-1}(t)), \quad i = 1, 2, \dots;$$

$$\tau^{-i}(t) = \tau^{-1}(\tau^{-(i-1)}(t)), \quad i = 2, 3, \dots;$$

$$H_0(t) = 1; \quad H_i(t) = h(t)h(\tau(t)) \cdots h(\tau^{i-1}(t)), \quad i = 1, 2, \dots,$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$. Since $\tau(t) < t$, we obtain

$$t < \tau^{-1}(t) < \tau^{-2}(t) < \cdots < \tau^{-i}(t) < \cdots, \quad t \geq t_0. \tag{1.15}$$

We note here that $\tau^{-p}(t) \to \infty$ as $p \to \infty$ for each fixed $t \geq t_0$. Otherwise, because of (1.15), there are a constant $c \geq t_0$ and a number $T \geq t_0$ such that $\lim_{p\to\infty} \tau^{-p}(T) = c$. Letting $p \to \infty$ in $\tau^{-p}(T) = \tau^{-1}(\tau^{-(p-1)}(T))$, we have $c = \tau^{-1}(c)$, which contradicts the assumption that $\tau(t) < t$ for $t \geq t_0$.

2. Difference equations

In this section we consider difference equations of the form

$$v(t) - p(t)v(\tau(t)) = q(t), \tag{2.1}$$

where the following conditions are assumed to hold: $T > 0$; $\tau \in C[T, \infty)$ is strictly increasing, $\lim_{t\to\infty} \tau(t) = \infty$ and $\tau(t) < t$ for $t \geq T$; $p \in C[\tau(T), \infty)$; $q \in C[T, \infty)$.

Take a function $Q \in C^n[T, \infty)$ such that $Q^{(n)}(t) = q(t)$ for $t \geq T$. If there exists a function $v \in C[\tau(T), \infty)$ satisfying

$$v(t) - p(t)v(\tau(t)) = Q(t),$$

then $v(t)$ is a solution of

$$\frac{d^n}{dt^n}[v(t) - p(t)v(\tau(t))] = q(t). \tag{2.2}$$

Hence, it is worthwhile to investigate difference equations of the type (2.1).

The following notation will be used:

$$P_0(t) = 1; \quad P_i(t) = p(t)p(\tau(t)) \cdots p(\tau^{i-1}(t)), \quad i = 1, 2, \dots. \tag{2.3}$$

Lemma 2.1. *Let $\varphi \in C[\tau(T), T]$ and $\varphi(T) - p(T)\varphi(\tau(T)) = q(T)$. Then the function $v(t)$ defined by*

$$v(t) = \begin{cases} \displaystyle\sum_{i=0}^{m} P_i(t)q(\tau^i(t)) + P_{m+1}(t)\varphi(\tau^{m+1}(t)), & \\ \qquad t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)], & m = 0, 1, \dots, \\ \varphi(t), & t \in [\tau(T), T], \end{cases} \tag{2.4}$$

is continuous on $[\tau(T), \infty)$ and gives the unique solution of the initial value problem

$$\begin{cases} v(t) - p(t)v(\tau(t)) = q(t), & t \geq T, \\ v(t) = \varphi(t), & t \in [\tau(T), T]. \end{cases} \tag{2.5}$$

Remark 2.1. There is a function $\varphi \in C[\tau(T), T]$ satisfying $\varphi(T) - p(T)\varphi(\tau(T)) = q(T)$. Indeed, for each $\gamma > 0$,

$$\varphi(t) = q(T)\left(\frac{t - \tau(T)}{T - \tau(T)}\right)^{\gamma}, \quad t \in [\tau(T), T],$$

is such a function. Thus, (2.1) always has a solution $v(t)$.

Proof of Lemma 2.1. It is easy to see that $v(t)$ is continuous on

$$[\tau(T), \infty) - \{\tau^{-m}(T) : m = 0, 1, 2, \dots\}.$$

We observe that

$$\lim_{t\to T-0} v(t) = \varphi(T) = q(T) + p(T)\varphi(\tau(T)) = \lim_{t\to T+0} v(t),$$

and that if $m \geq 1$, then

$$\begin{aligned}
&\lim_{t\to\tau^{-m}(T)-0} v(t) \\
&= \sum_{i=0}^{m-1} P_i(\tau^{-m}(T))q(\tau^{i-m}(T)) + P_m(\tau^{-m}(T))\varphi(T) \\
&= \sum_{i=0}^{m-1} P_i(\tau^{-m}(T))q(\tau^{i-m}(T)) + P_m(\tau^{-m}(T))[q(T) + p(T)\varphi(\tau(T))] \\
&= \sum_{i=0}^{m-1} P_i(\tau^{-m}(T))q(\tau^{i-m}(T)) + P_m(\tau^{-m}(T))q(T) \\
&\quad + P_{m+1}(\tau^{-m}(T))\varphi(\tau(T)) \\
&= \lim_{t\to\tau^{-m}(T)+0} v(t).
\end{aligned}$$

Consequently, $v(t)$ is continuous on $[\tau(T), \infty)$.

Now we show that $v(t)$ satisfies (2.5). Clearly, $v(t) = \varphi(t)$ for $t \in [\tau(T), T]$. For $t \in [T, \tau^{-1}(T)]$, we obtain

$$v(t) = q(t) + p(t)\varphi(\tau(t)) = q(t) + p(t)v(\tau(t)),$$

because of $\tau(t) \in [\tau(T), T]$. Now let $t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)]$, $m = 1, 2, \ldots$. Then $\tau(t) \in [\tau^{-(m-1)}(T), \tau^{-m}(T)]$. Since

$$P_i(t) = p(t)P_{i-1}(\tau(t)), \quad i = 1, 2, \ldots,$$

we find that

$$
\begin{aligned}
v(t) &= q(t) + \sum_{i=1}^{m} P_i(t) q(\tau^i(t)) + P_{m+1}(t)\varphi(\tau^{m+1}(t)) \\
&= q(t) + \sum_{i=1}^{m} p(t) P_{i-1}(\tau(t)) q(\tau^{i-1}(\tau(t))) \\
&\quad + p(t) P_m(\tau(t)) \varphi(\tau^m(\tau(t))) \\
&= q(t) + p(t) \left[\sum_{i=0}^{m-1} P_i(\tau(t)) q(\tau^i(\tau(t))) + P_m(\tau(t))\varphi(\tau^m(\tau(t))) \right] \\
&= q(t) + p(t) v(\tau(t)).
\end{aligned}
$$

Hence, $v(t) - p(t)v(\tau(t)) = q(t)$ for $t \geq T$.

The solution of (2.5) is unique. In fact, if $u \in C[\tau(T), \infty)$ is a solution of (2.5), then we have

$$
\begin{aligned}
u(t) &= q(t) + p(t) u(\tau(t)) \\
&= q(t) + p(t)[q(\tau(t)) + p(\tau(t)) u(\tau^2(t))] \\
&= P_0(t) q(t) + P_1(t) q(\tau(t)) + P_2(t) u(\tau^2(t)) \\
&\ \ \vdots \\
&= \sum_{i=0}^{m} P_i(t) q(\tau^i(t)) + P_{m+1}(t) u(\tau^{m+1}(t)) \\
&= \sum_{i=0}^{m} P_i(t) q(\tau^i(t)) + P_{m+1}(t) \varphi(\tau^{m+1}(t))
\end{aligned}
$$

for $t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)]$, $m = 0, 1, 2, \ldots$.

Lemma 2.2. *Suppose that $v \in C[\tau(T), \infty)$ satisfies* (2.1) *for $t \geq T$. Suppose that $|p(t)| \leq \lambda < 1$ on $[T, \infty)$ for some $\lambda > 0$. If $\lim_{t\to\infty} q(t) = 0$, then $\lim_{t\to\infty} v(t) = 0$.*

Proof. Let $\varepsilon > 0$ be arbitrary. There is a number $T_1 \geq T$ such that $|q(t)| < \varepsilon$ for $t \geq T_1$. We set $\varphi(t) = v(t)$ for $t \in [\tau(T_1), T_1]$. Then $\varphi(T_1) - p(T_1)\varphi(\tau(T_1)) = q(T_1)$. From Lemma 2.1 it follows that $v(t)$

satisfies

$$v(t) = \sum_{i=0}^{m} P_i(t)q(\tau^i(t)) + P_{m+1}(t)\varphi(\tau^{m+1}(t)),$$
$$t \in [\tau^{-m}(T_1), \tau^{-(m+1)}(T_1)], \quad m = 0, 1, \ldots .$$

Put $K = \max\{|\varphi(t)| : t \in [\tau(T_1), T_1]\}$. There exists an integer $N \geq 1$ such that $\lambda^{m+1}K < \varepsilon$ for $m \geq N$. We find that

$$|v(t)| \leq \sum_{i=0}^{m} \lambda^i \varepsilon + \lambda^{m+1}K \leq \sum_{i=0}^{\infty} \lambda^i \varepsilon + \varepsilon = \frac{\varepsilon}{1-\lambda} + \varepsilon = \frac{2-\lambda}{1-\lambda}\varepsilon,$$
$$t \in [\tau^{-m}(T_1), \tau^{-(m+1)}(T_1)], \quad m = N, N+1, \ldots .$$

Consequently, we have

$$|v(t)| \leq \frac{2-\lambda}{1-\lambda}\varepsilon, \quad t \geq \tau^{-N}(T_1),$$

which means $\lim_{t\to\infty} v(t) = 0$.

Now we shall be concerned with the asymptotic behavior of solutions of the difference equation

$$\omega(t) - p(t)\omega(\tau(t)) = 1. \tag{2.6}$$

We note here that if $q(t) \neq 0$ for all large t, then (2.1) becomes

$$\frac{u(t)}{q(t)} - p(t)\frac{q(\tau(t))}{q(t)}\frac{u(\tau(t))}{q(\tau(t))} = 1 \quad \text{for all large } t,$$

which is the same form as (2.6).

In Lemmas 2.3–2.10 below, we assume that $\omega \in C[\tau(T), \infty)$ satisfies (2.6) for $t \geq T$.

Lemma 2.3. *Suppose that $|p(t)| \leq \lambda < 1$ on $[T, \infty)$ for some $\lambda > 0$. Then*

$$\frac{1-2\lambda}{1-\lambda} \leq \liminf_{t\to\infty} \omega(t) \leq \limsup_{t\to\infty} \omega(t) \leq \frac{1}{1-\lambda}.$$

In particular, if $\lambda < 1/2$, then

$$\liminf_{t\to\infty} \omega(t) > 0.$$

Lemma 2.4. *Suppose that $|p(t)| \leq \lambda < 1$ and $p(t)p(\tau(t)) \geq 0$ on $[T, \infty)$ for some $\lambda > 0$. Then*

$$0 < 1 - \lambda \leq \liminf_{t \to \infty} \omega(t) \leq \limsup_{t \to \infty} \omega(t) \leq \frac{1}{1 - \lambda}.$$

Lemma 2.5. *Suppose that $0 \leq \mu \leq p(t) \leq \lambda < 1$ on $[T, \infty)$ for some $\mu \geq 0$ and $\lambda > 0$. Then*

$$0 < \frac{1}{1 - \mu} \leq \liminf_{t \to \infty} \omega(t) \leq \limsup_{t \to \infty} \omega(t) \leq \frac{1}{1 - \lambda}.$$

Lemma 2.6. *Suppose that $0 \leq \mu \leq -p(t) \leq \lambda < 1$ on $[T, \infty)$ for some $\mu \geq 0$ and $\lambda > 0$. Then*

$$0 < \frac{1 - \lambda}{1 - \mu^2} \leq \liminf_{t \to \infty} \omega(t) \leq \limsup_{t \to \infty} \omega(t) \leq \frac{1 - \mu}{1 - \lambda^2}.$$

Lemma 2.7. *Suppose that $1 < \mu \leq p(t) \leq \lambda$ on $[T, \infty)$ for some $\mu > 1$ and $\lambda > 1$. If $\omega(t)$ is bounded on $[T, \infty)$, then*

$$-\frac{1}{\mu - 1} \leq \liminf_{t \to \infty} \omega(t) \leq \limsup_{t \to \infty} \omega(t) \leq -\frac{1}{\lambda - 1} < 0.$$

Lemma 2.8. *Suppose that $1 < \mu \leq -p(t) \leq \lambda$ on $[T, \infty)$ for some $\mu > 1$ and $\lambda > 1$. If $\omega(t)$ is bounded on $[T, \infty)$, then*

$$0 < \frac{\mu - 1}{\lambda^2 - 1} \leq \liminf_{t \to \infty} \omega(t) \leq \limsup_{t \to \infty} \omega(t) \leq \frac{\lambda - 1}{\mu^2 - 1}.$$

Lemma 2.9. *Suppose that $\lim_{t \to \infty} |p(t)| = \infty$. If $\omega(t)$ is bounded on $[T, \infty)$, then*

$$\lim_{t \to \infty} \omega(t) p(\tau^{-1}(t)) = -1.$$

Lemma 2.10. *Suppose that $\lim_{t \to \infty} p(t) = l$ for some $l \in \mathbb{R}$. If either $1 < |l| < \infty$ and $\omega(t)$ is bounded on $[T, \infty)$ or $|l| < 1$, then*

$$\lim_{t \to \infty} \omega(t) = \frac{1}{1 - l}.$$

Proof of Lemma 2.3. Put $\varphi(t) = \omega(t)$ on $[\tau(T), T]$. Then Lemma 2.1 implies that

(2.7)
$$\omega(t) = \sum_{i=0}^{m} P_i(t) + P_{m+1}(t)\varphi(\tau^{m+1}(t)), \quad t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)],$$

for $m = 0, 1, 2, \ldots$. Let $K > 0$ be a constant such that $|\varphi(t)| \le K$ for $t \in [\tau(T), T]$. For any $\varepsilon > 0$, there is an integer $N \ge 1$ such that $\lambda^{m+1}K < \varepsilon$ for $m \ge N$. From (2.7) it follows that, for $t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)]$, $m = N, N+1, \ldots$,

$$\begin{aligned}\omega(t) &= 1 + \sum_{i=1}^{m} P_i(t) + P_{m+1}(t)\varphi(\tau^{m+1}(t))\\ &\ge 1 - \sum_{i=1}^{m} \lambda^i - \lambda^{m+1}K\\ &\ge 1 - \frac{\lambda}{1-\lambda} - \varepsilon = \frac{1-2\lambda}{1-\lambda} - \varepsilon,\end{aligned}$$

and

$$\omega(t) \le \sum_{i=0}^{m} \lambda^i + \lambda^{m+1}K \le \frac{1}{1-\lambda} + \varepsilon.$$

Therefore, we have

$$\frac{1-2\lambda}{1-\lambda} - \varepsilon \le \omega(t) \le \frac{1}{1-\lambda} + \varepsilon, \quad t \ge \tau^{-N}(T).$$

This completes the proof.

Proofs of Lemmas 2.4–2.6. Observe that

$$\begin{aligned}\omega(t) &= 1 + p(t)\omega(\tau(t))\\ &= 1 + p(t)[1 + p(\tau(t))\omega(\tau^2(t))]\\ &= 1 + p(t) + P_2(t)\omega(\tau^2(t)), \quad t \ge \tau^{-1}(T).\end{aligned}$$

Hence, we have

$$
\begin{aligned}
(2.8)\quad \omega(t) &= 1 + p(t) + P_2(t)[1 + p(\tau^2(t)) + P_2(\tau^2(t))\omega(\tau^4(t))] \\
&= 1 + p(t) + P_2(t)[1 + p(\tau^2(t))] + P_4(t)\omega(\tau^4(t)) \\
&\ \ \vdots \\
&= 1 + p(t) + \sum_{j=1}^{m-1} P_{2j}(t)[1 + p(\tau^{2j}(t))] + P_{2m}(t)\omega(\tau^{2m}(t)) \\
&= \sum_{j=0}^{m-1} P_{2j}(t)[1 + p(\tau^{2j}(t))] + P_{2m}(t)\omega(\tau^{2m}(t))
\end{aligned}
$$

for $t \geq \tau^{-(2m-1)}(T)$ and $m = 1, 2, \ldots$.

(Proof of Lemma 2.4) From Lemma 2.3, it follows that $|\omega(t)| \leq K$ on $[\tau(T), \infty)$ for some $K > 0$ and that $\limsup_{t\to\infty} \omega(t) \leq 1/(1-\lambda)$. Let $\varepsilon > 0$. There is an integer $N \geq 2$ such that $\lambda^{2N}K < \varepsilon$. Since $p(t)p(\tau(t)) \geq 0$ for $t \geq T$, we find that $P_{2j}(t) \geq 0$, $j = 1, 2, \ldots, N-1$, for $t \geq \tau^{-(2N-1)}(T)$, so that

$$P_{2j}(t)[1 + p(\tau^{2j}(t))] \geq P_{2j}(t)[1 - \lambda] \geq 0, \quad j = 1, 2, \ldots, N-1,$$

for $t \geq \tau^{-(2N-1)}(T)$. By (2.8) we obtain

$$\omega(t) \geq 1 + p(t) + P_{2N}(t)\omega(\tau^{2N}(t)) \geq 1 - \lambda - \varepsilon, \quad t \geq \tau^{-(2N-1)}(T),$$

which means $\liminf_{t\to\infty} \omega(t) \geq 1 - \lambda$.

(Proof of Lemma 2.5) By Lemma 2.3, we see that $|\omega(t)| \leq K$ on $[\tau(T), \infty)$ for some $K > 0$ and that $\limsup_{t\to\infty} \omega(t) \leq 1/(1-\lambda)$. For any $\varepsilon > 0$, there is an integer $N \geq 2$ such that

$$\lambda^{2N}K < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{\mu^{2N}}{1-\mu} < \frac{\varepsilon}{2}.$$

Using (2.8), we see that

$$
\begin{aligned}
\omega(t) \geq \sum_{j=0}^{N-1} \mu^{2j}(1+\mu) - \frac{\varepsilon}{2} = \frac{1-\mu^{2N}}{1-\mu^2}(1+\mu) - \frac{\varepsilon}{2} &= \frac{1-\mu^{2N}}{1-\mu} - \frac{\varepsilon}{2} \\
&\geq \frac{1}{1-\mu} - \varepsilon
\end{aligned}
$$

for $t \geq \tau^{-(2N-1)}(T)$. Hence, $\liminf_{t\to\infty} \omega(t) \geq 1/(1-\mu)$.

(Proof of Lemma 2.6) Lemma 2.3 implies that $|\omega(t)| \leq K$ on $[T, \infty)$ for some $K > 0$. Let $\varepsilon > 0$. There is an integer $N \geq 2$ such that

$$\lambda^{2N} K < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{\mu^{2N}(1-\lambda)}{1-\mu^2} < \frac{\varepsilon}{2}.$$

We note that $\mu^{2j} \leq P_{2j}(t) \leq \lambda^{2j}$ for $j = 1, 2, \ldots$. From (2.8) it follows that

$$\omega(t) \leq \sum_{j=0}^{N-1} \lambda^{2j}(1-\mu) + \frac{\varepsilon}{2} \leq \frac{1-\mu}{1-\lambda^2} + \frac{\varepsilon}{2},$$

and

$$\omega(t) \geq \sum_{j=0}^{N-1} \mu^{2j}(1-\lambda) - \frac{\varepsilon}{2} = (1-\lambda)\frac{1-\mu^{2N}}{1-\mu^2} - \frac{\varepsilon}{2} \geq \frac{1-\lambda}{1-\mu^2} - \varepsilon$$

for $t \geq \tau^{-(2N-1)}(T)$. This completes the proof.

Proof of Lemma 2.7. Since

$$p(t)\omega(\tau(t)) = -1 + \omega(t), \quad t \geq T,$$

we have

$$\omega(\tau(t)) = -\frac{1}{p(t)} + \frac{\omega(t)}{p(t)}, \quad t \geq T,$$

and hence

(2.9)

$$\begin{aligned}
\omega(t) &= -\frac{1}{p(\tau^{-1}(t))} + \frac{\omega(\tau^{-1}(t))}{p(\tau^{-1}(t))} \\
&= -\frac{1}{p(\tau^{-1}(t))} + \frac{1}{p(\tau^{-1}(t))}\left[-\frac{1}{p(\tau^{-2}(t))} + \frac{\omega(\tau^{-2}(t))}{p(\tau^{-2}(t))}\right] \\
&= -\frac{1}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-2}(t))} + \frac{\omega(\tau^{-2}(t))}{p(\tau^{-1}(t))p(\tau^{-2}(t))} \\
&= -[P_1(\tau^{-1}(t))]^{-1} - [P_2(\tau^{-2}(t))]^{-1} + [P_2(\tau^{-2}(t))]^{-1}\omega(\tau^{-2}(t)) \\
&\;\;\vdots \\
&= -\sum_{i=1}^{m}[P_i(\tau^{-i}(t))]^{-1} + [P_m(\tau^{-m}(t))]^{-1}\omega(\tau^{-m}(t))
\end{aligned}$$

for $t \geq T$ and $m = 1, 2, \ldots$. Let $K > 0$ be a constant such that $|\omega(t)| \leq K$ for $t \geq T$. For any $\varepsilon > 0$, there exists an integer $N \geq 1$ such that

$$K\mu^{-N} < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{\lambda^{-N}}{\lambda - 1} < \frac{\varepsilon}{2}.$$

In view of (2.9), we see that

$$\omega(t) \leq -\sum_{i=1}^{N} \lambda^{-i} + \frac{\varepsilon}{2} = -\frac{1 - \lambda^{-N}}{\lambda - 1} + \frac{\varepsilon}{2} \leq -\frac{1}{\lambda - 1} + \varepsilon,$$

and

$$\omega(t) \geq -\sum_{i=1}^{N} \mu^{-i} - \frac{\varepsilon}{2} \geq -\frac{1}{\mu - 1} - \frac{\varepsilon}{2}$$

for $t \geq T$. The proof is complete.

Proof of Lemma 2.8. From (2.9) it follows that

$$\begin{aligned}
\omega(t) = & -\sum_{j=1}^{m} \left[[P_{2j-1}(\tau^{-(2j-1)}(t))]^{-1} + [P_{2j}(\tau^{-2j}(t))]^{-1}\right] \\
& + [P_{2m}(\tau^{-2m}(t))]^{-1}\omega(\tau^{-2m}(t)) \\
= & -\sum_{j=1}^{m} [P_{2j}(\tau^{-2j}(t))]^{-1}[p(\tau^{-2j}(t)) + 1] \\
& + [P_{2m}(\tau^{-2m}(t))]^{-1}\omega(\tau^{-2m}(t)), \quad t \geq T, \quad m = 1, 2, \ldots.
\end{aligned}$$

Choose a constant $K > 0$ such that $|\omega(t)| \leq K$ for $t \geq T$. Let $\varepsilon > 0$. Then

$$K\mu^{-2N} < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{\lambda^{-2N}(\mu - 1)}{\lambda^2 - 1} < \frac{\varepsilon}{2} \quad \text{for some } N \in \mathbb{N}.$$

Since

$$0 < \mu - 1 \leq -[p(\tau^{-2j}(t)) + 1] \leq \lambda - 1, \quad t \geq T, \quad j = 1, 2, \ldots,$$

we conclude that

$$\omega(t) \leq \sum_{j=1}^{N} \mu^{-2j}(\lambda - 1) + \frac{\varepsilon}{2} \leq \frac{\lambda - 1}{\mu^2 - 1} + \frac{\varepsilon}{2},$$

and

$$\omega(t) \geq \sum_{j=1}^{N} \lambda^{-2j}(\mu-1) - \frac{\varepsilon}{2} = \frac{1-\lambda^{-2N}}{\lambda^2-1}(\mu-1) - \frac{\varepsilon}{2} \geq \frac{\mu-1}{\lambda^2-1} - \varepsilon$$

for $t \geq T$. This completes the proof.

Proof of Lemma 2.9. From (2.6) it follows that

$$\omega(\tau(t)) = \frac{\omega(t)-1}{p(t)} \quad \text{for all large } t \geq T. \tag{2.10}$$

Since $\omega(t)$ is bounded, we see that $\lim_{t\to\infty}[\omega(t)-1]/p(t) = 0$, so that $\lim_{t\to\infty}\omega(t) = 0$, by (2.10). Letting $t \to \infty$ in (2.6), we obtain $\lim_{t\to\infty} p(t)\omega(\tau(t)) = -1$. This completes the proof.

Proof of Lemma 2.10. First we assume that $l = 0$. Let $\varepsilon \in (0, 1/2)$ be arbitrary. There is a number $T_0 \geq T$ such that $|p(t)| < \varepsilon < 1/2$ for $t \geq T_0$. From Lemma 2.3 it follows that

$$\frac{1-2\varepsilon}{1-\varepsilon} \leq \liminf_{t\to\infty}\omega(t) \leq \limsup_{t\to\infty}\omega(t) \leq \frac{1}{1-\varepsilon}. \tag{2.11}$$

Letting $\varepsilon \to 0$ in (2.11), we have $\lim_{t\to\infty}\omega(t) = 1 = 1/(1-l)$. In exactly the same way, it can be shown that $\lim_{t\to\infty} p(t) = 1/(1-l)$ for the cases $0 < l < 1$, $-1 < l < 0$, $1 < l < \infty$ and $-\infty < l < -1$, by using Lemmas 2.5–2.8.

3. Fundamental operators

In this section we are concerned with the mapping $\Phi : C[T_*, \infty) \longrightarrow C[T_*, \infty)$ such that

$$\Phi[u](t) - p(t)\Phi[u](\tau(t)) = u(t), \quad u \in C[T_*, \infty).$$

We assume throughout this section that $t_0 > 0$; $\tau \in C[t_0, \infty)$ is strictly increasing, $\lim_{t\to\infty}\tau(t) = \infty$ and $\tau(t) < t$ for $t \geq t_0$; $p \in C[t_0, \infty)$.

We use the notation (2.3).

Proposition 3.1. *Let T_* and T be numbers with $t_0 \leq T_* \leq \tau(T)$. Suppose that the following condition* (3.1) *holds:*

(3.1) *$p(t)$ is bounded on $[T,\infty)$ and there are $N \in \mathbb{N}$ and $\lambda > 0$ such that $|P_N(t)| \le \lambda < 1$ for $t \ge \tau^{-(N-1)}(T)$.*

Then there exists a mapping $\Phi : C[T_*,\infty) \longrightarrow C[T_*,\infty)$ which has the following properties:

(i) *the mapping Φ is continuous in the $C[T_*,\infty)$-topology;*

(ii) *for each $u \in C[T_*,\infty)$, Φ satisfies $\Phi[u](t) - p(t)\Phi[u](\tau(t)) = u(t)$ for $t \ge T$;*

(iii) *if $u \in C[T_*,\infty)$ and $\lim_{t\to\infty} u(t) = 0$, then $\lim_{t\to\infty} \Phi[u](t) = 0$.*

Remark 3.1. If $|p(t)| \le \lambda < 1$ on $[T,\infty)$ for some $\lambda > 0$, then (3.1) holds.

Proof of Proposition 3.1. For each $u \in C[T_*,\infty)$, we assign the function $\Phi[u]$ by

(3.2)

$$\Phi[u](t) = \begin{cases} \displaystyle\sum_{i=0}^{m} P_i(t)u(\tau^i(t)) + P_{m+1}(t)\frac{u(T)}{T-\tau(T)}(\tau^{m+1}(t) - \tau(T)), \\ \qquad\qquad t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)], \quad m = 0,1,\dots, \\ \displaystyle\frac{u(T)}{T-\tau(T)}(t - \tau(T)), \qquad\qquad t \in [T_*, T]. \end{cases}$$

From Lemma 2.1 it follows that $\Phi[u](t)$ is continuous on $[T_*,\infty)$ for each $u \in C[T_*,\infty)$ and satisfies the property (ii).

Now we show that Φ has the properties (i) and (iii).

(i) It suffices to prove that if $\{u_j\}_{j=1}^{\infty}$ is a sequence in $C[T_*,\infty)$ converging to $u \in C[T_*,\infty)$ uniformly on every compact subinterval of $[T_*,\infty)$, then $\Phi[u_j]$ converges to $\Phi[u]$ uniformly on every compact subinterval of $[T_*,\infty)$. It is clear that $\Phi[u_j]$ converges to $\Phi[u]$ uniformly on $[T_*,T]$. We claim that $\Phi[u_j] \to \Phi[u]$ uniformly on $I_m \equiv [\tau^{-m}(T), \tau^{-(m+1)}(T)]$, $m = 0,1,2,\dots$. Then we easily see that $\Phi[u_j]$ converges to $\Phi[u]$ uniformly on every compact subinterval of $[T_*,\infty)$.

We have

$$
\begin{aligned}
&\sup_{t\in I_m} |\Phi[u_j](t) - \Phi[u](t)| \\
&\le \sum_{i=0}^{m} K^i \sup_{t\in I_m} |u_j(\tau^i(t)) - u(\tau^i(t))| \\
&\quad + K^{m+1} \sup_{t\in I_m} \left| \frac{u_j(T) - u(T)}{T - \tau(T)} (\tau^{m+1}(t) - \tau(T)) \right| \\
&\le \sum_{i=0}^{m} K^i \sup_{t\in I_{m-i}} |u_j(t) - u(t)| + K^{m+1} |u_j(T) - u(T)|,
\end{aligned}
$$

for $m = 0, 1, 2, \dots$, where $K = \max\{|p(t)| : t \in [T_*, \tau^{-(m+1)}(T)]\}$. Hence, we conclude that

$$
\sup_{t\in I_m} |\Phi[u_j](t) - \Phi[u](t)| \to 0 \quad (j \to \infty), \quad m = 0, 1, 2, \dots,
$$

so that $\Phi[u_j]$ converges to $\Phi[u]$ uniformly on I_m for $m = 0, 1, 2, \dots$.

(iii) Let $u \in C[T_*, \infty)$ such that $\lim_{t\to\infty} u(t) = 0$. From (ii) it follows that

$$
\begin{aligned}
\Phi[u](t) &= u(t) + p(t)\Phi[u](\tau(t)) \\
&= u(t) + p(t)[u(\tau(t)) + p(\tau(t))\Phi[u](\tau^2(t))] \\
&= P_0(t)u(t) + P_1(t)u(\tau(t)) + P_2(t)\Phi[u](\tau^2(t)) \\
&\ \ \vdots \\
&= \sum_{i=0}^{N-1} P_i(t)u(\tau^i(t)) + P_N(t)\Phi[u](\tau^N(t)), \quad t \ge \tau^{-(N-1)}(T).
\end{aligned}
$$

Since $p(t)$ is bounded, we have $\lim_{t\to\infty} \sum_{i=0}^{N-1} P_i(t)u(\tau^i(t)) = 0$. Applying Lemma 2.2 with $v(t)$, $p(t)$, $\tau(t)$ and $q(t)$ replaced by $\Phi[u](t)$, $P_N(t)$, $\tau^N(t)$ and

$$
\sum_{i=0}^{N-1} P_i(t)u(\tau^i(t)),
$$

we conclude that $\lim_{t\to\infty} \Phi[u](t) = 0$.

Proposition 3.2. *Let T_* and T be numbers with $t_0 \le T_* \le \tau(T)$. Suppose that the following condition* (3.3) *holds:*

(3.3) $|p(t)| > 0$ *for* $t \geq T$, $|p(t)|^{-1}$ *is bounded on* $[T, \infty)$, *and there are* $N \in \mathbb{N}$ *and* $\mu > 0$ *such that* $|P_N(t)| \geq \mu > 1$ *for* $t \geq \tau^{-(N-1)}(T)$.

Let $M > 0$, *and define*

$$B(M) = \{u \in C[T_*, \infty) : |u(t)| \leq M, \ t \geq T\}.$$

Then there exists a mapping $\Psi : B(M) \longrightarrow C[T_*, \infty)$ *which has the following properties*:

(i) *the mapping* Ψ *is continuous in the* $C[T_*, \infty)$*-topology*;

(ii) *for each* $u \in B(M)$, Ψ *satisfies* $\Psi[u](t) - p(t)\Psi[u](\tau(t)) = u(t)$ *for* $t \geq T$;

(iii) $\Psi[u](t)$ *is bounded on* $[T_*, \infty)$ *for each* $u \in B(M)$;

(iv) *if* $u \in B(M)$ *satisfies* $\lim_{t\to\infty} u(t) = 0$, *then*

$$\Psi[u](t) = o\left(\frac{1}{|p(\tau^{-1}(t))|}\right) \quad (t \to \infty).$$

(v) *if* $u \in B(M)$, $u(t) > 0$ *for* $t \geq T_*$ *and*

$$\limsup_{t\to\infty} \frac{u(t)}{u(\tau(t))} \leq 1,$$

then $\Psi[u](t)/u(t)$ *is bounded on* $[T_*, \infty)$.

Remark 3.2. If $|p(t)| \geq \mu > 1$ on $[T, \infty)$ for some $\mu > 0$, then (3.3) holds.

Remark 3.3. The mapping Ψ described in Proposition 3.2 satisfies

$$\Psi[u](t) = o(1) \quad (t \to \infty)$$

for each $u \in B(M)$ with $\lim_{t\to\infty} u(t) = 0$. This follows from (3.3) and the property (iv) in Proposition 3.2.

Proof of Proposition 3.2. For each $u \in B(M)$, we define the function $\Psi[u]$ as follows:

$$\Psi[u](t) = \begin{cases} -\sum_{i=1}^{\infty}[P_i(\tau^{-i}(t))]^{-1}u(\tau^{-i}(t)), & t \geq \tau(T), \\ \Psi[u](\tau(T)), & t \in [T_*, \tau(T)]. \end{cases}$$

Using (3.3), there is a constant $K > 1$ such that $|p(t)|^{-1} \leq K$ for $t \geq T$. For each $i \in \mathbb{N}$, we take integers $q(i) \geq 0$ and $r(i)$ such that $i = q(i)N + r(i)$ and $0 \leq r(i) \leq N-1$. Then we have

$$\begin{aligned} P_i(t) = P_N(t)P_N(\tau^N(t)) \cdots P_N(\tau^{(q(i)-1)N}(t)) \\ \times p(\tau^{q(i)N}(t))p(\tau^{q(i)N+1}(t)) \cdots p(\tau^{q(i)N+r(i)-1}(t)) \end{aligned}$$

for $t \geq \tau^{-(i-1)}(T)$ and $i = 1, 2, \ldots$. Then, from (3.3) it follows that

$$|[P_i(\tau^{-i}(t))]^{-1}| \leq \mu^{-q(i)}K^{r(i)} \leq \mu^{-q(i)}K^{N-1}, \quad t \geq \tau(T), \quad i = 1, 2, \ldots. \tag{3.4}$$

Since

$$\sum_{i=1}^{\infty} \mu^{-q(i)}K^{N-1} \equiv L < \infty,$$

we see that Ψ is well-defined and that, for each $u \in B(M)$, $\Psi[u]$ is continuous on $[T_*, \infty)$ and satisfies

$$|\Psi[u](t)| \leq \sum_{i=1}^{\infty} \mu^{-q(i)}K^{N-1} \sup_{s \geq \tau^{-1}(t)} |u(s)| = L \sup_{s \geq \tau^{-1}(t)} |u(s)| \leq LM \tag{3.5}$$

for $t \geq \tau(T)$. This implies that Ψ satisfies the property (iii).

Now we show that Ψ satisfies the properties (i), (ii), (iv) and (v).

(i) Let $\{u_j\}_{j=1}^{\infty}$ be a sequence in $C[T_*, \infty)$ converging to $u \in C[T_*, \infty)$ uniformly on every compact subinterval of $[T_*, \infty)$. Let $[\alpha, \beta]$ be an arbitrary compact subinterval of $[\tau(T), \infty)$. For any $\varepsilon > 0$, there is an integer $p \geq 1$ such that

$$MK^{N-1} \sum_{i=p+1}^{\infty} \mu^{-q(i)} < \frac{\varepsilon}{4}.$$

Since

$$\begin{aligned}
&\lim_{j\to\infty} \left[\sup\left\{|u_j(\tau^{-i}(t)) - u(\tau^{-i}(t))| : t \in [\alpha, \beta]\right\}\right] \\
&= \lim_{j\to\infty} \left[\sup\left\{|u_j(s) - u(s)| : s \in [\tau^{-i}(\alpha), \tau^{-i}(\beta)]\right\}\right] = 0
\end{aligned}$$

for $i = 1, 2, \ldots, p$, there exists an integer j_0 such that

$$\sup\left\{\sum_{i=1}^{p} K^i |u_j(\tau^{-i}(t)) - u(\tau^{-i}(t))| : t \in [\alpha, \beta]\right\} < \frac{\varepsilon}{2}, \quad j \geq j_0.$$

By virtue of (3.4), we see that

$$\begin{aligned}
&|\Psi[u_j](t) - \Psi[u](t)| \\
&\leq \sum_{i=1}^{p} |[P_i(\tau^{-i}(t))]^{-1}| |u_j(\tau^{-i}(t)) - u(\tau^{-i}(t))| \\
&\quad + \sum_{i=p+1}^{\infty} |[P_i(\tau^{-i}(t))]^{-1}| |u_j(\tau^{-i}(t))| \\
&\quad + \sum_{i=p+1}^{\infty} |[P_i(\tau^{-i}(t))]^{-1}| |u(\tau^{-i}(t))| \\
&\leq \sum_{i=1}^{p} K^i |u_j(\tau^{-i}(t)) - u(\tau^{-i}(t))| + 2 \sum_{i=p+1}^{\infty} \mu^{-q(i)} K^{N-1} M \\
&\leq \frac{\varepsilon}{2} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon, \quad t \in [\alpha, \beta], \quad j \geq j_0,
\end{aligned}$$

which implies that $\Psi[u_j](t)$ converges to $\Psi[u](t)$ uniformly on $[\alpha, \beta]$. In view of the fact that $\Psi[u](t) = \Psi[u](\tau(T))$ for $t \in [T_*, \tau(T)]$, we find that $\Psi[u_j](t) \to \Psi[u](t)$ uniformly on $[T_*, \tau(T)]$. Consequently, Ψ is continuous on $B(M)$.

(ii) Since

$$
\begin{aligned}
&p(t)[P_i(\tau^{-(i-1)}(t))]^{-1} \\
&= p(t)[p(\tau^{-(i-1)}(t))p(\tau^{-(i-2)}(t))\cdots p(\tau^{-1}(t))p(t)]^{-1} \\
&= [P_{i-1}(\tau^{-(i-1)}(t))]^{-1}, \quad t \geq T, \quad i = 2, 3, \dots,
\end{aligned}
$$

we have

$$
\begin{aligned}
&p(t)\Psi[u](\tau(t)) \\
&= -p(t)\left[[P_1(t)]^{-1}u(t) + \sum_{i=2}^{\infty}[P_i(\tau^{-(i-1)}(t))]^{-1}u(\tau^{-(i-1)}(t))\right] \\
&= -u(t) - \sum_{i=2}^{\infty}[P_{i-1}(\tau^{-(i-1)}(t))]^{-1}u(\tau^{-(i-1)}(t)) \\
&= -u(t) + \Psi[u](t), \quad t \geq T.
\end{aligned}
$$

(iv) Let $u \in B(M)$ and $\lim_{t\to\infty} u(t) = 0$. From (3.5) it follows that $\Psi[u](t) \to 0$ as $t \to \infty$. Using (ii), we see that

$$
p(t)\Psi[u](\tau(t)) = \Psi[u](t) - u(t) \to 0 \quad \text{as } t \to \infty,
$$

so that Ψ satisfies (iv).

(v) Let $u \in B(M)$ such that $u(t) > 0$ for $t \geq T_*$ and

$$
\limsup_{t\to\infty} \frac{u(t)}{u(\tau(t))} \leq 1.
$$

Take $\delta > 0$ such that $(1+\delta)^{(N+1)}/\mu < 1$. There is a number $T_1 \geq T$ which satisfies

$$
\frac{u(\tau^{-1}(t))}{u(t)} \leq 1 + \delta, \quad t \geq T_1.
$$

Then

$$
\begin{aligned}
\left|\frac{u(\tau^{-i}(t))}{u(t)}\right| &= \left|\frac{u(\tau^{-1}(t))}{u(t)}\frac{u(\tau^{-2}(t))}{u(\tau^{-1}(t))}\cdots\frac{u(\tau^{-i}(t))}{u(\tau^{-(i-1)}(t))}\right| \\
&\leq (1+\delta)^i = (1+\delta)^{q(i)N+r(i)} \leq (1+\delta)^{(N+1)q(i)}, \quad t \geq T_1,
\end{aligned}
$$

for $i = 1, 2, \ldots$. In view of (3.4), we conclude that

$$\left|\frac{\Psi[u](t)}{u(t)}\right| \le \sum_{i=1}^{\infty}\left[\frac{(1+\delta)^{(N+1)}}{\mu}\right]^{q(i)} K^{N-1} < \infty, \quad t \ge T_1.$$

The proof is complete.

Proposition 3.2 implies the following result.

Lemma 3.1. *Let $T \ge \tau^{-1}(t_0)$. Suppose that* (3.3) *holds. Then there exists a bounded function $\omega \in C[\tau(T), \infty)$ satisfying* (2.6) *for $t \ge T$.*

From Propositions 3.1, 3.2 and Remark 3.3, we have the following result.

Lemma 3.2. *Let $T \ge \tau^{-1}(t_0)$. Suppose that $q \in C[T, \infty)$ and $\lim_{t\to\infty} q(t) = 0$. If* (3.1) *or* (3.3) *holds, then there exists a function $v \in C[\tau(T), \infty)$ satisfying* (2.1) *for $t \ge T$ and $\lim_{t\to\infty} v(t) = 0$.*

Proposition 3.3. *Let T_* and T be numbers such that $t_0 \le T_* \le \tau(T)$ and let $r \in C[T_*, \infty)$ with $r(t) > 0$ for $t \ge T_*$. Suppose that the following condition* (3.6) *holds:*

(3.6) *$p(t)[r(\tau(t))/r(t)]$ is bounded on $[T, \infty)$ and there are $N \in \mathbb{N}$ and $\lambda > 0$ such that $|P_N(t)[r(\tau^N(t))/r(t)]| \le \lambda < 1$ for $t \ge \tau^{-(N-1)}(T)$.*

Then there exists a mapping $\Phi : C[T_*, \infty) \longrightarrow C[T_*, \infty)$ which satisfies the following properties:

(i) *the mapping Φ is continuous in the $C[T_*, \infty)$-topology;*

(ii) *for each $u \in C[T_*, \infty)$, Φ satisfies $\Phi[u](t) - p(t)\Phi[u](\tau(t)) = u(t)$ for $t \ge T$;*

(iii) *if $u \in C[T_*, \infty)$ satisfies $u(t) = o(r(t))$ $(t \to \infty)$, then $\Phi[u](t) = o(r(t))$ $(t \to \infty)$.*

Proposition 3.4. *Let T_* and T be numbers such that $t_0 \le T_* \le \tau(T)$ and let $r \in C[T_*, \infty)$ with $r(t) > 0$ for $t \ge T_*$. Suppose that the following condition* (3.7) *holds:*

(3.7) $|p(t)| > 0$ *for* $t \geq T$, $|p(t)[r(\tau(t))/r(t)]|^{-1}$ *is bounded on* $[T, \infty)$, *and there are* $N \in \mathbb{N}$ *and* $\mu > 0$ *such that* $|P_N(t)|[r(\tau^N(t))/r(t)] \geq \mu > 1$ *for* $t \geq \tau^{-(N-1)}(T)$.

Define

$$U = \{u \in C[T_*, \infty) : |u(t)| \leq r(t),\ t \geq T\}.$$

Then there exists a mapping $\Psi : U \longrightarrow C[T_*, \infty)$ *which satisfies the following properties:*

(i) *the mapping* Ψ *is continuous in the* $C[T_*, \infty)$*-topology;*

(ii) *for each* $u \in U$, Ψ *satisfies* $\Psi[u](t) - p(t)\Psi[u](\tau(t)) = u(t)$ *for* $t \geq T$;

(iii) *if* $u \in U$ *satisfies* $u(t) = o(r(t))$ $(t \to \infty)$, *then*

$$\Psi[u](t) = o\left(\frac{r(\tau^{-1}(t))}{|p(\tau^{-1}(t))|}\right) \quad (t \to \infty);$$

(iv) *if* $u \in U$, $u(t) > 0$ *for* $t \geq T_*$ *and*

$$\limsup_{t\to\infty} \frac{u(t)r(\tau(t))}{u(\tau(t))r(t)} \leq 1,$$

then $\Psi[u](t)/u(t)$ *is bounded on* $[T_*, \infty)$.

Remark 3.4. If $|p(t)|[r(\tau(t))/r(t)] \leq \lambda < 1$ for $t \geq T$, then (3.6) holds. If $|p(t)|[r(\tau(t))/r(t)] \geq \mu > 1$ for $t \geq T$, then (3.7) holds.

Proofs of Propositions 3.3 and 3.4. We give the proof of Proposition 3.4 only. In exactly the same way, we can show Proposition 3.3.

By applying Proposition 3.2 with $p(t)$ and M replaced by $p(t)[r(\tau(t))/r(t)]$ and 1, respectively, there exists a mapping $\Psi_1 : B(1) \longrightarrow C[T_*, \infty)$ such that (a) the mapping Ψ_1 is continuous in the $C[T_*, \infty)$-topology; (b) for each $v \in B(1)$, Ψ_1 satisfies

$$\Psi_1[v](t) - p(t)\frac{r(\tau(t))}{r(t)}\Psi_1[v](\tau(t)) = v(t), \quad t \geq T;$$

(c) if $v \in B(1)$ satisfies $\lim_{t\to\infty} v(t) = 0$, then

$$\Psi_1[v](t) = o\left(\frac{1}{|p(\tau^{-1}(t))|[r(t)/r(\tau^{-1}(t))]}\right) \quad (t \to \infty);$$

and (d) if $v \in B(1)$, $v(t) > 0$ for $t \geq T_*$ and

$$\limsup_{t\to\infty} \frac{v(t)}{v(\tau(t))} \leq 1,$$

then $\Psi_1[v](t)/v(t)$ is bounded on $[T_*, \infty)$. For each $u \in U$, we define $\Psi[u]$ by $\Psi[u](t) = r(t)\Psi_1[u(\cdot)/r(\cdot)](t)$, $t \geq T_*$. Then we easily see that Ψ maps U into $C[T_*, \infty)$ and satisfies (i)–(iv) in Proposition 3.4.

4. Existence theorems

In this section we show that if f is small enough in some sense, then (1.1) has a solution $x(t)$ which behaves like the solution $\omega(t)$ of the unperturbed equation

$$\frac{d^n}{dt^n}[\omega(t) + h(t)\omega(\tau(t))] = q(t). \tag{4.1}$$

We always assume that (1.2)–(1.8) and the following condition hold:

(4.2) there exists a function $F \in C([t_0, \infty) \times [0, \infty))$ such that $F(t, u)$ is nondecreasing in $u \in [0, \infty)$ for each fixed $t \geq t_0$ and satisfies

$$|f(t, u)| \leq F(t, |u|), \quad (t, u) \in [t_0, \infty) \times \mathbb{R}.$$

The main results of this section are as follows.

Theorem 4.1. *Let $k \in \{0, 1, 2, \ldots, n-1\}$, and let $\omega(t)$ be a solution of the unperturbed equation* (4.1). *Suppose that either*

(4.3) *$h(t)[\tau(t)/t]^k$ is bounded on $[t_0, \infty)$ and there are $N \in \mathbb{N}$ and $\lambda > 0$ such that $|H_N(t)|[\tau^N(t)/t]^k \leq \lambda < 1$ for $t \geq \tau^{-(N-1)}(t_0)$*

or

(4.4) *$|h(t)| > 0$ for $t \geq t_0$, $|h(t)[\tau(t)/t]^k|^{-1}$ is bounded on $[t_0, \infty)$, and there are $N \in \mathbb{N}$ and $\mu > 0$ such that $|H_N(t)|[\tau^N(t)/t]^k \geq \mu > 1$ for $t \geq \tau^{-(N-1)}(t_0)$.*

If

$$(4.5) \qquad \int^{\infty} t^{n-k-1} F(t, |\omega(g(t))| + \varepsilon[g(t)]^k) dt < \infty \quad \textit{for some } \varepsilon > 0,$$

then (1.1) *possesses a solution* $x(t)$ *satisfying*

$$(4.6) \qquad x(t) = \omega(t) + o(t^k) \quad (t \to \infty).$$

Theorem 4.2. *Let* $k \in \{0, 1, 2, \dots, n-1\}$, *and let* $\omega(t)$ *be a solution of the unperturbed equation* (4.1). *Suppose that* (4.4) *holds. If*

$$(4.7) \qquad \int^{\infty} t^{n-k-1} F\left(t, |\omega(g(t))| + \frac{\varepsilon[\tau^{-1}(g(t))]^k}{|h(\tau^{-1}(g(t)))|}\right) dt < \infty$$

for some $\varepsilon > 0$,

then there exists a solution $x(t)$ *of* (1.1) *satisfying*

$$(4.8) \qquad x(t) = \omega(t) + o\left(\frac{[\tau^{-1}(t)]^k}{|h(\tau^{-1}(t))|}\right) \quad (t \to \infty).$$

Remark 4.1. In view of the boundedness of $|h(t)[\tau(t)/t]^k|^{-1}$ in condition (4.4), we find that the solution $x(t)$ obtained in Theorem 4.2 satisfies (4.6). Further, if $|h(t)[\tau(t)/t]^k|^{-1}$ is bounded, then (4.5) implies (4.7). Thus, Theorem 4.2 implies Theorem 4.1 for the case (4.4).

In particular, for the case $k = 0$, Theorems 4.1 and 4.2 give the following results.

Corollary 4.1. *Let* $\omega(t)$ *be a solution of the unperturbed equation* (4.1). *Suppose that either*

(4.9) $h(t)$ *is bounded on* $[t_0, \infty)$ *and there are* $N \in \mathbb{N}$ *and* $\lambda > 0$ *such that* $|H_N(t)| \le \lambda < 1$ *for* $t \ge \tau^{-(N-1)}(t_0)$

or

(4.10) $|h(t)| > 0$ *for* $t \ge t_0$, $|h(t)|^{-1}$ *is bounded on* $[t_0, \infty)$, *and there are* $N \in \mathbb{N}$ *and* $\mu > 0$ *such that* $|H_N(t)| \ge \mu > 1$ *for* $t \ge \tau^{-(N-1)}(t_0)$.

If

$$\int^{\infty} t^{n-1}F(t, |\omega(g(t))| + \varepsilon)dt < \infty \quad \textit{for some } \varepsilon > 0,$$

then there exists a solution $x(t)$ of (1.1) *such that*

$$x(t) = \omega(t) + o(1) \quad (t \to \infty).$$

Corollary 4.2. *Let $\omega(t)$ be a solution of the unperturbed equation* (4.1). *Suppose that* (4.10) *holds. If*

$$\int^{\infty} t^{n-1}F\left(t, |\omega(g(t))| + \frac{\varepsilon}{|h(\tau^{-1}(g(t)))|}\right)dt < \infty \quad \textit{for some } \varepsilon > 0,$$

then (1.1) *has a solution $x(t)$ satisfying*

$$x(t) = \omega(t) + o\left(\frac{1}{|h(\tau^{-1}(t))|}\right) \quad (t \to \infty).$$

Proofs of Theorems 4.1 and 4.2. From Remark 4.1, it is sufficient to give the proof of Theorem 4.1 for the case (4.3) and the proof of Theorem 4.2. Assume that either (4.3) and (4.5) or (4.4) and (4.7) hold. Let $T_0 \geq t_0$ be a number such that $\omega(t)$ satisfies (4.1) and is continuous on $[T_0, \infty)$. We can take a number $T \geq 1$ satisfying

$$T_* \equiv \min\{\tau(T), \inf\{g(t) : t \geq T\}\} \geq T_0$$

and

$$\int_T^{\infty} t^{n-k-1}F(|\omega(g(t))| + \varepsilon\,\psi(g(t)))dt < 1,$$

where

$$\psi(t) = \begin{cases} t^k & \text{for the case that (4.3) holds,} \\ [\tau^{-1}(t)]^k/|h(\tau^{-1}(t))| & \text{for the case that (4.4) holds.} \end{cases}$$

Put $F(t) = F(t, |\omega(g(t))| + \varepsilon\,\psi(g(t)))$ and

$$\eta(t) = \begin{cases} \displaystyle\int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!}F(s)ds, & k = 0, \\ \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!}\int_s^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!}F(r)drds, & k \neq 0, \end{cases}$$

for $t \geq T$. Consider the set Y of all functions $y \in C[T_*, \infty)$ satisfying

$$y(t) = y(T) \quad \text{for } t \in [T_*, T] \quad \text{and} \quad |y(t)| \leq \eta(t) \quad \text{for } t \geq T.$$

Obviously, Y is a closed convex subset of $C[T_*, \infty)$. It can be shown that

$$Y \subset \{u \in C[T_*, \infty) : |u(t)| \leq t^k \text{ for } t \geq T\}.$$

In fact, if $k = 0$, then $\eta(t) \leq 1$ for $t \geq T$, and if $k \neq 0$, then

$$\begin{aligned}
\eta(t) &\leq \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} ds \times \int_T^\infty r^{n-k-1} F(r) dr \\
&\leq \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} ds = \frac{(t-T)^k}{k!} \leq t^k, \quad t \geq T.
\end{aligned}$$

In view of the fact that $\lim_{t\to\infty} \eta(t)/t^k = 0$, we have $\lim_{t\to\infty} |y(t)|/t^k = 0$ for each $y \in Y$. We use Proposition 3.3 or 3.4 with $p(t) = -h(t)$ and $r(t) = t^k$. Then there exists a continuous mapping $\Lambda : Y \longrightarrow C[T_*, \infty)$ such that, for each $y \in Y$,

$$\Lambda[y](t) + h(t)\Lambda[y](\tau(t)) = y(t), \quad t \geq T, \tag{4.11}$$

and

$$\Lambda[y](t) = o(\psi(t)) \quad (t \to \infty). \tag{4.12}$$

Define the mapping $\mathcal{F} : Y \longrightarrow C[T_*, \infty)$ as follows:

$$(\mathcal{F}y)(t) = \begin{cases}
\displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \overline{f}(s, \omega(g(s)) + (-1)^{n-1}\Lambda[y](g(s))) ds, \\
\hfill k = 0, \quad t \geq T, \\
\displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \\
\quad \times \overline{f}(r, \omega(g(r)) + (-1)^{n-k-1}\Lambda[y](g(r))) dr ds, \\
\hfill k \neq 0, \quad t \geq T, \\
(\mathcal{F}y)(T), \qquad t \in [T_*, T],
\end{cases}$$

where

$$\overline{f}(t,u) = \begin{cases} f(t, \omega(g(t)) + \varepsilon\,\psi(g(t))), & u \geq \omega(g(t)) + \varepsilon\,\psi(g(t)), \\ f(t,u), & |u - \omega(g(t))| \leq \varepsilon\,\psi(g(t)), \\ f(t, \omega(g(t)) - \varepsilon\,\psi(g(t))), & u \leq \omega(g(t)) - \varepsilon\,\psi(g(t)). \end{cases}$$

Note that $|\overline{f}(t,u)| \leq F(t)$ for all $u \in \mathbb{R}$. Then it is easy to see that $\mathcal{F}$ is well defined on Y and maps Y into itself. Since Λ is continuous on Y, the Lebesgue dominated convergence theorem shows that $\mathcal{F}$ is continuous on Y.

Now we claim that $\mathcal{F}(Y)$ is relatively compact. We note that $\mathcal{F}(Y)$ is uniformly bounded on every compact subinterval of $[T_*, \infty)$, because of $\mathcal{F}(Y) \subset Y$. By the Ascoli-Arzelà theorem, it suffices to verify that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*, \infty)$. If $k = 0$ and $n = 1$, then

$$|(\mathcal{F}y)'(t)| \leq F(t), \quad t \geq T.$$

If either $k = 0$ and $n \geq 2$ or $k = 1$, then

$$|(\mathcal{F}y)'(t)| \leq \int_T^\infty s^{n-2}F(s)ds \leq 1, \quad t \geq T.$$

If $k \geq 2$, then

$$\begin{aligned} |(\mathcal{F}y)'(t)| &\leq \int_T^t \frac{(t-s)^{k-2}}{(k-2)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} F(r)dr \\ &\leq \int_T^t \frac{(t-s)^{k-2}}{(k-2)!} ds \times \int_T^\infty r^{n-k-1}F(r)dr \\ &\leq \frac{(t-T)^{k-1}}{(k-1)!}, \quad t \geq T. \end{aligned}$$

Let I be an arbitrary compact subinterval of $[T, \infty)$. Then we see that $\{(\mathcal{F}y)'(t) : y \in Y\}$ is uniformly bounded on I. The mean value theorem implies that $\mathcal{F}(Y)$ is equicontinuous on I. Since $|(\mathcal{F}y)(t_1) - (\mathcal{F}y)(t_2)| = 0$ for $t_1, t_2 \in [T_*, T]$, we conclude that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*, \infty)$.

Applying the Schauder-Tychonoff fixed point theorem to the operator $\mathcal{F}$, we find that there exists a $\widetilde{y} \in Y$ such that $\widetilde{y} = \mathcal{F}\widetilde{y}$. Set

$$x(t) = \omega(t) + (-1)^{n-k-1}\Lambda[\widetilde{y}](t).$$

From (4.12) it follows that $x(t)$ satisfies $x(t) = \omega(t) + o(\psi(t))$ $(t \to \infty)$, and hence there exists a number $\widetilde{T} \geq T$ such that $|x(g(t)) - \omega(g(t))| \leq \varepsilon\,\psi(g(t))$ for $t \geq \widetilde{T}$. Then $\overline{f}(t, x(g(t))) = f(t, x(g(t)))$ for $t \geq \widetilde{T}$. By virtue of (4.11), we observe that

$$\begin{aligned}
(4.13)\qquad & x(t) + h(t)x(\tau(t)) \\
&= \omega(t) + h(t)\omega(\tau(t)) + (-1)^{n-k-1}[\Lambda[\widetilde{y}](t) + h(t)\Lambda[\widetilde{y}](\tau(t))] \\
&= \omega(t) + h(t)\omega(\tau(t)) + (-1)^{n-k-1}\widetilde{y}(t) \\
&= \omega(t) + h(t)\omega(\tau(t)) + (-1)^{n-k-1}(\mathcal{F}\widetilde{y})(t)
\end{aligned}$$

for $t \geq \widetilde{T}$. By differentiation of (4.13), we see that $x(t)$ is a solution of (1.1). The proof is complete.

Now we assume that (4.1) has a positive solution $\omega(t)$. Consider the equation

$$(4.14)\qquad \frac{d^n}{dt^n}[x(t) + h(t)x(\tau(t))] + \varphi(t, x(g(t))) = q(t),$$

where (1.2)–(1.6), (1.8) and the following condition are assumed to hold:

(4.15) $\varphi \in C([t_0, \infty) \times (0, \infty))$ and there exists a continuous function $F \in C([t_0, \infty) \times (0, \infty))$ such that $F(t, u)$ is nondecreasing in $u \in (0, \infty)$ for each fixed $t \geq t_0$ and satisfies

$$|\varphi(t, u)| \leq F(t, u), \quad (t, u) \in [t_0, \infty) \times (0, \infty).$$

Theorem 4.3. *Let $k \in \{0, 1, 2, \ldots, n-1\}$, and let $\omega(t)$ be a solution of the unperturbed equation* (4.1) *satisfying*

$$\liminf_{t \to \infty} \frac{\omega(t)}{t^k} > 0.$$

Suppose that (4.3) *or* (4.4) *holds. If*

$$\int^{\infty} t^{n-k-1} F(t, \omega(g(t)) + \varepsilon[g(t)]^k) dt < \infty \quad \textit{for some } \varepsilon > 0,$$

then (4.14) *has a positive solution* $x(t)$ *satisfying* (4.6).

Theorem 4.4. *Let* $k \in \{0, 1, 2, \ldots, n-1\}$, *and let* $\omega(t)$ *be a solution of the unperturbed equation* (4.1) *satisfying*

$$\liminf_{t \to \infty} \frac{\omega(t)|h(\tau^{-1}(t))|}{[\tau^{-1}(t)]^k} > 0.$$

Suppose that (4.4) *holds. If*

$$\int^{\infty} t^{n-k-1} F\left(t, \omega(g(t)) + \frac{\varepsilon[\tau^{-1}(g(t))]^k}{|h(\tau^{-1}(g(t)))|}\right) dt < \infty \quad \textit{for some } \varepsilon > 0,$$

then there exists a positive solution $x(t)$ *of* (4.14) *satisfying* (4.8).

Proofs of Theorems 4.3 and 4.4. We may assume without loss of generality that $\varepsilon > 0$ is sufficiently small. Apply the proofs of Theorems 4.1 and 4.2 with f and T_0 replaced by φ and a large number T_1 such that $\omega(t)$ is continuous and satisfies (4.1) on $[T_1, \infty)$, and $\omega(t) > \varepsilon\psi(t)$ for $t \geq T_1$.

5. Existence of positive solutions

In this section we derive various sufficient conditions and necessary conditions for the neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + h(t)x(\tau(t))] + f(t, x(g(t))) = 0 \tag{5.1}$$

to have certain positive solutions. It is assumed throughout this section that (1.2)–(1.6) and the following condition (5.2) hold:

(5.2) $f \in C([t_0, \infty) \times (0, \infty))$ and there exists a continuous function $F \in C([t_0, \infty) \times (0, \infty))$ such that $F(t, u)$ is nondecreasing in $u \in (0, \infty)$ for each fixed $t \geq t_0$ and satisfies

$$|f(t, u)| \leq F(t, u), \quad (t, u) \in [t_0, \infty) \times (0, \infty).$$

We note here that the unperturbed equation of (5.1) is

$$\frac{d^n}{dt^n}[\omega(t) + h(t)\omega(\tau(t))] = 0. \tag{5.3}$$

Theorem 5.1. *Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that one of the following conditions* (5.4)–(5.6) *holds:*

$$|h(t)|[\tau(t)/t]^k \le \lambda < 1/2, \quad t \ge t_0, \quad \textit{for some } \lambda > 0; \tag{5.4}$$

$$|h(t)|[\tau(t)/t]^k \le \lambda < 1 \quad \textit{and} \quad h(t)h(\tau(t)) \ge 0, \quad t \ge \tau^{-1}(t_0), \quad \textit{for some } \lambda > 0; \tag{5.5}$$

$$1 < \mu \le |h(t)|[\tau(t)/t]^k \le \lambda, \quad t \ge t_0, \quad \textit{for some } \lambda > 0 \textit{ and } \mu > 0. \tag{5.6}$$

If

$$\int^{\infty} t^{n-k-1} F(t, c[g(t)]^k)dt < \infty \quad \textit{for some } c > 0, \tag{5.7}$$

then (5.1) *has a positive solution $x(t)$ satisfying*

$$0 < \liminf_{t\to\infty} \frac{x(t)}{t^k} \le \limsup_{t\to\infty} \frac{x(t)}{t^k} < \infty. \tag{5.8}$$

Proof. If (5.4) or (5.6) holds, then (4.3) or (4.4) holds, respectively. If (5.5) holds, then (4.3) with t_0 replaced by $\tau^{-1}(t_0)$ holds. Lemmas 2.1, 2.3, 2.4, 2.7, 2.8 and 3.1 imply that there exists a function $\theta \in C[\tau(T), \infty)$ satisfying

$$\theta(t) + h(t)\left[\frac{\tau(t)}{t}\right]^k \theta(\tau(t)) = 1, \quad t \ge T, \tag{5.9}$$

and

$$0 < \liminf_{t\to\infty} |\theta(t)| \le \limsup_{t\to\infty} |\theta(t)| < \infty, \tag{5.10}$$

where T is a sufficiently large number. We note that $\theta(t)$ is eventually positive or eventually negative. For any $a > 0$, we put $\omega(t) = a\, t^k|\theta(t)|$.

In view of (5.9) and (5.10), we see that $\omega(t)$ is a solution of the unperturbed equation (5.3) and satisfies

$$0 < \liminf_{t\to\infty} \frac{\omega(t)}{t^k} \le \limsup_{t\to\infty} \frac{\omega(t)}{t^k} < \infty.$$

If $a > 0$ is sufficiently small, then there are a constant $\varepsilon > 0$ and a number $T_1 \ge T$ such that

$$\omega(g(t)) + \varepsilon[g(t)]^k < c[g(t)]^k, \quad t \ge T_1.$$

By virtue of Theorem 4.3, (5.1) has a positive solution $x(t)$ satisfying (5.8).

Theorem 5.2. *Let $k \in \{0, 1, 2, \dots, n-1\}$. Suppose that*

$$\lim_{t\to\infty} h(t) \left[\frac{\tau(t)}{t}\right]^k = l \quad \textit{for some } l \in \mathbb{R} \textit{ with } |l| \neq 1. \tag{5.11}$$

If (5.7) holds, then (5.1) has a positive solution $x(t)$ satisfying

$$\lim_{t\to\infty} \frac{x(t)}{t^k} \quad \textit{exists and is a positive finite value.} \tag{5.12}$$

Proof. We easily see that (4.3) or (4.4) holds for some large number t_0. Therefore, by the same arguments as in the proof of Theorem 5.1, the conclusion follows from Theorem 4.3 and Lemma 2.10.

Theorem 5.3. *Let $k \in \{0, 1, 2, \dots, n-1\}$. Suppose that*

$$\lim_{t\to\infty} |h(t)|[\tau(t)/t]^k = \infty.$$

If

$$\int^{\infty} t^{n-k-1} F\left(t, c\frac{[\tau^{-1}(g(t))]^k}{|h(\tau^{-1}(g(t)))|}\right) dt < \infty \quad \textit{for some } c > 0, \tag{5.13}$$

then (5.1) has a positive solution $x(t)$ satisfying

$$\lim_{t\to\infty} x(t) \frac{|h(\tau^{-1}(t))|}{[\tau^{-1}(t)]^k} \quad \textit{exists and is a positive finite value.} \tag{5.14}$$

Proof. It is easy to check that (4.4) holds for some large number t_0. From Lemmas 2.9 and 3.1 it follows that there exists a function $\theta \in C[\tau(T), \infty)$ satisfying (5.9) and

$$\lim_{t\to\infty} |\theta(t)||h(\tau^{-1}(t))|[t/\tau^{-1}(t)]^k = 1,$$

where T is sufficiently large. Notice that $\theta(t)$ is eventually positive or eventually negative. There are constants $a > 0$, $\varepsilon > 0$ and a number $T_1 \geq T$ such that

$$a|\theta(t)|t^k + \varepsilon \frac{[\tau^{-1}(t)]^k}{|h(\tau^{-1}(t))|} < c\frac{[\tau^{-1}(t)]^k}{|h(\tau^{-1}(t))|}, \quad t \geq T_1.$$

Put $\omega(t) = a|\theta(t)|t^k$. Then we find that $\omega(t)$ is a solution of (5.3) and satisfies

$$\lim_{t\to\infty} \frac{\omega(t)|h(\tau^{-1}(t))|}{[\tau^{-1}(t)]^k} = a > 0. \tag{5.15}$$

Applying Theorem 4.4, we conclude that (5.1) has a solution $x(t)$ satisfying (4.8). By (5.15), $x(t)$ satisfies

$$\lim_{t\to\infty} x(t)\frac{|h(\tau^{-1}(t))|}{[\tau^{-1}(t)]^k} = a > 0.$$

This completes the proof.

Since $\tau(t) < t$ for $t \geq t_0$, the conditions

$$|h(t)| \leq \lambda < 1/2, \quad t \geq t_0, \quad \text{for some } \lambda > 0 \tag{5.16}$$

and

$$|h(t)| \leq \lambda < 1, \quad h(t)h(\tau(t)) \geq 0, \quad t \geq \tau^{-1}(t_0), \quad \text{for some } \lambda > 0 \tag{5.17}$$

imply (5.4) and (5.5), respectively. Thus, by Theorem 5.1, we have the following corollary.

Corollary 5.1. *Let $k \in \{0, 1, 2, \dots, n-1\}$, and suppose that (5.16) or (5.17) holds. If (5.7) holds, then (5.1) has a positive solution $x(t)$ satisfying (5.8).*

In particular, for the case $k = 0$, Theorems 5.1–5.3 give the following results.

Corollary 5.2. *Suppose that*

$$1 < \mu \le |h(t)| \le \lambda, \quad t \ge t_0, \quad \text{for some } \lambda > 0 \text{ and } \mu > 0. \tag{5.18}$$

If

$$\int^{\infty} t^{n-1} F(t, c)dt < \infty \quad \text{for some } c > 0, \tag{5.19}$$

then (5.1) *has a positive solution* $x(t)$ *satisfying*

$$0 < \liminf_{t \to \infty} x(t) \le \limsup_{t \to \infty} x(t) < \infty.$$

Corollary 5.3. *Suppose that* $\lim_{t\to\infty} h(t) = l$ *for some* $l \in \mathbb{R}$ *with* $|l| \neq 1$. *If* (5.19) *holds, then* (5.1) *has a positive solution* $x(t)$ *satisfying*

$$\lim_{t \to \infty} x(t) \quad \text{exists and is a positive finite value.}$$

Corollary 5.4. *Suppose that* $\lim_{t\to\infty} |h(t)| = \infty$. *If*

$$\int^{\infty} t^{n-1} F(t, c/|h(\tau^{-1}(g(t)))|)dt < \infty \quad \text{for some } c > 0,$$

then (5.1) *has a positive solution* $x(t)$ *such that*

$$\lim_{t \to \infty} x(t)|h(\tau^{-1}(t))| \quad \text{exists and is a positive finite value.}$$

Now we consider the case

$$\lim_{t \to \infty} \tau(t)/t = 1. \tag{5.20}$$

Corollary 5.5. *Let* $k \in \{0, 1, 2, \dots, n-1\}$. *Suppose that* (5.18) *and* (5.20) *hold. If* (5.7) *holds, then* (5.1) *has a positive solution* $x(t)$ *satisfying* (5.8).

Corollary 5.6. *Let* $k \in \{0, 1, 2, \dots, n-1\}$. *Suppose that* (5.20) *holds and that* $\lim_{t\to\infty} h(t) = l$ *for some* $l \in \mathbb{R}$ *with* $|l| \neq 1$. *If* (5.7) *holds, then* (5.1) *has a positive solution* $x(t)$ *satisfying* (5.12).

Corollary 5.7. *Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that* (5.20) *holds and that* $\lim_{t\to\infty} |h(t)| = \infty$. *If* (5.13) *holds, then* (5.1) *has a positive solution $x(t)$ satisfying* (5.14).

Proofs of Corollaries 5.5–5.7. We give the proof of Corollary 5.5 only. In a similar fashion, we can prove that Corollaries 5.6 and 5.7 follow. For the case $k = 0$, Corollary 5.5 follows immediately from Corollary 5.2. Assume that $k \neq 0$. If (5.18) and (5.20) hold, then (5.6) holds for some large number t_0. Hence, Theorem 5.1 shows that (5.1) has a positive solution $x(t)$ satisfying (5.8).

Now consider neutral differential equations of the form

$$\frac{d^n}{dt^n}[x(t) + h(t)x(\tau(t))] + \sigma f(t, x(g(t))) = 0, \tag{5.21}$$

where $\sigma = +1$ or -1. We establish necessary conditions for (5.21) to have a positive solution $x(t)$ satisfying (5.8) or (5.14). For equation (5.21) we assume the following:

$$\begin{aligned} &f \in C([t_0,\infty)\times(0,\infty)),\ f(t,u) \geq 0 \text{ for } (t,u) \in [t_0,\infty)\times(0,\infty) \\ &\text{and } f(t,u) \text{ is nondecreasing in } u \in (0,\infty) \text{ for each fixed } t \geq t_0. \end{aligned} \tag{5.22}$$

Theorem 5.4. *Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that* (5.22) *holds and that $h(t)[\tau(t)/t]^k$ is bounded on $[t_0, \infty)$. If there exists a positive solution $x(t)$ of* (5.21) *which satisfies* (5.8), *then*

$$\int^{\infty} t^{n-k-1} f(t, c[g(t)]^k)dt < \infty \quad \textit{for some } c > 0. \tag{5.23}$$

Proof. Put $y(t) = x(t) + h(t)x(\tau(t))$. We get

$$\frac{y(t)}{t^k} = \frac{x(t)}{t^k} + h(t)\left[\frac{\tau(t)}{t}\right]^k \frac{x(\tau(t))}{[\tau(t)]^k},$$

which implies that $y(t)/t^k$ is bounded. From (5.21), we have

$$\sigma y^{(n)}(t) = -f(t, x(g(t))) \leq 0 \quad \text{for all large } t. \tag{5.24}$$

We see that $y^{(i)}(t)$ $(i = 0, 1, 2, \dots, n-1)$ are eventually monotonic, so that $\lim_{t\to\infty} y^{(i)}(t)$ $(i = 0, 1, 2, \dots, n-1)$ exist in $\mathbb{R} \cup \{-\infty, \infty\}$. Since $y(t)/t^k$ is bounded, we find that $\lim_{t\to\infty} y^{(k)}(t) = l$ for some $l \in \mathbb{R}$ and $\lim_{t\to\infty} y^{(i)}(t) = 0$ for $i = k+1, \dots, n-1$. Repeated integration of (5.24) yields

$$y^{(k)}(t) = l + (-1)^{n-k-1}\sigma \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, x(g(s)))ds$$

for all large t. Consequently we obtain

$$\int_T^\infty s^{n-k-1} f(s, x(g(s)))ds < \infty \quad \text{for some } T \geq t_0.$$

By virtue of (5.8) and the monotonicity of f, we conclude that (5.23) holds.

Theorem 5.5. *Let* $k \in \{0, 1, 2, \dots, n-1\}$. *Suppose that* (5.22) *holds and that* $\lim_{t\to\infty} |h(t)|[\tau(t)/t]^k = \infty$. *If* (5.21) *has a positive solution* $x(t)$ *satisfying* (5.14), *then*

$$\text{(5.25)} \qquad \int^\infty t^{n-k-1} f\left(t, c\frac{[\tau^{-1}(g(t))]^k}{|h(\tau^{-1}(g(t)))|}\right) dt < \infty \quad \textit{for some } c > 0,$$

Proof. Observe that

$$\frac{x(\tau(t))}{[\tau(t)]^k} = \frac{x(\tau(t))h(t)}{t^k} \frac{1}{h(t)} \left[\frac{t}{\tau(t)}\right]^k,$$

so that $\lim_{t\to\infty} x(t)/t^k = 0$. Put $y(t) = x(t) + h(t)x(\tau(t))$. Then

$$\lim_{t\to\infty} \frac{y(t)}{t^k} = \lim_{t\to\infty} \frac{x(t)}{t^k} + \lim_{t\to\infty} \frac{h(t)x(\tau(t))}{t^k} = L$$

for some constant $L \neq 0$. By the same argument as in the proof of Theorem 5.4, the conclusion follows.

By Theorems 5.1–5.5, we obtain necessary and sufficient conditions for (5.21) to have a positive solution $x(t)$ satisfying (5.8) or (5.12) or (5.14).

Theorem 5.6. *Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that* (5.22) *holds and that one of conditions* (5.4)–(5.6) *holds. Then* (5.21) *has a positive solution $x(t)$ satisfying* (5.8) *if and only if* (5.23) *holds.*

Theorem 5.7. *Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that* (5.11) *and* (5.22) *hold. Then* (5.21) *has a positive solution $x(t)$ satisfying* (5.12) *if and only if* (5.23) *holds.*

Theorem 5.8. *Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that* (5.22) *holds and that $\lim_{t\to\infty} |h(t)|[\tau(t)/t]^k = \infty$. Then* (5.21) *has a positive solution $x(t)$ satisfying* (5.14) *if and only if* (5.25) *holds.*

6. Existence of positive solutions for the case $h(t) \leq 0$

In this section we consider the neutral differential equation (1.14) for the case $h(t) \leq 0$. It is convenient to rewrite (1.14) in the form

$$\frac{d^n}{dt^n}[x(t) - p(t)x(\tau(t))] + f(t, x(g(t))) = 0. \tag{6.1}$$

For equation (6.1), conditions (1.2), (1.3), (1.5), (1.6), (5.2) and the following condition are assumed to hold: $p \in C[t_0, \infty)$ and $p(t) \geq 0$ for $t \geq t_0$.

The notation (2.3) will be used.

Theorem 6.1. *Let $k \in \{0, 1, 2, \ldots, n-1\}$ and let $u_k(t)$ be a positive continuous function satisfying*

(6.2)
$$0 < \liminf_{t\to\infty} \frac{u_k(t) - p(t)u_k(\tau(t))}{t^k} \leq \limsup_{t\to\infty} \frac{u_k(t) - p(t)u_k(\tau(t))}{t^k} < \infty.$$

If

$$\int^{\infty} t^{n-k-1} F(t, u_k(g(t)))dt < \infty, \tag{6.3}$$

then (6.1) *has a positive solution $x(t)$ such that*

$$0 < \liminf_{t\to\infty} \frac{x(t)}{u_k(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{u_k(t)} < \infty \tag{6.4}$$

and

$$\lim_{t\to\infty} \frac{x(t) - p(t)x(\tau(t))}{t^k} = const > 0.$$

Theorem 6.2. *Suppose that $p(t) > 0$ for $t \geq t_0$. Let k be an integer with $0 \leq k \leq n-1$, and assume that there exists a positive continuous function $u_k(t)$ such that*

(6.5)

$$-\infty < \liminf_{t\to\infty} \frac{u_k(t) - p(t)u_k(\tau(t))}{t^k} \leq \limsup_{t\to\infty} \frac{u_k(t) - p(t)u_k(\tau(t))}{t^k} < 0.$$

Let $w(t)$ be a positive continuous function satisfying

$$w(t) - p(t)w(\tau(t)) = 0, \quad t \geq t_0. \tag{6.6}$$

If

$$\int^{\infty} t^{n-k-1} F(t, w(g(t)))dt < \infty, \tag{6.7}$$

then (6.1) *has a positive solution $x(t)$ satisfying*

$$0 < \liminf_{t\to\infty} \frac{x(t)}{w(t)} \leq \limsup_{t\to\infty} \frac{x(t)}{w(t)} < \infty \tag{6.8}$$

and

$$\lim_{t\to\infty} \frac{x(t) - p(t)x(\tau(t))}{t^k} = const < 0. \tag{6.9}$$

Remark 6.1. It should be emphasized that restrictive condition on $p(t)$, such as $p(t) \leq \lambda$, is not assumed in Theorems 6.1 and 6.2.

Remark 6.2. It should be noted here that there always exists a positive continuous function $u_k(t)$ which satisfies (6.2), and the integral condition (6.3) and the asymptotic condition (6.4) do not depend on the choice of the function $u_k(t)$. (See Lemma 6.2 below.)

Remark 6.3. The integral condition (6.7) and the asymptotic condition (6.8) are independent of the choice of the function $w(t)$. (See Lemma 6.4.)

First let us prove Theorem 6.1. We need the following lemmas.

Lemma 6.1. *Assume that $u, v \in C[\tau(t_0), \infty)$ satisfy*

$$\begin{cases} u(t) - p(t)u(\tau(t)) \geq v(t) - p(t)v(\tau(t)), & t \geq t_0, \\ \qquad\qquad u(t) \geq v(t), & t \in [\tau(t_0), t_0]. \end{cases}$$

Then $u(t) \geq v(t)$ for $t \geq \tau(t_0)$.

Proof. Put $w(t) = u(t) - v(t)$. Then

$$\begin{cases} w(t) - p(t)w(\tau(t)) \geq 0, & t \geq t_0, \\ \qquad w(t) \geq 0, & t \in [\tau(t_0), t_0]. \end{cases}$$

It is sufficient to show that $w(t) \geq 0$ for $t \geq \tau(t_0)$. Obviously, $w(t) \geq 0$ for $t \in [\tau(t_0), t_0]$. Assume that $w(t) \geq 0$ for $t \in [\tau^{-(m-1)}(t_0), \tau^{-m}(t_0)]$, $m = 0, 1, 2, \ldots$. If $t \in [\tau^{-m}(t_0), \tau^{-(m+1)}(t_0)]$, then

$$w(t) \geq p(t)w(\tau(t)) \geq 0,$$

because of $\tau^{-(m-1)}(t_0) \leq \tau(t) \leq \tau^{-m}(t_0)$. By induction, we conclude that $w(t) \geq 0$ for $t \geq \tau(t_0)$.

Lemma 6.2. *There exists a function $v_k \in C[\tau(t_0), \infty)$ such that $v_k(t) > 0$ for $t \geq t_0$ and*

$$v_k(t) - p(t)v_k(\tau(t)) = t^k, \quad t \geq t_0. \tag{6.10}$$

In addition, for each positive continuous function u_k satisfying (6.2), *there are constants $c_* > 0$, $c^* > 0$ and a number $T \geq t_0$ such that*

$$c_* v_k(t) \leq u_k(t) \leq c^* v_k(t), \quad t \geq \tau(T). \tag{6.11}$$

Proof. Lemma 2.1 together with Remark 2.1 implies that the function

$$v_k(t) = \begin{cases} \displaystyle\sum_{i=0}^{m} P_i(t)[\tau^i(t)]^k + P_{m+1}(t)\frac{\tau^{m+1}(t) - \tau(t_0)}{t_0 - \tau(t_0)} t_0^k, \\ \qquad t \in [\tau^{-m}(t_0), \tau^{-(m+1)}(t_0)], \quad m = 0, 1, \ldots, \\ \displaystyle\frac{t - \tau(t_0)}{t_0 - \tau(t_0)} t_0^k, \qquad t \in [\tau(t_0), t_0], \end{cases}$$

satisfies $v_k(t) - p(t)v_k(\tau(t)) = t^k$ for $t \geq t_0$. It is easy to see that $v_k(t) \geq t^k > 0$ for $t \geq t_0$.

We can choose a sufficiently large $T \geq t_0$, a sufficiently small $c_* > 0$ and a sufficiently large $c^* > 0$ such that

$$c_* t^k \leq u_k(t) - p(t)u_k(\tau(t)) \leq c^* t^k, \quad t \geq T$$

and

$$c_* v_k(t) \leq u_k(t) \leq c^* v_k(t), \quad t \in [\tau(T), T].$$

By (6.10), we have

$$u_k(t) - p(t)u_k(\tau(t)) \geq c_* t^k = c_* v_k(t) - p(t)[c_* v_k(\tau(t))], \quad t \geq T.$$

Using Lemma 6.1, we obtain $u_k(t) \geq c_* v_k(t)$ for $t \geq \tau(T)$. In the same way, we have $u_k(t) \leq c^* v_k(t)$ for $t \geq \tau(T)$. The proof is complete.

Proof of Theorem 6.1. Let $v_k \in C[\tau(t_0), \infty)$ satisfy (6.10) and $v_k(t) > 0$ for $t \geq t_0$. In view of Lemma 6.2, we can take a number $T \geq t_0$ and constants $c_* > 0$, $c^* > 0$ satisfying (6.11),

$$T_* = \min\{\tau(T), \inf\{g(t) : t \geq T\}\} \geq t_0,$$

and

$$\int_T^\infty t^{n-k-1} F(t, c_* v_k(g(t)))dt < \frac{c_*}{4}. \tag{6.12}$$

Consider the set Y of all functions $y \in C[T_*, \infty)$ satisfying

$$y(t) = y(T) \quad \text{for } t \in [T_*, T] \quad \text{and} \quad c_* t^k/2 \leq y(t) \leq c_* t^k \quad \text{for } t \geq T.$$

Clearly, Y is a nonempty, closed and convex subset of $C[T_*, \infty)$. For $y \in Y$, we assign the function $\Phi[y]$ on $[T_*, \infty)$ by

(6.13)

$$\Phi[y](t) = \begin{cases} \displaystyle\sum_{i=0}^{m} P_i(t) y(\tau^i(t)) + P_{m+1}(t)\varphi(\tau^{m+1}(t)), \\ \qquad\qquad t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)], \quad m = 0, 1, \dots, \\ \varphi(t), \qquad\qquad t \in [T_*, T], \end{cases}$$

where

$$\varphi(t) = \frac{y(T)}{T^k} v_k(t), \quad t \in [T_*, T].$$

By the same arguments as in the proof of Proposition 3.1, we see that $\Phi : Y \longrightarrow C[T_*, \infty)$ is continuous in the $C[T_*, \infty)$-topology, and satisfies

$$\Phi[y](t) - p(t)\Phi[y](\tau(t)) = y(t), \quad t \ge T, \quad y \in Y. \tag{6.14}$$

We define the operator $\mathcal{F} : Y \longrightarrow C[T_*, \infty)$ by

$$(\mathcal{F}y)(t) = \begin{cases} \dfrac{3}{4}c_* + (-1)^{n-1} \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, \Phi[y](g(s))) ds, \\ \qquad\qquad k = 0, \quad t \ge T, \\ \dfrac{3}{4}c_* t^k + (-1)^{n-k-1} \times \\ \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, \Phi[y](g(r))) dr ds, \\ \qquad\qquad k \ne 0, \quad t \ge T, \\ (\mathcal{F}y)(T), \qquad t \in [T_*, T]. \end{cases}$$

We claim that $\mathcal{F}$ is well defined. By virtue of (6.10) and (6.14), we find that

$$c_* v_k(t) - p(t) c_* v_k(\tau(t)) = c_* t^k \ge y(t) = \Phi[y](t) - p(t)\Phi[y](\tau(t))$$

for $t \ge T$ and $y \in Y$, and

$$c_* v_k(t) \ge \frac{y(T)}{T^k} v_k(t) = \Phi[y](t), \quad t \in [T_*, T], \quad y \in Y.$$

In view of of Lemma 6.1, we have $\Phi[y](t) \leq c_* v_k(t)$ for $t \geq T_*$ and $y \in Y$. Likewise, we obtain $\Phi[y](t) \geq c_* v_k(t)/2 > 0$ for $t \geq T_*$ and $y \in Y$. Consequently, $\mathcal{F}$ is well defined.

We conclude that $\mathcal{F}$ maps Y into itself. In fact, from (6.12) it follows that

$$\begin{aligned}&\left|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, \Phi[y](g(s)))ds\right| \\ &\leq \int_T^\infty s^{n-1} F(s, c_* v_k(g(s)))ds < \frac{c_*}{4}\end{aligned}$$

for $k = 0$, $t \geq T$, $y \in Y$, and

$$\begin{aligned}&\left|\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, \Phi[y](g(r)))drds\right| \\ &\leq \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} ds \cdot \int_T^\infty r^{n-k-1} F(r, c_* v_k(g(s)))dr < \frac{c_*}{4} t^k\end{aligned}$$

for $k \neq 0$, $t \geq T$ and $y \in Y$. As in the proofs of Theorems 4.1 and 4.2, we see that $\mathcal{F}$ is continuous on Y and $\mathcal{F}(Y)$ is relatively compact. Using the Schauder-Tychonoff fixed point theorem, we have $\widetilde{y} = \mathcal{F}\widetilde{y}$ for some $\widetilde{y} \in Y$. Then it is easy to verify that $x(t) = \Phi[\widetilde{y}](t)$ is a positive solution of (6.1) satisfying

$$c_* v_k(t)/2 \leq x(t) \leq c_* v_k(t), \quad t \geq T_*,$$

and

$$\lim_{t\to\infty} \frac{x(t) - p(t)x(\tau(t))}{t^k} = \frac{3}{4} c_* > 0.$$

This completes the proof.

For the proof of Theorem 6.2, we need some lemmas.

Lemma 6.3. *Suppose that* $p(t) > 0$ *for* $t \geq t_0$. *There exists a positive continuous function* $u_k(t)$ *satisfying* (6.5) *if and only if there exists a positive continuous function* $v_k(t)$ *satisfying*

(6.15) $$v_k(t) - p(t)v_k(\tau(t)) = -t^k \quad \textit{for all large } t.$$

Proof. The "if" part is clear. We show the "only if" part. There are a number T and a constant $c > 0$ such that

$$u_k(t) - p(t)u_k(\tau(t)) \le -ct^k, \quad t \ge T.$$

Put

$$\varphi(t) = \left(\frac{K + T^k}{p(T)} - K\right)\frac{T - t}{T - \tau(T)} + K, \quad t \in [\tau(T), T],$$

where $K > 0$. It is easy to see that

$$\varphi(T) - p(T)\varphi(\tau(T)) = -T^k$$

and

$$\begin{aligned}\min\{\varphi(t) : t \in [\tau(T), T]\} &= \min\{\varphi(\tau(T)), \varphi(T)\} \\ &= \min\left\{\frac{K + T^k}{p(T)}, K\right\}.\end{aligned}$$

Hence, we can choose $K > 0$ so large that

$$c^{-1}u_k(t) \le \varphi(t), \quad t \in [\tau(T), T].$$

Let $v_k \in C[\tau(T), \infty)$ be a solution of the initial value problem

$$\begin{cases} v_k(t) - p(t)v_k(\tau(t)) = -t^k, & t \ge T, \\ v_k(t) = \varphi(t), & t \in [\tau(T), T]. \end{cases}$$

(See Lemma 2.1.) We see from Lemma 6.2 that $v_k(t) \ge c^{-1}u_k(t) > 0$ for $t \ge \tau(T)$. This completes the proof.

Lemma 6.4. *Suppose that $p(t) > 0$ for $t \ge t_0$. Then there exists a function $w \in C[\tau(t_0), \infty)$ satisfying* (6.6) *and $w(t) > 0$ for $t \ge \tau(t_0)$. In addition, if $w_1, w_2 \in C[\tau(t_0), \infty)$ are positive solutions of* (6.6)*, then there exist constants c_* and c^* such that*

$$(6.16) \qquad c_* w_2(t) \le w_1(t) \le c^* w_2(t), \quad t \ge \tau(t_0).$$

Proof. Put

$$\varphi(t) = \frac{p(t_0) - 1}{t_0 - \tau(t_0)}(t - \tau(t_0)) + 1, \quad t \in [\tau(t_0), t_0].$$

Then we easily find that $\varphi(t)$ satisfies $\varphi(t) > 0$ for $t \in [\tau(t_0), t_0]$ and $\varphi(t_0) - p(t_0)\varphi(\tau(t_0)) = 0$. From Lemma 2.1 it follows that the function

$$w(t) = \begin{cases} P_{m+1}(t)\varphi(\tau^{m+1}(t)), & t \in [\tau^{-m}(t_0), \tau^{-(m+1)}(t_0)], \\ & m = 0, 1, \dots, \\ \varphi(t), & t \in [\tau(t_0), t_0], \end{cases}$$

satisfies (6.6). Since $p(t) > 0$ and $\varphi(t) > 0$, we have $w(t) > 0$ for $t \geq \tau(t_0)$.

Let $w_1, w_2 \in C[\tau(t_0), \infty)$ be positive solutions of (6.6). Take constants $c_* > 0$ and $c^* > 0$ such that

$$c_* w_2(t) \leq w_1(t) \leq c^* w_2(t), \quad t \in [\tau(t_0), t_0].$$

Then, from Lemma 6.1, we obtain (6.16).

Lemma 6.5. *Let $w(t)$ and $u_k(t)$ be positive continuous functions on $[\tau(t_0), \infty)$ which satisfy (6.6) and (6.5), respectively, where k is an integer with $0 \leq k \leq n-1$. Then*

$$\limsup_{t \to \infty} \frac{u_k(t)}{w(t)} < \infty.$$

Proof. By (6.5), there is a $T \geq t_0$ such that

$$u_k(t) - p(t)u_k(\tau(t)) < 0, \quad t \geq T.$$

Take a sufficiently large number $c > 0$ such that $u_k(t) \leq c\,w(t)$ for $t \in [\tau(T), T]$. In view of Lemma 6.1, we obtain $u_k(t) \leq c\,w(t)$ for $t \geq \tau(T)$. This completes the proof.

Proof of Theorem 6.2. By Lemma 6.3, there exists a positive continuous function $v_k(t)$ satisfying (6.15). Using Lemma 6.5 applied to the case of $u_k(t) = v_k(t)$, we find that

$$c_1[v_k(t) + w(t)] \leq w(t), \quad t \geq t_1, \tag{6.17}$$

for some $c_1 > 0$ and $t_1 \geq t_0$, so that

$$\int^{\infty} t^{n-k-1} F(t, c_1[v_k(g(t)) + w(g(t))])dt < \infty.$$

By Lemma 6.5 again there exist $t_2 > t_1$ and $c_2 > c_1$ such that

$$(6.18) \qquad (c_2 - c_1)v_k(t) \leq \frac{1}{3}c_1 w(t), \quad t \geq t_2.$$

Choose $T \geq t_0$ so large that

$$T_* = \min\{\tau(T), \inf\{g(t) : t \geq T\}\} \geq t_2$$

and

$$\int_T^{\infty} t^{n-k-1} F(t, c_1[v_k(g(t)) + w(g(t))])dt < \frac{c_2 - c_1}{2}.$$

Consider the set Y of all functions $y \in C[T_*, \infty)$ satisfying

$$y(t) = y(T) \quad \text{for } t \in [T_*, T] \quad \text{and} \quad c_1 t^k \leq y(t) \leq c_2 t^k \quad \text{for } t \geq T.$$

For $y \in Y$, we define the function $\Phi[y]$ by (6.13) with

$$\varphi(t) = -\left[\frac{y(T)}{T^k}v_k(t) + \frac{2}{3}c_1 w(t)\right], \quad t \in [T_*, T].$$

Using the same arguments as in the proof of Proposition 3.1, we find that $\Phi : Y \longrightarrow C[T_*, \infty)$ is continuous in the $C[T_*, \infty)$-topology and satisfies (6.14). Let $\widetilde{c} = (c_1 + c_2)/2$, and define the operator $\mathcal{F}$ by

$$(\mathcal{F}y)(t) = \begin{cases} \widetilde{c} - (-1)^{n-1} \displaystyle\int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} f(s, -\Phi[y](g(s)))ds, \\ \hfill k = 0, \quad t \geq T, \\ \widetilde{c}t^k - (-1)^{n-k-1} \times \\ \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, -\Phi[y](g(r)))drds, \\ \hfill k \neq 0, \quad t \geq T, \\ (\mathcal{F}y)(T), \qquad t \in [T_*, T]. \end{cases}$$

It can be shown that $\mathcal{F}$ is well defined. In fact, from (6.18) we observe that

$$c_2 v_k(t) + \frac{1}{3}c_1 w(t) \leq -\varphi(t) \leq c_1 v_k(t) + c_1 w(t), \quad t \in [T_*, T],$$

so that

$$0 < c_2 v_k(t) + \frac{1}{3} c_1 w(t) \le -\Phi[y](t) \le c_1 v_k(t) + c_1 w(t), \quad t \ge T_*,$$

by Lemma 6.1. By the same arguments as in the proof of Theorem 6.1, the Schauder-Tychonoff fixed point theorem shows that $\widetilde{y} = \mathcal{F}\widetilde{y}$ for some $\widetilde{y} \in Y$. In view of (6.17), we easily see that $x(t) = -\Phi[\widetilde{y}](t)$ is a positive solution of (6.1) and satisfies

$$\frac{1}{3} c_1 w(t) \le c_2 v_k(t) + \frac{1}{3} c_1 w(t) \le x(t) \le c_1 v_k(t) + c_1 w(t) \le w(t)$$

for $t \ge T_*$ and

$$\lim_{t\to\infty} \frac{x(t) - p(t)x(\tau(t))}{t^k} = -\widetilde{c}.$$

The proof is complete.

7. The case $h(t) = 1$

We consider the neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + x(t-\tau)] + f(t, x(g(t))) = q(t), \tag{7.1}$$

where $\tau > 0$, and (1.2), (1.3), (1.6) and (1.8) are assumed to hold.

Theorem 7.1. *Suppose that* (1.7) *and* (4.2) *hold. Let* $k \in \{0, 1, 2, \dots, n-1\}$, *and let* $\omega(t)$ *be a solution of the unperturbed equation*

$$\frac{d^n}{dt^n}[\omega(t) + \omega(t-\tau)] = q(t). \tag{7.2}$$

If

$$\int^{\infty} t^{n-k-1} F(t, |\omega(g(t))| + \varepsilon[g(t)]^k) dt < \infty \quad \textit{for some } \varepsilon > 0,$$

then (7.1) *has a solution* $x(t)$ *satisfying*

$$x(t) = \omega(t) + o(t^k) \quad (t \to \infty). \tag{7.3}$$

Theorem 7.2. *Suppose that (5.2) holds. Let $k \in \{0, 1, 2, \ldots, n-1\}$, and let $\omega(t)$ be a positive solution of the unperturbed equation (7.2) such that*

$$\liminf_{t \to \infty} \frac{\omega(t)}{t^k} > 0.$$

If

$$\int^{\infty} t^{n-k-1} F(t, \omega(g(t)) + \varepsilon [g(t)]^k) dt < \infty \quad \textit{for some } \varepsilon > 0,$$

then (7.1) has a positive solution $x(t)$ satisfying (7.3).

Now consider the equation

$$\frac{d^n}{dt^n}[x(t) + x(t - \tau)] + f(t, x(g(t))) = 0, \tag{7.4}$$

where $\tau > 0$ and (1.2), (1.3) and (1.6) are assumed to hold. Let $\omega_+ \in C[t_0, \infty)$ and $\omega_+(t + \tau) = -\omega_+(t)$ for $t \geq t_0$. Then, for any $c \in \mathbb{R}$, $\omega_+(t) + c$ is a solution of

$$\frac{d^n}{dt^n}[\omega(t) + \omega(t - \tau)] = 0. \tag{7.5}$$

For $c > 0$, the functions $p_k(t) = c\, t^k$ $(k = 0, 1, 2, \ldots, n-1)$ are positive solutions of (7.5). Hence, by Theorems 7.1 and 7.2, we have the following results.

Corollary 7.1. *Assume that (1.7) and (4.2) hold and that*

$$\int^{\infty} t^{n-1} F(t, a) dt < \infty \quad \textit{for some } a > 0. \tag{7.6}$$

Then, for each $\omega_+ \in C[t_0, \infty)$ and $c \in \mathbb{R}$ such that $\omega_+(t + \tau) = -\omega_+(t)$ for $t \geq t_0$ and $\max_t |\omega_+(t)| + |c| < a$, equation (7.4) has a solution $x(t)$ satisfying

$$x(t) = \omega_+(t) + c + o(1) \quad (t \to \infty).$$

Corollary 7.2. *Let $k \in \{0, 1, 2, \dots, n-1\}$. Suppose that* (5.2) *holds and that*

$$\int^{\infty} t^{n-k-1} F(t, a[g(t)]^k)dt < \infty \quad \textit{for some } a > 0.$$

Then (7.4) *has a positive solution $x(t)$ satisfying*

$$\lim_{t\to\infty} \frac{x(t)}{t^k} \quad \textit{exists and is a positive finite value.}$$

Remark 7.1. The solution obtained in Corollary 7.1 is oscillatory or nonoscillatory according to whether the function $\omega_+(t) + c$ is oscillatory or nonoscillatory. Since condition (7.6) is independent of the choice of the function $\omega_+(t) + c$, equation (7.4) possesses both oscillatory solutions and nonoscillatory solutions if (7.6) holds.

We consider the equation

$$\frac{d^n}{dt^n}[x(t) + x(t-\tau)] + \sigma f(t, x(g(t))) = 0, \tag{7.7}$$

where $\sigma = +1$ or -1, $\tau > 0$ and (1.2), (1.3), (1.6) and (5.22) are assumed to hold. From Theorem 5.4 and Corollary 7.2, we obtain the following result.

Corollary 7.3. *Let $k \in \{0, 1, 2, \dots, n-1\}$. Assume that* (5.22) *holds. Then* (7.7) *has a positive solution $x(t)$ satisfying*

$$\lim_{t\to\infty} \frac{x(t)}{t^k} \quad \textit{exists and is a positive finite value}$$

if and only if

$$\int^{\infty} t^{n-k-1} f(t, a[g(t)]^k)dt < \infty \quad \textit{for some } a > 0.$$

Now we prepare the next proposition for the proof of Theorem 7.1.

Proposition 7.1. *Let T and T_* be constants with $T - \tau \geq T_* \geq t_0$. Suppose that $\eta \in C[T-\tau, \infty)$ such that $\eta(t) \geq 0$ for $t \geq T - \tau$ and $\lim_{t\to\infty} \eta(t) = 0$ and define*

$$U = \left\{ u \in C[T_*, \infty) : \left| \sum_{i=1}^{\infty} (-1)^{i+1} u(t + i\tau) \right| \leq \eta(t), \quad t \geq T - \tau \right\},$$

and

$$\Phi[u](t) = \begin{cases} \displaystyle\sum_{i=1}^{\infty}(-1)^{i+1}u(t+i\tau), & t \geq T-\tau, \\ \Phi[u](T-\tau), & t \in [T_*, T-\tau], \end{cases}$$

for each $u \in U$. Then Φ maps U into $C[T_*,\infty)$ and has the following properties:

(i) *the mapping Φ is continuous in the $C[T_*,\infty)$-topology;*

(ii) *for each $u \in U$, Φ satisfies $\Phi[u](t) + \Phi[u](t-\tau) = u(t)$ for $t \geq T$ and $\lim_{t\to\infty} \Phi[u](t) = 0$.*

Proof. It can be shown that Φ is well defined and that $\Phi[u](t)$ is continuous on $[T_*,\infty)$ for each $u \in U$. In fact, if $u \in U$, then

(7.8)
$$\begin{aligned}\sup_{t\in[T-\tau,\infty)}\left|\sum_{i=p+1}^{\infty}(-1)^{i+1}u(t+i\tau)\right| &= \sup_{t\in[T-\tau,\infty)}\left|\sum_{i=1}^{\infty}(-1)^{i+1}u(t+p\tau+i\tau)\right| \\ &\leq \sup_{t\in[T-\tau,\infty)} \eta(t+p\tau) \\ &= \sup_{t\in[T+(p-1)\tau,\infty)} \eta(t), \quad p = 0,1,2,\dots,\end{aligned}$$

which means that the series $\sum_{i=1}^{\infty}(-1)^{i+1}u(t+i\tau)$ converges uniformly on $[T-\tau,\infty)$.

Now we prove that the mapping Φ satisfies (i) and (ii).

(i) For any $\varepsilon > 0$, there is an integer $p \geq 1$ such that

$$\sup_{t\in[T+(p-1)\tau,\infty)} \eta(t) < \frac{\varepsilon}{3}. \tag{7.9}$$

Take an arbitrary compact subinterval I of $[T-\tau,\infty)$. Let $\{u_j\}_{j=1}^{\infty}$ be a sequence in U converging to $u \in U$ uniformly on every compact subinterval of $[T_*,\infty)$. There exists an integer $j_0 \geq 1$ such that

$$\sum_{i=1}^{p}|u_j(t+i\tau) - u(t+i\tau)| < \frac{\varepsilon}{3}, \quad t \in I, \quad j \geq j_0.$$

It follows from (7.8) and (7.9) that

$$
\begin{aligned}
&|\Phi[u_j](t) - \Phi[u](t)| \\
&\le \sum_{i=1}^{p} |u_j(t+i\tau) - u(t+i\tau)| \\
&\quad + \left| \sum_{i=p+1}^{\infty} (-1)^{i+1} u_j(t+i\tau) \right| + \left| \sum_{i=p+1}^{\infty} (-1)^{i+1} u(t+i\tau) \right| \\
&< \varepsilon, \quad t \in I, \quad j \ge j_0,
\end{aligned}
$$

which implies that $\Phi[u_j]$ converges $\Phi[u]$ uniformly on I. In view of the fact that $\Phi[u](t) = \Phi[u](T-\tau)$ for $t \in [T_*, T-\tau]$ and $u \in U$, we conclude that Φ is continuous on U.

(ii) Let $u \in U$. Observe that

$$
\begin{aligned}
\text{(7.10)} \qquad &\Phi[u](t) + \Phi[u](t-\tau) \\
&= \sum_{i=1}^{\infty} (-1)^{i+1} u(t+i\tau) + \sum_{i=1}^{\infty} (-1)^{i+1} u(t+(i-1)\tau) \\
&= \sum_{i=1}^{\infty} (-1)^{i+1} u(t+i\tau) - \sum_{i=0}^{\infty} (-1)^{i+1} u(t+i\tau) \\
&= u(t) \quad \text{for } t \ge T,
\end{aligned}
$$

and that

$$\lim_{t\to\infty} |\Phi[u](t)| \le \lim_{t\to\infty} \eta(t) = 0.$$

Thus, Φ satisfies (ii).

Proof of Theorem 7.1. We can take a number T such that

$$T_* \equiv \min\{T-\tau,\ \inf\{g(t) : t \ge T\}\} \ge t_0,$$

$\omega(t)$ is continuous and satisfies (7.2) on $[T_*, \infty)$, and

$$\text{(7.11)} \qquad \int_T^\infty s^{n-k-1} F(s) ds < \varepsilon,$$

where $F(t) = F(t, |\omega(g(t))| + \varepsilon[g(t)]^k)$. Let

$$G(t) = \begin{cases} \displaystyle\int_t^\infty \frac{(s-t)^{n-k-2}}{(n-k-2)!} F(s)ds, & k \neq n-1, \\ F(t), & k = n-1, \end{cases}$$

for $t \geq T$. Notice that

$$\int_t^\infty G(s)ds = \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} F(s)ds, \quad t \geq T. \tag{7.12}$$

We consider the set Y of all functions $y \in C[T_*, \infty)$ such that

$$y(t) = y(T) \quad \text{for } t \in [T_*, T], \quad |y(t)| \leq \int_t^\infty G(s)ds \quad \text{for } t \geq T$$

and

$$|y(t+\tau) - y(t)| \leq \int_t^{t+\tau} G(s)ds \quad \text{for } t \geq T.$$

Obviously, Y is a closed convex subset of $C[T_*, \infty)$.

Now we claim that if $y \in Y$, then

$$\left| \sum_{i=1}^m (-1)^{i+1} y(t+i\tau) \right| \leq \int_{t+\tau}^\infty G(s)ds, \quad t \geq T - \tau, \tag{7.13}$$

for $m = 1, 2, \ldots$. We see that if m is odd, then

$$\begin{aligned}
&\left| \sum_{i=1}^m (-1)^{i+1} y(t+i\tau) \right| \\
&= \left| \sum_{j=1}^{(m-1)/2} [y(t+(2j-1)\tau) - y(t+2j\tau)] + y(t+m\tau) \right| \\
&\leq \sum_{j=1}^{(m-1)/2} \int_{t+(2j-1)\tau}^{t+2j\tau} G(s)ds + \int_{t+m\tau}^\infty G(s)ds \\
&\leq \int_{t+\tau}^\infty G(s)ds, \quad t \geq T - \tau.
\end{aligned}$$

For the case where m is even, using the equality

$$\sum_{i=1}^m (-1)^{i+1} y(t+i\tau) = \sum_{j=1}^{m/2} [y(t+(2j-1)\tau) - y(t+2j\tau)], \quad t \geq T - \tau,$$

we can conclude (7.13).

According to (7.13), if $m \geq p \geq 1$ and $t \in [T-\tau,\infty)$, then

$$\begin{aligned}\left|\sum_{i=p}^{m}(-1)^{i+1}y(t+i\tau)\right| &= \left|\sum_{i=1}^{m-p+1}(-1)^{i+p}y(t+(i+p-1)\tau)\right| \\ &= \left|\sum_{i=1}^{m-p+1}(-1)^{i+1}y(t+(p-1)\tau+i\tau)\right| \\ &\leq \int_{t+p\tau}^{\infty}G(s)ds \to 0 \quad \text{as } p\to\infty.\end{aligned}$$

Hence, the series $\sum_{i=1}^{\infty}(-1)^{i+1}y(t+i\tau)$ converges for each fixed $t \in [T-\tau,\infty)$. Letting $m\to\infty$ in (7.13), we obtain

$$(7.14) \qquad \left|\sum_{i=1}^{\infty}(-1)^{i+1}y(t+i\tau)\right| \leq \int_{t+\tau}^{\infty}G(s)ds, \quad t \geq T-\tau,$$

for each $y \in Y$. By using Proposition 7.1 and by taking account of (7.11), (7.12) and (7.14), there exists a continuous mapping $\Phi : Y \longrightarrow C[T_*,\infty)$ such that

$$\Phi[y](t)+\Phi[y](t-\tau)=y(t), \quad t \geq T,$$

$$(7.15) \qquad \lim_{t\to\infty}\Phi[y](t)=0 \quad \text{and} \quad |\Phi[y](t)| \leq \varepsilon, \quad t \geq T_*,$$

for each $y \in Y$. For $y \in Y$, we set

$$\Phi_k[y](t)=\begin{cases}\Phi[y](t), & k=0,\\ \displaystyle\int_{T_*}^{t}\frac{(t-s)^{k-1}}{(k-1)!}\Phi[y](s)ds, & k\neq 1,\end{cases}$$

and

$$\Omega_k[y](t)=\omega(t)+(-1)^{n-k-1}\Phi_k[y](t).$$

Since

$$\frac{d^k}{dt^k}\Phi_k[y](t)=\Phi[y](t)\to 0 \quad \text{as } t\to\infty,$$

we find that

$$(7.16) \qquad \Omega_k[y](t)=\omega(t)+o(t^k) \quad (t\to\infty), \quad y \in Y.$$

From the second half of (7.15) we have $|\Phi_k[y](t)| \leq \varepsilon t^k$ for $t \geq T_*$.

We define the mapping $\mathcal{F} : Y \longrightarrow C[T_*, \infty)$ as follows:

$$(\mathcal{F}y)(t) = \begin{cases} \displaystyle\int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, \Omega_k[y](g(s)))ds, & t \geq T, \\ (\mathcal{F}y)(T), & t \in [T_*, T]. \end{cases}$$

Since $|\Omega_k[y](t)| \leq |\omega(t)| + \varepsilon t^k$ for $t \geq T_*$ and $y \in Y$, the mapping $\mathcal{F}$ is well defined. We find that $\mathcal{F}(Y) \subset Y$. In fact, if $t \geq T$ and $y \in Y$, then

$$|(\mathcal{F}y)(t)| \leq \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)} F(s)ds = \int_t^\infty G(s)ds,$$

and

$$\begin{aligned} |(\mathcal{F}y)(t+\tau) - (\mathcal{F}y)(t)| &= \left| \int_t^{t+\tau} f(s, \Omega_{n-1}[y](g(s)))ds \right| \\ &\leq \int_t^{t+\tau} F(s)ds = \int_t^{t+\tau} G(s)ds \end{aligned}$$

for the case $k = n-1$, and

$$\begin{aligned} &|(\mathcal{F}y)(t+\tau) - (\mathcal{F}y)(t)| \\ = &\left| \int_t^{t+\tau} \int_s^\infty \frac{(r-s)^{n-k-2}}{(n-k-2)!} f(r, \Omega_k[y](g(r)))drds \right| \\ \leq &\int_t^{t+\tau} \int_s^\infty \frac{(r-s)^{n-k-2}}{(n-k-2)!} F(r)drds = \int_t^{t+\tau} G(s)ds \end{aligned}$$

for the case $k \neq n-1$. By the same arguments as in the proofs of Theorems 4.1 and 4.2, we see that $\mathcal{F}$ is continuous on Y and $\mathcal{F}(Y)$ is relatively compact. Consequently, we are able to apply the Schauder-Tychonoff fixed point theorem to the operator $\mathcal{F}$ and conclude that there exists a $\widetilde{y} \in Y$ such that $\widetilde{y} = \mathcal{F}\widetilde{y}$. Set $x(t) = \Omega_k[\widetilde{y}](t)$. From (7.16) it follows that $x(t)$ satisfies $x(t) = \omega(t) + o(t^k)$ $(t \to \infty)$. We see that

$$(7.17)\quad \frac{d^k}{dt^k}[x(t)+x(t-\tau)]$$
$$= \frac{d^k}{dt^k}[\omega(t)+\omega(t-\tau)+(-1)^{n-k-1}\{\Phi_k[\widetilde{y}](t)+\Phi_k[\widetilde{y}](t-\tau)\}]$$
$$= \frac{d^k}{dt^k}[\omega(t)+\omega(t-\tau)]+(-1)^{n-k-1}[\Phi[\widetilde{y}](t)+\Phi[\widetilde{y}](t-\tau)]$$
$$= \frac{d^k}{dt^k}[\omega(t)+\omega(t-\tau)]+(-1)^{n-k-1}\widetilde{y}(t)$$
$$= \frac{d^k}{dt^k}[\omega(t)+\omega(t-\tau)]$$
$$+(-1)^{n-k-1}\int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!}f(s,x(g(s)))ds, \quad t \ge T,$$

so that

$$\frac{d^n}{dt^n}[x(t)+x(t-\tau)] = q(t) - f(t,x(g(t))), \quad t \ge T.$$

This implies that $x(t)$ is a solution of (7.1). The proof is complete.

Proof of Theorem 7.2. We may assume that $\varepsilon > 0$ so small that $\omega(t) > \varepsilon t^k$ for $t \ge T_0$ for some $T_0 \ge t_0$. Hence, the conclusion follows from the proof of Theorem 7.1 with $t_0 = T_0$, since

$$0 < \omega(t) - \varepsilon t^k \le \Omega_k[y](t) \le \omega(t) + \varepsilon t^k, \quad t \ge T_*.$$

8. The case $h(t) = -1$

In this section we consider neutral differential equation of the form

$$(8.1)\qquad \frac{d^n}{dt^n}[x(t)-x(t-\tau)] + f(t,x(g(t))) = q(t),$$

where $\tau > 0$, and (1.2), (1.3), (1.6) and (1.8) are assumed to hold.

Theorem 8.1. *Suppose that* (1.7) *and* (4.2) *hold. Let* $k \in \{0,1,2,\dots,n\}$*, and let* $\omega(t)$ *be a solution of the unperturbed equation*

$$(8.2)\qquad \frac{d^n}{dt^n}[\omega(t)-\omega(t-\tau)] = q(t).$$

Then (8.1) *has a solution* $x(t)$ *satisfying*

$$x(t) = \omega(t) + o(t^k) \quad (t \to \infty) \tag{8.3}$$

if

$$\int^{\infty} t^{n-k} F(t, |\omega(g(t))| + \varepsilon[g(t)]^k)dt < \infty \quad \textit{for some } \varepsilon > 0.$$

Theorem 8.2. *Suppose that* (5.2) *holds. Let* $k \in \{0, 1, 2, \ldots, n\}$, *and let* $\omega(t)$ *be a positive solution of the unperturbed equation* (8.2) *such that*

$$\liminf_{t \to \infty} \frac{\omega(t)}{t^k} > 0.$$

If

$$\int^{\infty} t^{n-k} F(t, \omega(g(t)) + \varepsilon[g(t)]^k)dt < \infty \quad \textit{for some } \varepsilon > 0,$$

then (8.1) *has a positive solution* $x(t)$ *satisfying* (8.3).

Before giving the proofs of Theorems 8.1 and 8.2, let us consider the unforced equation

$$\frac{d^n}{dt^n}[x(t) - x(t-\tau)] + f(t, x(g(t))) = 0, \tag{8.4}$$

where $\tau > 0$ and (1.2), (1.3) and (1.6) are assumed to hold. Obviously, continuous τ-periodic functions are solutions of the unperturbed equation

$$\frac{d^n}{dt^n}[\omega(t) - \omega(t-\tau)] = 0. \tag{8.5}$$

The functions $p_k(t) = c\,t^k$ $(k = 0, 1, 2, \ldots, n)$ are positive solutions of (8.5), where $c > 0$. From Theorems 8.1 and 8.2 we obtain the following corollaries.

Corollary 8.1. *Assume that* (1.7) *and* (4.2) *hold and that*

$$\int^{\infty} t^n F(t, a)dt < \infty \quad \textit{for some } a > 0. \tag{8.6}$$

Then, for each continuous τ-periodic function $\omega_-(t)$ such that $\max_t |\omega_-(t)| < a$, equation (8.4) *has a solution $x(t)$ satisfying*

$$x(t) = \omega_-(t) + o(1) \quad (t \to \infty).$$

Corollary 8.2. *Let $k \in \{0, 1, 2, \ldots, n\}$. Suppose that* (5.2) *holds and that*

$$\int^\infty t^{n-k} F(t, a[g(t)]^k) dt < \infty \quad \textit{for some } a > 0.$$

Then (8.4) *possesses a positive solution $x(t)$ satisfying*

$$\lim_{t\to\infty} \frac{x(t)}{t^k} \quad \textit{exists and is a positive finite value.} \tag{8.7}$$

Remark 8.1. The solution obtained in Corollary 8.1 is oscillatory or nonoscillatory according to whether the function $\omega_-(t)$ is oscillatory or nonoscillatory. Since condition (8.6) do not depend on the choice of the function of $\omega_-(t)$, equation (8.4) has both oscillatory solutions and nonoscillatory solutions if (8.6) holds.

We consider neutral differential equations of the form

$$\frac{d^n}{dt^n}[x(t) - x(t - \tau)] + \sigma f(t, x(g(t))) = 0, \tag{8.8}$$

where $\sigma = +1$ or -1, $\tau > 0$ and (1.2), (1.3), (1.6) and (5.22) are assumed to hold.

Theorem 8.3. *Let $k \in \{0, 1, 2, \ldots, n\}$. Suppose that* (5.22) *holds. Then* (8.8) *has a positive solution $x(t)$ satisfying* (8.7) *if and only if*

$$\int^\infty t^{n-k} f(t, a[g(t)]^k) dt < \infty \quad \textit{for some } a > 0. \tag{8.9}$$

Proof. The "if" part follows from Corollary 8.2 immediately. We prove the "only if" part.

Let $x(t)$ be a solution of (8.8) satisfying (8.7). Put $y(t) = x(t) - x(t - \tau)$. From (8.8) and (5.22) it follows that

$$\sigma y^{(n)}(t) = -f(t, x(g(t))) \leq 0 \quad \text{for all large } t. \tag{8.10}$$

Then we see that $y^{(i)}(t)$ $(i = 0, 1, 2, \dots, n-1)$ are eventually monotonic, so that $\lim_{t\to\infty} y^{(i)}(t)$ $(i = 0, 1, 2, \dots, n-1)$ exist in $\mathbb{R}\cup\{-\infty, \infty\}$. We observe that

$$\lim_{t\to\infty} \frac{y(t)}{t^k} = \lim_{t\to\infty} \left[\frac{x(t)}{t^k} - \frac{(t-\tau)^k}{t^k}\frac{x(t-\tau)}{(t-\tau)^k}\right] = 0.$$

Therefore, if $k \neq n$, then $\lim_{t\to\infty} y^{(i)}(t) = 0$ for $i = k, k+1, \dots, n-1$. Consequently, if $k \neq n$, then repeated integration of (8.10) yields

$$(8.11) \qquad y^{(k)}(t) = (-1)^{n-k-1}\sigma \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, x(g(s)))ds$$

for all large t.

We first assume that $y(t)$ is eventually nondecreasing. Let $M_* = \min x(t)$ on $[T-\tau, T]$ and $M^* = \max x(t)$ on $[T-\tau, T]$, where T is sufficiently large. By virtue of Lemma 2.1, we have

$$x(t) = \sum_{i=0}^{m} y(t-i\tau) + x(t-(m+1)\tau), \quad t \in [T+m\tau, T+(m+1)\tau],$$

$$m = 0, 1, 2, \dots .$$

Since $y(t)$ is nondecreasing, we obtain

$$\frac{1}{\tau}\int_{t-\tau}^{t} y(s)ds \le y(t) \le \frac{1}{\tau}\int_{t}^{t+\tau} y(s)ds.$$

We find that

$$\begin{aligned} x(t) &\ge \sum_{i=0}^{m} \frac{1}{\tau}\int_{t-(i+1)\tau}^{t-i\tau} y(s)ds + M_* \\ &= \frac{1}{\tau}\int_{t-(m+1)\tau}^{t} y(s)ds + M_* \\ &= \frac{1}{\tau}\int_{T}^{t} y(s)ds + \frac{1}{\tau}\int_{t-(m+1)\tau}^{T} y(s)ds + M_* \end{aligned}$$

for $t \in [T+m\tau, T+(m+1)\tau]$, $m = 0, 1, 2, \dots$, so that

$$x(t) \ge \frac{1}{\tau}\int_{T}^{t} y(s)ds + K_*, \quad t \ge T,$$

where

$$K_* = M_* + \frac{1}{\tau} \min_{s \in [T-\tau, T]} \int_s^T y(u)du.$$

Likewise, it can be shown that

$$x(t) \le \frac{1}{\tau} \int_T^{t+\tau} y(s)ds + K^*, \quad t \ge T,$$

where

$$K^* = M^* + \frac{1}{\tau} \max_{s \in [T, T+\tau]} \int_s^T y(u)du.$$

Then we conclude that

$$\tau[x(t-\tau) - K^*] \le \int_T^t y(s)ds \le \tau[x(t) - K_*], \quad t \ge T + \tau.$$

If $k = 0$, since $x(t)$ is bounded and $y(t)$ is positive or negative, we see that $\int_T^t y(s)ds$ is bounded and monotone on $[T, \infty)$, so that $y(t)$ is integrable on $[T, \infty)$. If $k \ne 0$, then

$$\begin{aligned}\lim_{t\to\infty} y^{(k-1)}(t) &= \lim_{t\to\infty} \frac{k!}{t^k} \int_T^t y(s)ds \\ &\le \lim_{t\to\infty} \frac{k!\tau[x(t) - K_*]}{t^k} = \tau k! \lim_{t\to\infty} \frac{x(t)}{t^k},\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty} y^{(k-1)}(t) &= \lim_{t\to\infty} \frac{k!}{(t+\tau)^k} \int_T^{t+\tau} y(s)ds \\ &\ge \lim_{t\to\infty} \frac{k!\tau[x(t) - K^*]}{(t+\tau)^k} = \tau k! \lim_{t\to\infty} \frac{x(t)}{t^k}.\end{aligned}$$

This shows that

$$\lim_{t\to\infty} y^{(k-1)}(t) = \tau k! \lim_{t\to\infty} \frac{x(t)}{t^k} \quad \text{if } k \ne 0.$$

Hence we conclude that $y^{(k)}(t)$ is integrable on $[T, \infty)$ for $k = 0, 1, 2, \dots, n$. Integrating (8.10) or (8.11) over $[T, \infty)$, we obtain

$$\int_T^\infty \frac{(s-T)^{n-k}}{(n-k)!} f(s, x(g(s)))ds < \infty,$$

which implies that (8.9) holds. In the same way, we can prove that (8.9) holds for the case where $y(t)$ is eventually nonincreasing. The proof is complete.

Now we make preparations for the proofs of Theorems 8.1 and 8.2.

Let T and T_* be constants with $T - \tau \geq T_* \geq t_0$. We denote by $S[T_*, \infty)$ the set of all functions $u \in C[T_*, \infty)$ such that the series

$$\sum_{i=1}^{\infty} |u(t + i\tau)|, \quad t \geq T - \tau, \tag{8.12}$$

converges uniformly on $[T - \tau, \infty)$. For each $u \in S[T_*, \infty)$, we define the function $\Psi[u]$ on $[T_*, \infty)$ by

$$\Psi[u](t) = \begin{cases} -\displaystyle\sum_{i=1}^{\infty} u(t + i\tau), & t \geq T - \tau, \\ \Psi[u](T - \tau), & t \in [T_*, T - \tau]. \end{cases}$$

Obviously, $\Psi[u](t)$ is continuous on $[T_*, \infty)$ for each $u \in S[T_*, \infty)$, and satisfies

$$\Psi[u](t) - \Psi[u](t - \tau) = u(t), \quad t \geq T, \quad u \in S[T_*, \infty). \tag{8.13}$$

Lemma 8.1. *Let $u \in C[T_*, \infty)$. Suppose that the series (8.12) converges for each fixed $t \in [T - \tau, \infty)$ and*

$$\lim_{t \to \infty} \sum_{i=1}^{\infty} |u(t + i\tau)| = 0.$$

Then $u \in S[T_*, \infty)$.

Proof. Put $\psi(t) = \sum_{i=1}^{\infty} |u(t + i\tau)|$. Let $m \in \mathbb{N}$. We find that

$$\begin{aligned} \sum_{i=m}^{\infty} |u(t + i\tau)| &= \sum_{i=1}^{\infty} |u(t + (m-1)\tau + i\tau)| \\ &= \psi(t + (m-1)\tau), \quad t \geq T - \tau, \end{aligned}$$

so that

$$\sup_{t\in[T-\tau,\infty)} \sum_{i=m}^{\infty} |u(t+i\tau)| = \sup_{t\in[T-\tau,\infty)} \psi(t+(m-1)\tau)$$

$$= \sup_{t\in[T+(m-2)\tau,\infty)} \psi(t) \to 0 \quad \text{as } m\to\infty.$$

This means that $u \in S[T_*,\infty)$.

Lemma 8.2. *Suppose that $\eta \in S[T_*,\infty)$ such that $\eta(t) \geq 0$ for $t \geq T$ and define*

$$U = \{u \in C[T_*,\infty) : |u(t)| \leq \eta(t), \quad t \geq T\}. \tag{8.14}$$

Then we have

(i) $U \subset S[T_*,\infty)$;

(ii) Ψ *is continuous on U in the $C[T_*,\infty)$-topology.*

Proof. It is clear that (i) holds. We prove that (ii) holds.

For every $\varepsilon > 0$, there is an integer $p \geq 1$ such that

$$\sum_{i=p+1}^{\infty} \eta(t+i\tau) < \frac{\varepsilon}{3}, \quad t \geq T-\tau.$$

Take an arbitrary compact subinterval I of $[T-\tau,\infty)$. Let $\{u_j\}_{j=1}^{\infty}$ be a sequence in U converging to $u \in U$ uniformly on every compact subinterval of $[T_*,\infty)$. There exists an integer $j_0 \geq 1$ such that

$$\sum_{i=1}^{p} |u_j(t+i\tau) - u(t+i\tau)| < \frac{\varepsilon}{3}, \quad t \in I, \quad j \geq j_0.$$

We see that

$$\begin{aligned}
&|\Psi[u_j](t) - \Psi[u](t)| \\
&\leq \sum_{i=1}^{p} |u_j(t+i\tau) - u(t+i\tau)| + \sum_{i=p+1}^{\infty} |u_j(t+i\tau)| + \sum_{i=p+1}^{\infty} |u(t+i\tau)| \\
&< \frac{\varepsilon}{3} + 2\sum_{i=p+1}^{\infty} \eta(t+i\tau) < \varepsilon, \quad t \in I, \quad j \geq j_0,
\end{aligned}$$

so that $\Psi[u_j]$ converges $\Psi[u]$ uniformly on I. For $t \in [T_*, T-\tau]$, we have $|\Psi[u_j](t) - \Psi[u](t)| = |\Psi[u_j](T-\tau) - \Psi[u](T-\tau)|$. Therefore, we can conclude that Ψ is continuous on U.

Lemma 8.3. *Let $p \in \mathbb{N} \cup \{0\}$. Suppose that $G \in C[T, \infty)$, $G(t) \geq 0$ for $t \geq T$ and*

$$\int_T^\infty t^{p+1} G(t) dt < \infty.$$

If $u \in C[T_*, \infty)$ satisfies

$$|u(t)| \leq \int_t^\infty (s-t)^p G(s) ds, \quad t \geq T,$$

then $u \in S[T_*, \infty)$ and

$$|\Psi[u](t)| \leq \frac{1}{\tau} \int_{t+\tau}^\infty s^{p+1} G(s) ds, \quad t \geq T - \tau.$$

Proof. We observe that

$$\begin{aligned}
\sum_{i=1}^\infty |u(t+i\tau)| &\leq \sum_{i=1}^\infty \int_{t+i\tau}^\infty (s-t-i\tau)^p G(s) ds \\
&\leq \sum_{i=1}^\infty \sum_{j=i}^\infty \int_{t+j\tau}^{t+(j+1)\tau} (s-t)^p G(s) ds \\
&= \sum_{j=1}^\infty j \int_{t+j\tau}^{t+(j+1)\tau} (s-t)^p G(s) ds, \quad t \geq T - \tau.
\end{aligned}$$

If $s \in [t+j\tau, t+(j+1)\tau]$, then $[(s-t)/\tau] - 1 \leq j \leq (s-t)/\tau$. Hence we have

$$\begin{aligned}
\sum_{i=1}^\infty |u(t+i\tau)| &\leq \sum_{j=1}^\infty \frac{1}{\tau} \int_{t+j\tau}^{t+(j+1)\tau} (s-t)^{p+1} G(s) ds \\
&= \frac{1}{\tau} \int_{t+\tau}^\infty (s-t)^{p+1} G(s) ds \\
&\leq \frac{1}{\tau} \int_{t+\tau}^\infty s^{p+1} G(s) ds, \quad t \geq T - \tau.
\end{aligned}$$

Then Lemma 8.1 implies that $u \in S[T_*.\infty)$. The proof is complete.

We are ready to prove Theorem 8.1 for the case $k \neq n$.

Proof of Theorem 8.1 ($k \neq n$). Let $k \in \{0, 1, 2, \ldots, n-1\}$. Take a number T such that

$$T_* \equiv \min\{T - \tau,\ \inf\{g(t) : t \geq T\}\} \geq t_0,$$

$\omega(t)$ is continuous and satisfies (8.2) on $[T_*, \infty)$, and

$$\frac{1}{\tau}\int_T^\infty s^{n-k} F(s)ds < \varepsilon,$$

where $F(t) = F(t, |\omega(g(t))| + \varepsilon[g(t)]^k)$. Consider the set Y of all functions $y \in C[T_*, \infty)$ such that

$$|y(t)| \leq \begin{cases} \displaystyle\int_t^\infty (s-t)^{n-k-1} F(s)ds, & t \geq T, \\ \displaystyle\int_T^\infty (s-T)^{n-k-1} F(s)ds, & t \in [T_*, T]. \end{cases}$$

Then Y is a closed convex subset of $C[T_*, \infty)$. Lemma 8.3 applied to the case $p = n-k-1$ and $G(t) = F(t)$ implies that $Y \subset S[T_*, \infty)$ and

$$|\Psi[y](t)| \leq \frac{1}{\tau}\int_{t+\tau}^\infty s^{n-k} F(s)ds < \varepsilon, \quad t \geq T - \tau, \quad y \in Y,$$

and hence $\lim_{t\to\infty} \Psi[y](t) = 0$ for each $y \in Y$.

For each $y \in Y$, we assign the functions $\Psi_k[y](t)$ and $\Omega_k[y](t)$ on $[T_*, \infty)$ by

$$\Psi_k[y](t) = \begin{cases} \Psi[y](t), & k = 0, \\ \displaystyle\int_{T_*}^t \frac{(t-s)^{k-1}}{(k-1)!}\Psi[y](s)ds, & k = 1, 2, \ldots, n-1, \end{cases}$$

and

$$\Omega_k[y](t) = \omega(t) + (-1)^{n-k-1}\Psi_k[y](t).$$

Then we find that (7.16) holds. In view of the fact that $|\Psi[y](t)| \leq \varepsilon$ for $t \geq T_*$, we obtain $|\Psi_k[y](t)| \leq \varepsilon t^k$ for $t \geq T_*$.

We define the mapping $\mathcal{F} : Y \longrightarrow C[T_*, \infty)$ by

$$(\mathcal{F}y)(t) = \begin{cases} \displaystyle\int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, \Omega_k[y](g(s)))ds, & t \geq T, \\ (\mathcal{F}y)(T), & t \in [T_*, T]. \end{cases}$$

By the same arguments as in the proof of Theorem 7.1, we conclude that $\mathcal{F}$ is well-defined and continuous and maps Y into itself, and that $\mathcal{F}(Y)$ is relatively compact. The Schauder-Tychonoff fixed point theorem shows that $\widetilde{y} = \mathcal{F}\widetilde{y}$ for some $\widetilde{y} \in Y$. Set $x(t) = \Omega_k[\widetilde{y}](t)$. By (7.16), $x(t)$ satisfies $x(t) = \omega(t) + o(t^k)$ $(t \to \infty)$. In a similar way to (7.17), we see that $x(t)$ is a solution of (8.1). This completes the proof.

We need a further preparation for the proof of Theorem 8.1 in the extreme case $k = n$.

Let T and T_* be constants such that $T - \tau \geq T_* \geq t_0$. For each $u \in C[T_*, \infty)$, we define the function $\Phi[u](t)$ on $[T_*, \infty)$ as follows:

$$\Phi[u](t) = \begin{cases} \displaystyle\sum_{i=0}^{m} u(t - i\tau) + \frac{u(T)}{\tau}(t - m\tau - T), & \\ \qquad\qquad t \in [T + m\tau, T + (m+1)\tau], \quad m = 0, 1, \dots, & \\ \displaystyle\frac{u(T)}{\tau}(t - T + \tau), & t \in [T - \tau, T], \\ 0, & t \in [T_*, T - \tau]. \end{cases}$$

By the same arguments as in the proof of Proposition 3.1, we find that Φ has the following properties:

(i) Φ maps $C[T_*, \infty)$ into itself and is continuous in the $C[T_*, \infty)$-topology;

(ii) for each $u \in C[T_*, \infty)$, Φ satisfies $\Phi[u](t) - \Phi[u](t - \tau) = u(t)$ for $t \geq T$.

Lemma 8.4. *Let $T > 2\tau$ and $T - \tau \geq T_* \geq t_0$. Suppose that $G \in C[T, \infty)$, $G(t) \geq 0$ for $t \geq T$ and*

$$\int_T^\infty G(s)ds < \infty.$$

Assume moreover that $u \in C[T_*, \infty)$ and

$$|u(t)| \leq \int_t^\infty G(s)ds, \quad t \geq T.$$

Then $\Phi[u](t) = o(t)$ $(t \to \infty)$ *and*

$$|\Phi[u](t)| \le \frac{t}{\tau}\int_T^\infty G(s)ds, \quad t \ge T_*.$$

Proof. If $t \in [T-\tau, T]$, we see that

$$\frac{t}{\tau} \ge \frac{T-\tau}{\tau} > \frac{2\tau-\tau}{\tau} = 1,$$

so that

$$|\Phi[u](t)| \le \frac{|u(T)|}{\tau}(t - T + \tau) \le |u(T)| \le \frac{t}{\tau}\int_T^\infty G(s)ds.$$

Thus we conclude that

$$|\Phi[u](t)| \le \frac{t}{\tau}\int_T^\infty G(s)ds, \quad t \in [T_*, T],$$

because of $\Phi[u](t) = 0$ for $t \in [T_*, T-\tau]$.

Let $t \in [T + m\tau, T + (m+1)\tau]$, $m = 0, 1, \ldots$. We observe that

$$\begin{aligned}
|\Phi[u](t)| &\le \sum_{i=0}^m |u(t - i\tau)| + \frac{|u(T)|}{\tau}(t - m\tau - T) \\
&\le \sum_{i=0}^m \int_{t-i\tau}^\infty G(s)ds + \int_T^\infty G(s)ds \\
&= (m+1)\int_t^\infty G(s)ds + \sum_{i=1}^m \int_{t-i\tau}^t G(s)ds + \int_T^\infty G(s)ds,
\end{aligned}$$

and that

$$\begin{aligned}
\sum_{i=1}^m \int_{t-i\tau}^t G(s)ds &= \sum_{i=1}^m \sum_{j=1}^i \int_{t-j\tau}^{t-(j-1)\tau} G(s)ds \\
&= \sum_{j=1}^m (m+1-j)\int_{t-j\tau}^{t-(j-1)\tau} G(s)ds \\
&\le \sum_{j=1}^m \int_{t-j\tau}^{t-(j-1)\tau} [m + 1 + (s-t)/\tau]G(s)ds \\
&= (m+1)\int_{t-m\tau}^t G(s)ds + \frac{1}{\tau}\int_{t-m\tau}^t (s-t)G(s)ds,
\end{aligned}$$

since if $s \in [t - j\tau, t - (j-1)\tau]$, then $[(s-t)/\tau] - 1 \le -j \le (s-t)/\tau$. Hence we obtain

$$|\Phi[u](t)|$$
$$\le (m+1)\int_{t-m\tau}^{\infty} G(s)ds + \frac{1}{\tau}\int_{t-m\tau}^{t}(s-t)G(s)ds + \int_{T}^{\infty} G(s)ds.$$

Since $t - m\tau \ge T$ and $m \le (t-T)/\tau$, we have

(8.15)

$$\begin{aligned}|\Phi[u](t)| \le\ & \left(\frac{t-T}{\tau}+1\right)\int_T^{\infty} G(s)ds + \frac{1}{\tau}\int_T^t (s-t)G(s)ds \\ & + \int_T^{\infty} G(s)ds \\ =\ & \left(2-\frac{T}{\tau}\right)\int_T^{\infty} G(s)ds + \frac{t}{\tau}\int_t^{\infty} G(s)ds + \frac{1}{\tau}\int_T^t s\,G(s)ds \\ \le\ & \frac{t}{\tau}\int_t^{\infty} G(s)ds + \frac{1}{\tau}\int_T^t s\,G(s)ds \equiv \varphi(t).\end{aligned}$$

Then

$$\varphi(t) \le \frac{t}{\tau}\int_t^{\infty} G(s)ds + \frac{t}{\tau}\int_T^t G(s)ds = \frac{t}{\tau}\int_T^{\infty} G(s)ds.$$

Further, from

$$\varphi'(t) = \frac{1}{\tau}\int_t^{\infty} G(s)ds \to 0 \quad \text{as } t \to \infty$$

it follows that $\Phi[u](t) = o(t)$ as $t \to \infty$. This completes the proof.

Proof of Theorem 8.1 ($k = n$). We can take a number $T > 2\tau$ such that

$$T_* \equiv \min\{T - \tau,\ \inf\{g(t) : t \ge T\}\} \ge t_0,$$

$\omega(t)$ is continuous and satisfies (8.2) on $[T_*, \infty)$, and

$$\frac{1}{\tau}\int_T^{\infty} F(s)ds \le \varepsilon,$$

where $F(t) = F(t, |\omega(g(t))| + \varepsilon[g(t)]^n)$. We consider the set Y of all functions $y \in C[T_*, \infty)$ satisfying

$$y(t) = y(T) \quad \text{for } t \in [T_*, T]$$

and

$$|y(t)| \leq \int_t^\infty F(s)ds \quad \text{for } t \geq T.$$

For each $y \in Y$, we set

$$\Phi_n[y](t) = \begin{cases} \Phi[y](t), & n = 1, \\ \displaystyle\int_{T_*}^t \frac{(t-s)^{n-2}}{(n-2)!} \Phi[y](s)ds, & n \geq 2, \end{cases}$$

and

$$\Omega_n[y](t) = \omega(t) + \Phi_n[y](t).$$

In view of Lemma 8.4, we find that

$$\Omega_n[y](t) = \omega(t) + o(t^n) \quad (t \to \infty),$$

and

$$|\Omega_n[y](t)| \leq |\omega(t)| + \varepsilon t^n, \quad t \geq T_*,$$

for each $y \in Y$.

Define the mapping $\mathcal{F} : Y \longrightarrow C[T_*, \infty)$ by

$$(\mathcal{F}y)(t) = \begin{cases} \displaystyle\int_t^\infty f(s, \Omega_n[y](g(s)))ds, & t \geq T, \\ (\mathcal{F}y)(T), & t \in [T_*, T]. \end{cases}$$

Using the same argument as in the proof of Theorem 8.1 for the case $k \neq n$, we conclude that $\mathcal{F}\widetilde{y} = \widetilde{y}$ for some $\widetilde{y} \in Y$, and that $x(t) = \Omega_n[\widetilde{y}](t)$ is a solution of (8.1) satisfying $x(t) = \omega(t) + o(t^n)$ $(t \to \infty)$. The proof is complete.

Proof of Theorem 8.2. By the same arguments as in the proof of Theorem 7.2, we can show Theorem 8.2.

9. The case $h(t) = \lambda$ with $|\lambda| \neq 1$

In this section we consider the neutral differential equations

$$(9.1)_+ \qquad \frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + f(t, x(g(t))) = 0,$$

and

$$(9.1)_- \qquad \frac{d^n}{dt^n}[x(t) - \lambda x(t-\tau)] + f(t, x(g(t))) = 0,$$

where $\lambda > 0$, $\lambda \neq 1$, $\tau > 0$, and (1.2), (1.3), (1.6), (1.7) and (4.2) are assumed to hold.

Let ω_+ and $\omega_- \in C[T, \infty)$ be functions satisfying $\omega_+(t+\tau) = -\omega_+(t)$ and $\omega_-(t+\tau) = \omega_-(t)$, respectively, for $t \geq T$. We easily see that $\lambda^{t/\tau}\omega_+(t)$ and $\lambda^{t/\tau}\omega_-(t)$ are solutions of the unperturbed equations

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] = 0 \quad \text{and} \quad \frac{d^n}{dt^n}[x(t) - \lambda x(t-\tau)] = 0,$$

respectively. Thus it is natural to expect that, if f is small enough in some sense, equation $(9.1)_+$ [resp. $(9.1)_-$] possesses a solution $x(t)$ behaving like the function $\lambda^{t/\tau}\omega_+(t)$ [resp. $\lambda^{t/\tau}\omega_-(t)$] as $t \to \infty$. Indeed, we have the following theorem.

Theorem 9.1. *Suppose that*

$$(9.2) \qquad \int^{\infty} \lambda^{-t/\tau} F(t, a\lambda^{g(t)/\tau})dt < \infty \quad \textit{for some } a > 0.$$

(i) *For each $\omega_+ \in C[t_0, \infty)$ such that $\omega_+(t+\tau) = -\omega_+(t)$ for $t \geq t_0$ and $\max_t |\omega_+(t)| < a$, equation $(9.1)_+$ has a solution $x_+(t)$ satisfying*

$$(9.3)_+ \qquad x_+(t) = \lambda^{t/\tau}[\omega_+(t) + o(1)] \quad (t \to \infty).$$

(ii) *For each $\omega_- \in C[t_0, \infty)$ such that $\omega_-(t+\tau) = \omega_-(t)$ for $t \geq t_0$ and $\max_t |\omega_-(t)| < a$, equation $(9.1)_-$ has a solution $x_-(t)$ satisfying*

$$(9.3)_- \qquad x_-(t) = \lambda^{t/\tau}[\omega_-(t) + o(1)] \quad (t \to \infty).$$

Remark 9.1. For the case $\omega_+(t) \not\equiv 0$, the solution of $(9.1)_+$ obtained in (i) of Theorem 9.1 is oscillatory. For the case $\omega_-(t) \not\equiv 0$, the solution of $(9.1)_-$ obtained in (ii) of Theorem 9.1 is oscillatory or nonoscillatory according to whether the function $\omega_-(t)$ is oscillatory or nonoscillatory. Since the condition (9.2) is independent of $\omega_-(t)$, $(9.1)_-$ has both oscillatory solutions and nonoscillatory solutions if (9.2) holds.

In what follows we superpose the plus sign $+$ and the minus sign $-$. For example, the two equalities $\omega_+(t+\tau) = -\omega_+(t)$ and $\omega_-(t+\tau) = \omega_-(t)$ are written as $\omega_\pm(t+\tau) = \mp\omega_\pm(t)$, and the two conditions $(9.3)_+$ and $(9.3)_-$ are written as

$$(9.3)_\pm \qquad x_\pm(t) = \lambda^{t/\tau}[\omega_\pm(t) + o(1)] \quad (t \to \infty).$$

Let T and T_* be constants with $T - \tau \geq T_* \geq t_0$. We denote by $S[T_*, \infty)$ the set of all functions $u \in C[T_*, \infty)$ such that the series (8.12) converges uniformly on $[T-\tau, \infty)$. For each $u \in S[T_*, \infty)$, we assign the function $\Psi_\pm[u]$ on $[T_*, \infty)$ by

$$\Psi_\pm[u](t) = \begin{cases} -\displaystyle\sum_{i=1}^{\infty}(\mp 1)^i u(t+i\tau), & t \geq T - \tau, \\ \Psi_\pm[u](T-\tau), & t \in [T_*, T-\tau]. \end{cases}$$

By the same argument as in Section 8, we have the following result.

Proposition 9.1. *Suppose that $\eta \in S[T_*, \infty)$ such that $\eta(t) \geq 0$ for $t \geq T$ and define the set U by (8.14). Then $\Psi_\pm$ maps $S[T_*, \infty)$ into $C[T_*, \infty)$ and satisfies*

$$\Psi_\pm[u](t) \pm \Psi_\pm[u](t-\tau) = u(t), \quad t \geq T, \quad u \in S[T_*, \infty),$$

and is continuous on U in the $C[T_*, \infty)$-topology.

We first prove the case $0 < \lambda < 1$ of Theorem 9.1. To this end, we need the next lemma.

Lemma 9.1. *Let* $0 < \lambda < 1$ *and* $k \in \mathbb{N} \cup \{0\}$*. Suppose that* $G \in C[T, \infty)$ *satisfies*

$$G(t) \geq 0 \quad \text{for } t \geq T \quad \text{and} \quad \int_T^\infty \lambda^{-t/\tau} G(t)dt < \infty. \tag{9.4}$$

Then

$$\sum_{i=1}^\infty \lambda^{-(t+i\tau)/\tau} \int_{t+i\tau}^\infty (s - t - i\tau)^k G(s)ds \leq \tau^k K \int_{t+\tau}^\infty \lambda^{-s/\tau} G(s)ds, \quad t \geq T - \tau, \tag{9.5}$$

where $K = \sum_{i=1}^\infty \lambda^{i-1} i^{k-1}$.

Proof. Let $t \geq T - \tau$ be fixed. Observe that

$$\begin{aligned} &\sum_{i=1}^\infty \lambda^{-(t+i\tau)/\tau} \int_{t+i\tau}^\infty (s - t - i\tau)^k G(s)ds \\ &= \sum_{i=1}^\infty \lambda^{-(t+i\tau)/\tau} \sum_{j=i}^\infty \int_{t+j\tau}^{t+(j+1)\tau} (s - t - i\tau)^k G(s)ds \\ &= \sum_{j=1}^\infty \int_{t+j\tau}^{t+(j+1)\tau} \sum_{i=1}^j \lambda^{(s-t-i\tau)/\tau} (s - t - i\tau)^k \lambda^{-s/\tau} G(s)ds. \end{aligned} \tag{9.6}$$

If $s \in [t + j\tau, t + (j+1)\tau]$, then $(j - i)\tau \leq s - t - i\tau \leq (j + 1 - i)\tau$. Hence we have

$$\begin{aligned} \sum_{i=1}^j \lambda^{(s-t-i\tau)/\tau} (s - t - i\tau)^k &\leq \tau^k \sum_{i=1}^j \lambda^{j-i} (j + 1 - i)^k \\ &\leq \tau^k \sum_{l=1}^j \lambda^{l-1} l^k \leq \tau^k K \end{aligned} \tag{9.7}$$

for $s \in [t + j\tau, t + (j+1)\tau]$. By (9.6) and (9.7), we obtain (9.5).

Proof of Theorem 9.1 ($0 < \lambda < 1$). Let $0 < \lambda < 1$. Put

$$c = \max_t |\omega_\pm(t)| \quad \text{and} \quad \psi(t) = \tau^{n-1} K \int_{t+\tau}^\infty \lambda^{-s/\tau} F(s)ds$$

for all large t, where $F(t) = F(t, a\lambda^{g(t)/\tau})$ and $K = \sum_{i=1}^\infty \lambda^{i-1} i^{n-1}$. We can choose a number $T \geq t_0$ such that

$$T_* \equiv \min\{T - \tau, \inf\{g(t) : t \geq T\}\} \geq t_0$$

and $\psi(t) < a - c$ for $t \geq T_*$. We define the function $\eta \in C[T_*, \infty)$ and the set Y by

$$\eta(t) = \begin{cases} \lambda^{-t/\tau} \displaystyle\int_t^\infty (s-t)^{n-1} F(s)ds, & t \geq T, \\ \lambda^{-T/\tau} \displaystyle\int_T^\infty (s-T)^{n-1} F(s)ds, & t \in [T_*, T], \end{cases}$$

and

$$Y = \{y \in C[T_*, \infty) : |y(t)| \leq \eta(t) \text{ for } t \geq T_*\}, \tag{9.8}$$

respectively. It is obvious that $\eta(t) \geq 0$ for $t \geq T_*$ and Y is closed and convex. Lemma 9.1 implies that

$$\sum_{i=1}^{\infty} \eta(t + i\tau) \leq \psi(t), \quad t \geq T - \tau. \tag{9.9}$$

In view of Lemma 8.1 and the fact that $\lim_{t\to\infty} \psi(t) = 0$, we see that $\eta \in S[T_*, \infty)$. From Proposition 9.1 it follows that $\Psi_\pm : Y \longrightarrow C[T_*, \infty)$ is continuous and satisfies

$$\Psi_\pm[y](t) \pm \Psi_\pm[y](t - \tau) = y(t), \quad t \geq T, \quad y \in Y. \tag{9.10}$$

Since $\psi(t) < a - c$ for $t \geq T_*$, from (9.9) we find that

$$|\Psi_\pm[y](t)| \leq \psi(t) \leq a - c, \quad t \geq T - \tau, \quad y \in Y. \tag{9.11}$$

We define the mapping $\mathcal{F}_\pm : Y \longrightarrow C[T_*, \infty)$ as follows:

$$(\mathcal{F}_\pm y)(t)$$
$$= \begin{cases} (-1)^{n-1}\lambda^{-t/\tau} \\ \quad \times \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, [\omega_\pm(g(s)) + \Psi_\pm[y](g(s))]\lambda^{g(s)/\tau})ds, \\ \hfill t \geq T, \\ (\mathcal{F}_\pm y)(T), \qquad\qquad t \in [T_*, T]. \end{cases}$$

From (9.11), we have

$$|\omega_\pm(t) + \Psi_\pm[y](t)| \leq |\omega_\pm(t)| + |\Psi_\pm[y](t)| \leq c + (a - c) = a, \quad t \geq T_*,$$

for each $y \in Y$, so that $\mathcal{F}_\pm$ is well defined and maps Y into itself. By using the same arguments as in the proofs of Theorems 4.1 and 4.2, we conclude that $\mathcal{F}_\pm$ is continuous and $\mathcal{F}_\pm(Y)$ is relatively compact. Application of the Schauder-Tychonoff fixed point theorem shows that there exist $\widetilde{y}_\pm \in Y$ such that $\widetilde{y}_\pm = \mathcal{F}_\pm \widetilde{y}_\pm$. Put $x_\pm(t) = [\omega_\pm(t) + \Psi_\pm[\widetilde{y}_\pm](t)]\lambda^{t/\tau}$. Then we obtain

$$x_\pm(t) \pm \lambda x_\pm(t-\tau) = (-1)^{n-1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x_\pm(g(s)))ds$$

for $t \geq T$, which implies that $x_\pm(t)$ are solutions of $(9.1)_\pm$. From (9.11) it follows that $x_\pm(t)$ satisfy $(9.3)_\pm$. This completes the proof of Theorem 9.1 for the case $0 < \lambda < 1$.

Now we turn to the the case $\lambda > 1$ of Theorem 9.1.

Lemma 9.2. *Let $\lambda > 1$ and $k \in \mathbb{N} \cup \{0\}$. Suppose that $G \in C[T, \infty)$ satisfies* (9.4). *Then*

$$\begin{aligned} (9.12) \qquad & \sum_{i=1}^\infty \lambda^{-(t+i\tau)/\tau} \int_T^{t+i\tau} (t + i\tau - s)^k G(s)ds \\ & \leq L \int_{t+\tau}^\infty \lambda^{-s/\tau} G(s)ds + 2^k L \lambda^{-t/\tau} \int_T^{t+\tau} G(s)ds \\ & \quad + \frac{2^k}{\lambda - 1} \lambda^{-t/\tau} \int_T^{t+\tau} (t + \tau - s)^k G(s)ds \end{aligned}$$

for $t \geq T - \tau$, where $L = \tau^k \sum_{i=1}^\infty \lambda^{-i+1} i^k$.

Proof. Let $t \geq T - \tau$ be fixed. We observe that

$$\begin{aligned}
(9.13) \qquad & \sum_{i=1}^{\infty} \lambda^{-(t+i\tau)/\tau} \int_T^{t+i\tau} (t+i\tau-s)^k G(s)ds \\
&= \sum_{i=1}^{\infty} \lambda^{-(t+i\tau)/\tau} \int_T^{t+\tau} (t+i\tau-s)^k G(s)ds \\
&\quad + \sum_{i=2}^{\infty} \lambda^{-(t+i\tau)/\tau} \sum_{j=1}^{i-1} \int_{t+j\tau}^{t+(j+1)\tau} (t+i\tau-s)^k G(s)ds \\
&= \lambda^{-t/\tau} \sum_{i=1}^{\infty} \lambda^{-i} \int_T^{t+\tau} (t+i\tau-s)^k G(s)ds \\
&\quad + \sum_{j=1}^{\infty} \int_{t+j\tau}^{t+(j+1)\tau} \sum_{i=j+1}^{\infty} \lambda^{-(t+i\tau-s)/\tau} (t+i\tau-s)^k \lambda^{-s/\tau} G(s)ds.
\end{aligned}$$

We have

$$\begin{aligned}
(t+i\tau-s)^k &= [(t+\tau-s)+(i-1)\tau]^k \\
&\leq 2^k[(t+\tau-s)^k + (i-1)^k\tau^k]
\end{aligned}$$

for $s \in [T, t+\tau]$, because of $(u+v)^k \leq 2^k(u^k+v^k)$ for $u \geq 0$ and $v \geq 0$. Then we see that

$$\begin{aligned}
(9.14) \qquad & \sum_{i=1}^{\infty} \lambda^{-i} \int_T^{t+\tau} (t+i\tau-s)^k G(s)ds \\
&\leq 2^k \sum_{i=1}^{\infty} \lambda^{-i} \int_T^{t+\tau} (t+\tau-s)^k G(s)ds \\
&\quad + 2^k\tau^k \sum_{i=1}^{\infty} \lambda^{-i}(i-1)^k \int_T^{t+\tau} G(s)ds \\
&\leq \frac{2^k}{\lambda-1} \int_T^{t+\tau} (t+\tau-s)^k G(s)ds + 2^k L \int_T^{t+\tau} G(s)ds.
\end{aligned}$$

If $s \in [t+j\tau, t+(j+1)\tau]$, then $(i-j-1)\tau \leq t+i\tau-s \leq (i-j)\tau$. Thus,

$$\begin{aligned}
(9.15) \qquad \sum_{i=j+1}^{\infty} \lambda^{-(t+i\tau-s)/\tau}(t+i\tau-s)^k &\leq \tau^k \sum_{i=j+1}^{\infty} \lambda^{-(i-j-1)}(i-j)^k \\
&= \tau^k \sum_{l=1}^{\infty} \lambda^{-l+1} l^k = L
\end{aligned}$$

for $s \in [t+j\tau, t+(j+1)\tau]$. Combining (9.13)–(9.15), we obtain (9.12).

Lemma 9.3. *Let $\lambda > 1$ and $k \in \mathbb{N} \cup \{0\}$. Suppose that $G \in [T, \infty)$ satisfies* (9.4). *Then*

$$\lim_{t\to\infty} \lambda^{-t/\tau} \int_T^{t+\tau} (t+\tau-s)^k G(s)ds = 0. \tag{9.16}$$

Proof. It suffices to give the proof for the case $k = 0$. In fact, if

$$\lim_{t\to\infty} \lambda^{-t/\tau} \int_T^{t+\tau} G(s)ds = 0,$$

then for $k \neq 0$ we have

$$\begin{aligned} &\lim_{t\to\infty} \lambda^{-t/\tau} \int_T^{t+\tau} (t+\tau-s)^k G(s)ds \\ &= \lim_{t\to\infty} \frac{d^k}{dt^k} \int_T^{t+\tau} (t+\tau-s)^k G(s)ds \Bigg/ \frac{d^k}{dt^k} \lambda^{t/\tau} \\ &= \lim_{t\to\infty} k! \left[\frac{\tau}{\log\lambda}\right]^k \lambda^{-t/\tau} \int_T^{t+\tau} G(s)ds = 0. \end{aligned}$$

Put $\varphi(t) = \lambda^{-t/\tau} \int_T^t G(s)ds$. An easy computation shows that

$$\int_T^t \varphi(s)ds = \frac{\tau}{\log\lambda}\left[\int_T^t \lambda^{-s/\tau} G(s)ds - \varphi(t)\right], \quad t \geq T. \tag{9.17}$$

Then we have

$$0 \leq \int_T^t \varphi(s)ds \leq \frac{\tau}{\log\lambda} \int_T^\infty \lambda^{-s/\tau} G(s)ds, \quad t \geq T,$$

which implies that φ is integrable on $[T, \infty)$. It follows from (9.17) that $l \equiv \lim_{t\to\infty} \varphi(t)$ exists and is a nonnegative finite value. Since φ is integrable on $[T, \infty)$, it is impossible that $l > 0$. Consequently, (9.16) holds for the case $k = 0$. This completes the proof.

Proof of Theorem 9.1 ($\lambda > 1$). We assume that $\lambda > 1$. Set $F(t) = F(t, a\lambda^{g(t)/\tau})$, $c = \max_t |\omega_\pm(t)|$,

$$L = \tau^{n-1} \sum_{i=1}^\infty \lambda^{-i+1} i^{n-1}, \quad M = 2^{n-1} \max\left\{L, \frac{1}{\lambda - 1}\right\},$$

and

$$\psi(t) = L\int_{t+\tau}^{\infty} \lambda^{-s/\tau}F(s)ds + M\lambda^{-t/\tau}\int_{T_0}^{t+\tau}[1+(t+\tau-s)^{n-1}]F(s)ds$$

for $t \geq T_0 - \tau$, where T_0 is a sufficiently large number. By Lemma 9.3, we have $\lim_{t\to\infty}\psi(t) = 0$. Hence, we can take a number $T \geq T_0$ such that

$$T_* \equiv \min\{T-\tau, \inf\{g(t) : t \geq T\}\} \geq T_0$$

and $\psi(t) < a - c$ for $t \geq T_*$. We put

$$\eta(t) = \begin{cases} \lambda^{-t/\tau}\displaystyle\int_T^t (t-s)^{n-1}F(s)ds, & t \geq T, \\ 0, & t \in [T_*, T], \end{cases}$$

and define Y by (9.8). In view of Lemma 9.2, we see that (9.9) holds, so that $\eta \in S[T_*, \infty)$ by Lemma 8.1. Proposition 9.1 implies that $\Psi_\pm : Y \longrightarrow C[T_*, \infty)$ is continuous and satisfies (9.10) and (9.11). By the same arguments as in the proof of Theorem 9.1 for the case $0 < \lambda < 1$, there exist $\widetilde{y}_\pm \in Y$ such that $\widetilde{y}_\pm = \mathcal{F}_\pm \widetilde{y}_\pm$, where $\mathcal{F}_\pm : Y \longrightarrow C[T_*, \infty)$ is the mapping defined by

$$(\mathcal{F}_\pm y)(t) = \begin{cases} -\lambda^{-t/\tau}\displaystyle\int_T^t \frac{(t-s)^{n-1}}{(n-1)!}f(s, [\omega_\pm(g(s)) + \Psi_\pm[y](g(s))]\lambda^{g(s)/\tau})ds, \\ \hfill t \geq T, \\ 0, \qquad t \in [T_*, T]. \end{cases}$$

Let $x_\pm(t) = [\omega_\pm(t) + \Psi_\pm[\widetilde{y}_\pm](t)]\lambda^{t/\tau}$. Then we find that

$$x_\pm(t) \pm \lambda x_\pm(t-\tau) = -\int_T^t \frac{(t-s)^{n-1}}{(n-1)!}f(s, x_\pm(g(s)))ds, \quad t \geq T,$$

so that $x_\pm(t)$ are solutions of $(9.1)_\pm$. From (9.11), we conclude that $x_\pm(t)$ satisfy $(9.3)_\pm$. This completes the proof of Theorem 9.1 for the case $\lambda > 1$.

10. The case $h(t) = h(\tau^N(t)) + o(1)$

We consider the equation

$$\frac{d^n}{dt^n}[x(t) + h(t)x(\tau(t))] + f(t, x(g(t))) = 0 \tag{10.1}$$

for the following case:

$$\lim_{t\to\infty}[h(t) - h(\tau^N(t))] = 0 \tag{10.2}$$

for some $N \in \mathbb{N}$. It is assumed throughout this section that (1.2)–(1.7) and (4.2) hold.

Pairs of the functions

$$h(t) = (1/2)\sin t + o(1) \quad (t \to \infty), \quad \tau(t) = t - (2\pi/N);$$

$$h(t) = 3 + \sin(2\pi \log t) + o(1) \quad (t \to \infty), \quad \tau(t) = e^{-1/N}t,$$

give typical examples satisfying (10.2).

Now we suppose that $(-1)^N H_N(t) \neq 1$ for $t \geq \tau^{-(N-1)}(t_0)$ and define the function $\omega_N(t)$ by

$$\omega_N(t) = \left[\sum_{i=0}^{N-1}(-1)^i H_i(t)\right] \Big/ [1 - (-1)^N H_N(t)], \quad t \geq \tau^{-(N-1)}(t_0).$$

It is easy to check that if $h(t) = h(\tau^N(t))$ for $t \geq \tau^{-(N-1)}(t_0)$, then

$$\omega_N(t) + h(t)\omega_N(\tau(t)) = 1, \quad t \geq \tau^{-N}(t_0).$$

Thus it is natural to expect that, if (10.2) holds, then there exists a continuous function $\omega(t)$ which satisfies

$$\omega(t) + h(t)\omega(\tau(t)) = 1 \tag{10.3}$$

for all large t and behaves like the function $\omega_N(t)$ as $t \to \infty$. In fact, we have the following result.

Lemma 10.1. *Suppose that $h(t)$ is bounded on $[t_0, \infty)$ and that either* (4.9) *or* (4.10) *holds. Assume moreover that* (10.2) *holds. Then* *there exists a continuous function* $\omega(t)$ *satisfying* (10.3) *for all large* t *and*

$$\omega(t) = \omega_N(t) + o(1) \quad (t \to \infty).$$

Proof. Put $p(t) = -h(t)$ and use the notation (2.3). Then

$$\omega_N(t) = \left[\sum_{i=0}^{N-1} P_i(t)\right] \Big/ [1 - P_N(t)].$$

First we claim that

$$\lim_{t\to\infty} [\omega_N(t) - p(t)\omega_N(\tau(t))] = 1. \tag{10.4}$$

We observe that

$$\begin{aligned}
&[\omega_N(t) - p(t)\omega_N(\tau(t))][1 - P_N(t)][1 - P_N(\tau(t))] \\
&\quad = \sum_{i=0}^{N-1} P_i(t)[1 - P_N(\tau(t))] - \sum_{i=0}^{N-1} p(t)P_i(\tau(t))[1 - P_N(t)] \\
&\quad = \sum_{i=0}^{N-1} P_i(t)[1 - P_N(\tau(t))] - \sum_{i=1}^{N} P_i(t)[1 - P_N(t)] \\
&\quad = \sum_{i=1}^{N-1} P_i(t)[1 - P_N(\tau(t)) - 1 + P_N(t)] + 1 - P_N(\tau(t)) \\
&\qquad - P_N(t)[1 - P_N(t)] \\
&\quad = \sum_{i=1}^{N-1} P_i(t)[P_N(t) - P_N(\tau(t))] + [1 - P_N(\tau(t))][1 - P_N(t)] \\
&\qquad + P_N(t)[P_N(t) - P_N(\tau(t))] \\
&\quad = \sum_{i=1}^{N} P_i(t)[P_N(t) - P_N(\tau(t))] + [1 - P_N(\tau(t))][1 - P_N(t)],
\end{aligned}$$

so that

$$\begin{aligned}
&\omega_N(t) - p(t)\omega_N(\tau(t)) \\
&\qquad = \frac{P_{N-1}(\tau(t)) \sum_{i=1}^{N} P_i(t)}{[1 - P_N(t)][1 - P_N(\tau(t))]} [p(t) - p(\tau^N(t))] + 1
\end{aligned}$$

for all large t. Since $p(t)$ is bounded and since either $|P_N(t)| \leq \lambda < 1$ or $|P_N(t)| \geq \mu > 1$, there is a constant $M > 0$ such that

$$|\omega_N(t) - p(t)\omega_N(\tau(t)) - 1| \leq M|p(t) - p(\tau^N(t))|$$

for all large t. From (10.2) it follows that (10.4) holds.

By applying Lemma 3.2 with $q(t) = 1 - \omega_N(t) + p(t)\omega_N(\tau(t))$, there is a continuous function $v(t)$ satisfying

$$v(t) - p(t)v(\tau(t)) = 1 - \omega_N(t) + p(t)\omega_N(\tau(t))$$

for all large t and $\lim_{t\to\infty} v(t) = 0$. Set $\omega(t) = \omega_N(t) + v(t)$. Then we easily see that $\omega(t)$ satisfies (2.6) for all large t and $\omega(t) = \omega_N(t) + o(1)$ $(t \to \infty)$. This completes the proof.

Lemma 10.2. *Let $k \in \mathbb{N} \cup \{0\}$. Suppose that $h(t)$ is bounded on $[t_0, \infty)$ and that either (4.9) or (4.10) holds. Assume moreover that (10.2) holds and $\lim_{t\to\infty} \tau(t)/t = 1$. Then there exists a continuous function $\omega(t)$ satisfying*

$$\omega(t) + h(t)\omega(\tau(t)) = t^k \tag{10.5}$$

for all large t and

$$\omega(t) = [\omega_N(t) + o(1)]t^k \quad (t \to \infty). \tag{10.6}$$

Remark 10.1. For the case where $h(t)$ is bounded, if either (4.9) or (4.10) holds and $\lim_{t\to\infty} \tau(t)/t = 1$, then either (4.3) or (4.4) holds with t_0 replaced by sufficiently large number T_1.

Proof of Lemma 10.2. Notice that

$$\begin{aligned}
&\lim_{t\to\infty}\left[h(t)\left[\frac{\tau(t)}{t}\right]^k - h(\tau^N(t))\left[\frac{\tau^{N+1}(t)}{\tau^N(t)}\right]^k\right] \\
&= \lim_{t\to\infty}[h(t)-h(\tau^N(t))]\left[\frac{\tau(t)}{t}\right]^k \\
&\quad + \lim_{t\to\infty} h(\tau^N(t))\left[\left[\frac{\tau(t)}{t}\right]^k - \left[\frac{\tau^{N+1}(t)}{\tau^N(t)}\right]^k\right] \\
&= 0.
\end{aligned}$$

Applying Lemma 10.1 with $h(t)$ replaced by $h(t)[\tau(t)/t]^k$, we see that there exists a continuous function $\theta(t)$ such that

$$\theta(t) + h(t)\left[\frac{\tau(t)}{t}\right]^k \theta(\tau(t)) = 1$$

for all large t and

$$\theta(t) = \left[\sum_{i=0}^{N-1}(-1)^i H_i(t)\left[\frac{\tau^i(t)}{t}\right]^k\right] \Big/ \left[1-(-1)^N H_N(t)\left[\frac{\tau^N(t)}{t}\right]^k\right] + o(1) \quad (t\to\infty).$$

It is not difficult to verify that

$$\left[\sum_{i=0}^{N-1}(-1)^i H_i(t)\left[\frac{\tau^i(t)}{t}\right]^k\right] \Big/ \left[1-(-1)^N H_N(t)\left[\frac{\tau^N(t)}{t}\right]^k\right] = \omega_N(t) + o(1) \quad (t\to\infty).$$

Thus, $\omega(t)\equiv t^k\theta(t)$ satisfies (10.5) for all large t and (10.6).

From Theorem 4.1, Lemmas 10.1 and 10.2, we obtain the following results.

Theorem 10.1. *Suppose that $h(t)$ is bounded on $[t_0,\infty)$ and that either* (4.9) *or* (4.10) *holds. Assume moreover that* (10.2) *holds. Then* (10.1) *has a solution $x(t)$ satisfying*

$$x(t) = c\,\omega_N(t) + o(1) \quad (t\to\infty) \quad \textit{for some } c\neq 0,$$

provided

$$\text{(10.7)} \qquad \int^{\infty} t^{n-1}F(t,a)dt < \infty \quad \textit{for some } a > 0.$$

Theorem 10.2. *Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that $h(t)$ is bounded on $[t_0, \infty)$ and that either (4.9) or (4.10) holds. Assume moreover that (10.2) holds and $\lim_{t\to\infty} \tau(t)/t = 1$. Then (10.1) has a solution $x(t)$ satisfying*

$$x(t) = c[\omega_N(t) + o(1)]t^k \quad (t \to \infty) \quad \textit{for some } c \neq 0,$$

provided

$$\text{(10.8)} \qquad \int^{\infty} t^{n-k-1}F(t, a[g(t)]^k)dt < \infty \quad \textit{for some } a > 0.$$

In particular, for the cases $N = 1$ and $N = 2$, Theorems 10.1 and 10.2 give the following results.

Corollary 10.1. *Suppose that*

(10.9) *$h(t)$ is bounded on $[t_0, \infty)$, $\lim_{t\to\infty}[h(t) - h(\tau(t))] = 0$, and either $|h(t)| \le \lambda < 1$ for some $\lambda > 0$ or $|h(t)| \ge \mu > 1$ on $[t_0, \infty)$ for some $\mu > 0$.*

If (10.7) holds, then (10.1) has a nonoscillatory solution $x(t)$ satisfying

$$x(t) = \frac{c}{1 + h(t)} + o(1) \quad (t \to \infty) \quad \textit{for some } c \neq 0.$$

Corollary 10.2. *Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that (10.9) holds and $\lim_{t\to\infty} \tau(t)/t = 1$. If (10.8) holds, then (10.1) possesses a nonoscillatory solution $x(t)$ satisfying*

$$x(t) = c\left[\frac{1}{1 + h(t)} + o(1)\right]t^k \quad (t \to \infty) \quad \textit{for some } c \neq 0.$$

Corollary 10.3. *Suppose that*

(10.10) *$h(t)$ is bounded on $[t_0, \infty)$, $\lim_{t\to\infty}[h(t) - h(\tau^2(t))] = 0$, and either $|h(t)h(\tau(t))| \le \lambda < 1$ on $[\tau^{-1}(t_0), \infty)$ for some $\lambda > 0$ or $|h(t)|^{-1}$ is bounded on $[t_0, \infty)$ and $|h(t)h(\tau(t))| \ge \mu > 1$ on $[\tau^{-1}(t_0), \infty)$ for some $\mu > 0$.*

If (10.7) *holds, then* (10.1) *has a solution* $x(t)$ *satisfying*

$$x(t) = \frac{c[1-h(t)]}{1-h(t)h(\tau(t))} + o(1) \quad (t \to \infty) \quad \textit{for some } c \neq 0.$$

Corollary 10.4. *Let* $k \in \{0, 1, 2, \ldots, n-1\}$. *Suppose that* (10.10) *holds and* $\lim_{t\to\infty} \tau(t)/t = 1$. *If* (10.8) *holds, then* (10.1) *has a solution* $x(t)$ *satisfying*

$$x(t) = c\left[\frac{1-h(t)}{1-h(t)h(\tau(t))} + o(1)\right] t^k \quad (t \to \infty) \quad \textit{for some } c \neq 0.$$

Now assume that

(10.11) $\lim_{t\to\infty}[h(t) + h(\tau(t))] = 0$ and $|h(t)| \leq \lambda < 1$ on $[\tau^{-1}(t_0), \infty)$ for some $\lambda > 0$.

Then we easily see that

$$\lim_{t\to\infty}[h(t) - h(\tau^2(t))] = \lim_{t\to\infty}[h(t) + h(\tau(t))] - \lim_{t\to\infty}[h(\tau(t)) + h(\tau^2(t))] = 0$$

and

$$\frac{1-h(t)}{1-h(t)h(\tau(t))} = \frac{1-h(t)}{1+[h(t)]^2} + o(1) \quad (t \to \infty).$$

Consequently, from Corollaries 10.3 and 10.4, we have the following results.

Corollary 10.5. *Suppose that* (10.11) *holds. If* (10.7) *holds, then* (10.1) *has a nonoscillatory solution* $x(t)$ *satisfying*

$$x(t) = \frac{c[1-h(t)]}{1+[h(t)]^2} + o(1) \quad (t \to \infty) \quad \textit{for some } c \neq 0.$$

Corollary 10.6. *Let* $k \in \{0, 1, 2, \ldots, n-1\}$. *Assume that* (10.11) *holds and* $\lim_{t\to\infty} \tau(t)/t = 1$. *If* (10.8) *holds, then* (10.1) *has a nonoscillatory solution* $x(t)$ *satisfying*

$$x(t) = c\left[\frac{1-h(t)}{1+[h(t)]^2} + o(1)\right] t^k \quad (t \to \infty) \quad \textit{for some } c \neq 0.$$

Remark 10.2. The solutions obtained in Corollaries 10.3 and 10.4 are oscillatory or nonoscillatory. Indeed, this is confirmed by the following example.

Example 10.1. We consider the equation

$$\frac{d^n}{dt^n}[x(t) + \lambda(1 - \sin t)x(t - \pi)] + f(t, x(g(t))) = 0, \tag{10.12}$$

where $|\lambda| < 1$. Here, $h(t) = \lambda(1 - \sin t)$ and $\tau(t) = t - \pi$. It is easy to check that (10.10) holds and $\lim_{t\to\infty} \tau(t)/t = 1$. Let $k \in \{0, 1, 2, \dots, n-1\}$. Applying Corollary 10.4 to equation (10.12), we see that if (10.8) holds, then (10.12) has a solution $x(t)$ satisfying

$$x(t) = c\left[\frac{1 + \lambda(\sin t - 1)}{1 + \lambda^2(\sin^2 t - 1)} + o(1)\right] t^k \quad (t \to \infty) \quad \text{for some } c \neq 0.$$

This solution $x(t)$ is oscillatory if $1/2 < \lambda < 1$ and is nonoscillatory if $-1 < \lambda < 1/2$.

11. Positive solutions for the case $h(t) = h(\tau(t))$

In this section we shall be concerned with the existence of positive solutions of the equation

$$\frac{d^n}{dt^n}[x(t) + h(t)x(\tau(t))] + \sigma f(t, x(g(t))) = 0, \tag{11.1}$$

where $\sigma = +1$ or -1, (1.2)–(1.6), (5.22) and the following condition hold:

$$h(t) = h(\tau(t)) \quad \text{and} \quad h(t) > -1 \quad \text{for } t \geq \tau^{-1}(t_0). \tag{11.2}$$

We obtain the following theorem, in which the "if" part is an analogue of Corollary 10.1.

Theorem 11.1. *Equation* (11.1) *has a positive solution* $x(t)$ *satisfying*

$$x(t) = \frac{c}{1 + h(t)} + o(1) \quad (t \to \infty) \quad \textit{for some } c > 0 \tag{11.3}$$

if and only if

$$\int^{\infty} t^{n-1} f(t,a)dt < \infty \quad \textit{for some } a > 0. \tag{11.4}$$

It should be emphasized that neither $|h(t)| \le \lambda < 1$ nor $|h(t)| \ge \lambda > 1$ is necessary in Theorem 11.1.

We give an example illustrating the above theorem.

Example 11.1. We consider the neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] + \sigma e^{-t}[P(g(t))]^{-\gamma}[x(g(t))]^{\gamma} = 0, \tag{11.5}$$

where $n \ge 1$, $\sigma = +1$ or -1, $\gamma > 0$, $\tau = \log(4/3)$, $g \in C[t_0, \infty)$, $\lim_{t\to\infty} g(t) = \infty$, $g(t) \ge 0$ for $t \ge t_0$, $h(t) = 1 + (3/2)\sin(2\pi t/\tau)$, and

$$\begin{aligned} P(t) &= \frac{11}{1+h(t)} + \sigma(-1)^{n-1}\frac{3e^{-t}}{3+4h(t)} \\ &= \frac{22}{4+3\sin(2\pi t/\tau)} + \sigma(-1)^{n-1}\frac{3e^{-t}}{7+6\sin(2\pi t/\tau)}, \quad t \ge 0. \end{aligned}$$

Clearly, $h(t) > -1$, $h(t) = h(t-\tau)$ for $t \ge t_0$ and $P(t) \ge 1/7$ for $t \ge 0$. Then it is easy to check that

$$\int_{t_0}^{\infty} t^{n-1} e^{-t}[P(g(t))]^{-\gamma} a^{\gamma} dt < \infty \quad (a > 0).$$

By Theorem 11.1, we conclude that (11.5) has a positive solution $x(t)$ satisfying

$$x(t) = \frac{c}{4+3\sin(2\pi t/\tau)} + o(1) \quad (t \to \infty) \quad \text{for some } c > 0.$$

Indeed, $x(t) = P(t)$ is such a positive solution.

Suppose that (11.2) holds. Then note that

$$[t_0, \infty) = \cup_{p=0}^{\infty}[\tau^{-p}(t_0), \tau^{-(p+1)}(t_0)]$$

and that the range of $h(t)$ for $t \in [t_0, \tau^{-1}(t_0)]$ is identical to the range of $h(t)$ $(= h(\tau^p(t)))$ for $t \in [\tau^{-p}(t_0), \tau^{-(p+1)}(t_0)]$, $p = 0, 1, 2, \ldots$. Let $\mu = \min h(t)$ on $[t_0, \tau^{-1}(t_0)]$ and $\lambda = \max h(t)$ on $[t_0, \tau^{-1}(t_0)]$. Then we find that $-1 < \mu \leq h(t) \leq \lambda < \infty$ for all $t \geq t_0$, so that the asymptotic condition (11.3) for $x(t)$ implies

$$0 < \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) < \infty.$$

Consequently, the "only if" part of Theorem 11.1 follows from Theorem 5.4.

Now we give the proof of "if" part of Theorem 11.1.

It is possible to take sufficiently large numbers $T \geq t_0$ and $T_* \geq t_0$ such that

$$h(T) = \max\{h(t) : t \in [t_0, \infty)\},$$

and

$$t_0 \leq T_* \leq \min\{\tau(T), \inf\{g(t) : t \geq T\}\}.$$

Let $\eta \in C[T, \infty)$ be fixed such that $\eta(t) \geq 0$ for $t \geq T$ and $\lim \eta(t) = 0$ as $t \to \infty$. We consider the set Y of all functions $y \in C[T_*, \infty)$ which is nonincreasing on $[T, \infty)$ and satisfies

$$y(t) = y(T) \quad \text{for } t \in [T_*, T], \quad 0 \leq y(t) \leq \eta(t) \quad \text{for } t \geq T.$$

It is easy to see that Y is a closed convex subset of $C[T_*, \infty)$.

To prove Theorem 11.1, the following proposition is used.

Proposition 11.1. *Let $\eta \in C[T, \infty)$ with $\eta(t) \geq 0$ for $t \geq T$ and $\lim_{t\to\infty} \eta(t) = 0$. For this η, define Y as above. Then there exists a mapping $\Phi : Y \longrightarrow C[T_*, \infty)$ which possesses the following properties:*

(i) *For each $y \in Y$, $\Phi[y]$ satisfies*

$$\Phi[y](t) + h(t)\Phi[y](\tau(t)) = y(t), \quad t \geq T \quad \textit{and} \quad \lim_{t\to\infty} \Phi[y](t) = 0;$$

(ii) *Φ is continuous on Y in the $C[T_*, \infty)$-topology.*

The proof of Proposition 11.1 is deferred to Section 13.

Proof of the "if" part of Theorem 11.1. Put

$$\eta(t) = \int_t^\infty s^{n-1} f(s, a) ds, \quad t \geq T.$$

We use Proposition 11.1 for this η. We can take constants $c > 0$, $\delta > 0$ and $\varepsilon > 0$ such that

$$0 < \delta + \varepsilon \leq \frac{c}{1 + h(t)} \leq a - \varepsilon, \quad t \geq T_*.$$

Define the mapping $\mathcal{F} : Y \longrightarrow C[T_*, \infty)$ as follows:

$$(\mathcal{F}y)(t) = \begin{cases} \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \overline{f}(s, \omega(g(s)) + \sigma(-1)^{n-1}\Phi[y](g(s)))ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \geq T, \\ (\mathcal{F}y)(T), \qquad\qquad\qquad t \in [T_*, T], \end{cases}$$

where $\omega(t) = c/[1 + h(t)]$ and

$$\overline{f}(t, u) = \begin{cases} f(t, a), & u \geq a, \\ f(t, u), & \delta \leq u \leq a, \\ f(t, \delta), & u \leq \delta. \end{cases}$$

It is easy to see that $\mathcal{F}$ is well defined on Y, maps Y into itself and is continuous on Y, and that $\mathcal{F}(Y)$ is relatively compact. Consequently, we are able to apply the Schauder-Tychonoff fixed point theorem to the operator $\mathcal{F}$ and we conclude that there exists a $\widetilde{y} \in Y$ such that $\widetilde{y} = \mathcal{F}\widetilde{y}$. Set

$$x(t) = \omega(t) + \sigma(-1)^{n-1}\Phi[\widetilde{y}](t).$$

Proposition 11.1 implies that $x(t)$ satisfies (11.3) and that there exists a number $\widetilde{T} \geq T$ such that $\delta \leq x(g(t)) \leq a$ for $t \geq \widetilde{T}$. Then

$\overline{f}(t, x(g(t))) = f(t, x(g(t)))$ for $t \geq \widetilde{T}$. Observe that

(11.6)

$$
\begin{aligned}
&x(t) + h(t)x(\tau(t)) \\
&= \frac{c}{1+h(t)} + h(t)\frac{c}{1+h(\tau(t))} + \sigma(-1)^{n-1}[\Phi[\widetilde{y}](t) + h(t)\Phi[\widetilde{y}](\tau(t))] \\
&= c + \sigma(-1)^{n-1}\widetilde{y}(t) \\
&= c + \sigma(-1)^{n-1}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x(g(s)))ds, \quad t \geq \widetilde{T}.
\end{aligned}
$$

By differentiation of (11.6), we see that $x(t)$ is a solution of (11.1). The proof is complete.

12. Positive solutions for the case where $h(t)$ is periodic

In this section we consider the neutral differential equation

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] + \sigma f(t, x(g(t))) = 0. \tag{12.1}$$

For equation (12.1), we always assume that $\sigma = +1$ or -1, $\tau > 0$, and that (1.2)–(1.4), (1.6), (5.22) and the following condition (12.2) hold:

$$h(t) = h(t-\tau) \quad \text{and} \quad h(t) > -1, \quad t \geq t_0. \tag{12.2}$$

We define the functions $\omega_k(t)$ $(k = 0, 1, 2, \ldots)$ on the interval $[t_0, \infty)$ by

$$
\omega_k(t) = \begin{cases} \dfrac{1}{1+h(t)}, & k = 0, \\[2ex] \dfrac{t^k}{1+h(t)} - \dfrac{h(t)}{1+h(t)}\displaystyle\sum_{i=0}^{k-1}\binom{k}{i}(-\tau)^{k-i}\omega_i(t), & k = 1, 2, \ldots. \end{cases}
$$

By induction, we easily see that $\omega_k(t)$ satisfies

$$\omega_k(t) + h(t)\omega_k(t-\tau) = t^k, \quad t \geq t_0 + \tau,$$

and

$$\omega_k(t) = \left[\frac{1}{1+h(t)} + o(1)\right]t^k \quad \text{as } t \to \infty. \tag{12.3}$$

Thus, if $k \in \{0, 1, 2, \dots, n-1\}$, a positive constant multiple of $\omega_k(t)$ is a positive solution of the unperturbed equation

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] = 0.$$

In this section we have the following theorem, in which the "if" part is an analogue of Corollary 10.2.

Theorem 12.1. *Let $k \in \{0, 1, 2, \dots, n-1\}$. Then (12.1) has a solution $x(t)$ satisfying*

$$x(t) = \left[\frac{c}{1+h(t)} + o(1)\right] t^k \quad (t \to \infty) \quad \textit{for some } c > 0 \tag{12.4}$$

if and only if

$$\int^{\infty} t^{n-k-1} f(t, a[g(t)]^k) dt < \infty \quad \textit{for some } a > 0. \tag{12.5}$$

We note here that the conditions $|h(t)| \leq \lambda < 1$ and $|h(t)| \geq \lambda > 1$ are not required in Theorem 12.1.

Theorem 12.1 for the case $k = 0$ is already shown as Theorem 11.1. Therefore, in what follows, we assume that $k \in \{1, 2, \dots, n-1\}$. Suppose that (12.2) and (12.5) hold. We can take a sufficiently large number $T \geq t_0 + k\tau$ and positive constants c_1 and c_2 such that

$$h(T) = \max\{h(t) : t \in [t_0, \infty)\},$$

$$T_* \equiv \min\{T - \tau, \inf\{g(t) : t \geq T - k\tau\}\} \geq t_0,$$

and

$$0 < c_1 \leq \frac{\omega_k(t)}{t^k} \leq c_2, \quad t \geq T_*. \tag{12.6}$$

Because of (12.3), it is possible to take $c_1, c_2 > 0$ satisfying (12.6). Put

$$\eta(t) = \tau^k \int_{t-k\tau}^{\infty} s^{n-k-1} f(s, a[g(s)]^k) ds, \quad t \geq T.$$

Then $\eta \in C[T, \infty)$ with $\eta(t) \geq 0$ for $t \geq T$ and $\lim_{t\to\infty} \eta(t) = 0$. We consider the set Y_0 of all functions $y \in C[T_*, \infty)$ which is nonincreasing on $[T, \infty)$ and satisfies

$$y(t) = y(T) \quad \text{for } t \in [T_*, T], \quad 0 \leq y(t) \leq \eta(t) \quad \text{for } t \geq T.$$

It is easy to check that Y_0 is a closed convex subset of $C[T_*, \infty)$. From Proposition 11.1, there exists a mapping $\Phi_0 : Y_0 \longrightarrow C[T_*, \infty)$ which is continuous on Y_0 and satisfies

$$\Phi_0[y](t) + h(t)\Phi_0[y](t - \tau) = y(t), \quad t \geq T \quad \text{and} \quad \lim_{t\to\infty} \Phi_0[y](t) = 0$$

for each $y \in Y_0$.

We use the notation:

$$\Delta[u](t) = u(t) - u(t - \tau);$$
$$\Delta^0[u] = u; \quad \Delta^j[u] = \Delta^{j-1}[\Delta[u]], \quad j = 1, 2, \dots .$$

Define the sets Y_i $(i = 1, 2, \dots, k)$ inductively as follows:

$$Y_i = \{y \in C[T_* - i\tau, \infty) : \Delta[y] \in Y_{i-1}\}, \quad i = 1, 2, \dots, k.$$

We see that

$$Y_i = \{y \in C[T_* - i\tau, \infty) : \Delta^i[y] \in Y_0\}, \quad i = 1, 2, \dots, k, \tag{12.7}$$

and that Y_i $(i = 1, 2, \dots, k)$ are closed convex subsets of $C[T_* - i\tau, \infty)$. For each $y \in Y_i$, we define the functions $\Phi_i[y]$ $(i = 1, 2, \dots, k)$ on $[T_*, \infty)$ by

$$\Phi_i[y](t) = \begin{cases} \dfrac{y(t)}{1 + h(t)} + \dfrac{h(t)}{1 + h(t)} \Phi_{i-1}[\Delta[y]](t), & t \geq T - \tau, \\ \Phi_i[y](T - \tau), & t \in [T_*, T - \tau], \end{cases}$$
$$i = 1, 2, \dots, k.$$

The method of induction shows that, for every $i \in \{1, 2, \dots, k\}$, Φ_i is well-defined on Y_i and maps Y_i into $C[T_*, \infty)$, and continuous on Y_i in the $C[T_* - i\tau, \infty)$-topology and satisfies

$$\Phi_i[y](t) + h(t)\Phi_i[y](t-\tau) = y(t), \quad t \geq T, \quad y \in Y_i.$$

We need the following lemmas.

Lemma 12.1. *Let $y \in Y_k$. Assume that $\lim_{t\to\infty} y(t)/t^k = 0$. Then $\lim_{t\to\infty} \Phi_k[y](t)/t^k = 0$.*

Proof. We observe that

$$\begin{aligned}
\Phi_k[y](t) &= \frac{y(t)}{1+h(t)} + \frac{h(t)}{1+h(t)}\Phi_{k-1}[\Delta[y]](t) \\
&= \frac{y(t)}{1+h(t)} + \frac{h(t)}{[1+h(t)]^2}\Delta[y](t) + \left[\frac{h(t)}{1+h(t)}\right]^2 \Phi_{k-2}[\Delta^2[y]](t) \\
&= \frac{y(t)}{1+h(t)} + \frac{h(t)}{[1+h(t)]^2}\Delta[y](t) + \frac{[h(t)]^2}{[1+h(t)]^3}\Delta^2[y](t) \\
&\quad + \left[\frac{h(t)}{1+h(t)}\right]^3 \Phi_{k-3}[\Delta^3[y]](t) \\
&\ \ \vdots \\
&= \sum_{i=0}^{k-1} \frac{[h(t)]^i}{[1+h(t)]^{i+1}}\Delta^i[y](t) + \left[\frac{h(t)}{1+h(t)}\right]^k \Phi_0[\Delta^k[y]](t).
\end{aligned}$$

Since $\lim_{t\to\infty} y(t)/t^k = 0$, we have

$$\lim_{t\to\infty} \frac{\Delta^i[y](t)}{t^k} = 0, \quad i = 0, 1, 2, \ldots, k-1.$$

Using the fact that

$$\lim_{t\to\infty} \Phi_0[\Delta^k[y]](t) = 0,$$

we conclude that $\lim_{t\to\infty} \Phi_k[y](t)/t^k = 0$.

Lemma 12.2. *Suppose that $u \in C^k[t_1 - k\tau, \infty)$. Then, for every $t \in [t_1, \infty)$, there is a number $\alpha \in (t - k\tau, t)$ such that $\Delta^k[u](t) = \tau^k u^{(k)}(\alpha)$.*

Proof. Let $t \geq t_1$ be arbitrary. Note that $(\Delta^i[u])'(t) = (\Delta^i[u'])(t)$ for $i = 0, 1, 2, \ldots, k-1$. By the mean value theorem, there is a number $\alpha_1 \in (t - \tau, t)$ such that

$$\begin{aligned}\Delta^k[u](t) = \Delta^{k-1}[u](t) - \Delta^{k-1}[u](t-\tau) &= \tau(\Delta^{k-1}[u])'(\alpha_1) \\ &= \tau\Delta^{k-1}[u'](\alpha_1).\end{aligned}$$

In exactly the same way, we obtain

$$\Delta^{k-1}[u'](\alpha_1) = \tau\Delta^{k-2}[u''](\alpha_2)$$

for some $\alpha_2 \in (\alpha_1 - \tau, \alpha_1)$, and there are numbers $\alpha_3, \alpha_4, \ldots, \alpha_k$ such that $\alpha_i \in (\alpha_{i-1} - \tau, \alpha_{i-1})$ and

$$\Delta^{k-(i-1)}[u^{(i-1)}](\alpha_{i-1}) = \tau\Delta^{k-i}[u^{(i)}](\alpha_i)$$

for $i = 3, 4, \ldots, k$. Consequently, we have

$$\begin{aligned}\Delta^k[u](t) = \tau\Delta^{k-1}[u'](\alpha_1) = \tau^2\Delta^{k-2}[u''](\alpha_2) = \cdots& \\ \cdots = \tau^k\Delta^{k-k}[u^{(k)}](\alpha_k) = \tau^k u^{(k)}(\alpha_k).&\end{aligned}$$

Since

$$\begin{aligned}(\alpha_{k-1} - \tau, \alpha_{k-1}) \subset (\alpha_{k-2} - 2\tau, \alpha_{k-2}) \subset \cdots& \\ \cdots \subset (\alpha_1 - (k-1)\tau, \alpha_1) \subset (t - k\tau, t),&\end{aligned}$$

we see that $\alpha_k \in (t - k\tau, t)$. This completes the proof.

Lemma 12.3. *Let $\overline{T} \geq \overline{T}_*$. Suppose that $u \in C^k[\overline{T}_* - k\tau, \infty)$.*

(i) *If $u^{(k)}(t) \geq 0$ for $t \geq \overline{T} - k\tau$, then $\Delta^k[u](t) \geq 0$ for $t \geq \overline{T}$.*

(ii) *Assume that $v \in C[\overline{T}, \infty)$. If $u^{(k)}(t) \leq \tau^{-k}v(t+k\tau)$ for $t \geq \overline{T} - k\tau$ and $v(t)$ is nonincreasing on $[\overline{T}, \infty)$, then $\Delta^k[u](t) \leq v(t)$ for $t \geq \overline{T}$.*

(iii) *If $u^{(k)}(t)$ is nonincreasing on $[\overline{T} - k\tau, \infty)$, then $\Delta^k[u](t)$ is nonincreasing on $[\overline{T}, \infty)$.*

(iv) *If* $u^{(k)}(t) = u^{(k)}(\overline{T})$ *for* $t \in [\overline{T}_* - k\tau, \overline{T}]$, *then* $\Delta^k[u](t) = \Delta^k[u](\overline{T})$ *for* $t \in [\overline{T}_*, \overline{T}]$.

Proof. (i) The conclusion follows from Lemma 12.2.

(ii) Put

$$V(t) = \tau^{-k} \int_{\overline{T}}^{t+k\tau} \frac{(t + k\tau - s)^{k-1}}{(k-1)!} v(s)ds \quad \text{for } t \geq \overline{T} - k\tau$$

and $w(t) = V(t) - u(t)$ for $t \geq \overline{T} - k\tau$. Then $w^{(k)}(t) = \tau^{-k}v(t + k\tau) - u^{(k)}(t) \geq 0$ for $t \geq \overline{T} - k\tau$. In view of (i), we have $\Delta^k[w](t) \geq 0$ for $t \geq \overline{T}$. We note that $\Delta^k[w](t) = \Delta^k[V](t) - \Delta^k[u](t)$. Lemma 12.2 implies that

$$\Delta^k[u](t) \leq \Delta^k[V](t) = \tau^k V^{(k)}(\alpha) = v(\alpha + k\tau), \quad t \geq \overline{T},$$

for some $\alpha \in (t - k\tau, t)$. Since v is nonincreasing on $[\overline{T}, \infty)$, we get

$$\Delta^k[u](t) \leq v(t), \quad t \geq \overline{T}.$$

(iii) Let $\varepsilon > 0$ be arbitrary. Set $z(t) = u(t) - u(t + \varepsilon)$. Since $u^{(k)}(t)$ is nonincreasing on $[\overline{T} - k\tau, \infty)$, $z^{(k)}(t) = u^{(k)}(t) - u^{(k)}(t + \varepsilon) \geq 0$ for $t \geq \overline{T} - k\tau$. From (i), we obtain

$$\Delta^k[u](t) - \Delta^k[u](t + \varepsilon) = \Delta^k[z](t) \geq 0, \quad t \geq \overline{T}.$$

Consequently, $\Delta^k[u](t)$ is nonincreasing on $[\overline{T}, \infty)$.

(iv) Let $t \in [\overline{T}_*, \overline{T}]$. By virtue of Lemma 12.2, there is a number $\alpha \in (t - k\tau, t)$ such that $\Delta^k[u](t) = \tau^k u^{(k)}(\alpha)$. Since $u^{(k)}(s) = u^{(k)}(\overline{T})$ for $s \in [\overline{T}_* - k\tau, \overline{T}]$ and $\alpha \in [\overline{T}_* - k\tau, \overline{T}]$, we have $\Delta^k[u](t) = \tau^k u^{(k)}(\overline{T})$. In particular, $\Delta^k[u](\overline{T}) = \tau^k u^{(k)}(\overline{T})$. Hence, $\Delta^k[u](t) = \Delta^k[u](\overline{T})$ for $t \in [\overline{T}_*, \overline{T}]$.

Lemma 12.4. *Let* $u \in C^k[T_* - k\tau, \infty)$. *Assume that* $u^{(k)}(t)$ *is nonincreasing on* $[T - k\tau, \infty)$ *and satisfies*

$$0 \leq u^{(k)}(t) \leq \tau^{-k}\eta(t + k\tau), \quad t \geq T - k\tau,$$

and

$$u^{(k)}(t) = u^{(k)}(T), \quad t \in [T_* - k\tau, T].$$

Then $u \in Y_k$.

Proof. Applying Lemma 12.3, we easily see that $\Delta^k[u] \in Y_0$. From (12.7) it follows that $u \in Y_k$.

Proof of Theorem 12.1. The "only if" part follows immediately from Theorem 5.4. Moreover, as stated before, the case $k = 0$ is already shown in Theorem 11.1. We therefore prove the "if" part of Theorem 12.1 for $k \neq 0$.

Let $k \in \{1, 2, \ldots, n-1\}$. From (12.6), there are constants $c > 0$, $\delta > 0$ and $\varepsilon > 0$ such that

$$0 < (\delta + \varepsilon)t^k \leq c\,\omega_k(t) \leq (a - \varepsilon)t^k, \quad t \geq T_*.$$

Here, a is a number in the integral condition (12.5). For each $y \in Y_k$, we denote the function $\Psi[y](t)$ by

$$\Psi[y](t) = c\,\omega_k(t) + (-1)^{n-k-1}\sigma\Phi_k[y](t), \quad t \geq T_*. \tag{12.8}$$

We note here that Ψ is continuous on Y_k in the $C[T_* - k\tau, \infty)$-topology and that, for each $y \in Y_k$,

$$\Psi[y](t) + h(t)\Psi[y](t - \tau) = ct^k + (-1)^{n-k-1}\sigma y(t), \quad t \geq T. \tag{12.9}$$

Define the mapping $\mathcal{F} : Y_k \longrightarrow C[T_* - k\tau, \infty)$ as follows:

$$(\mathcal{F}y)(t)$$
$$= \begin{cases} \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \overline{f}(r, \Psi[y](g(r)))drds, & t \geq T, \\ \displaystyle\frac{(t-T)^k}{k!} \int_T^\infty \frac{(r-T)^{n-k-1}}{(n-k-1)!} \overline{f}(r, \Psi[y](g(r)))dr, & \\ & t \in [T_* - k\tau, T], \end{cases}$$

where

$$\overline{f}(t,u) = \begin{cases} f(t, a[g(t)]^k), & u \geq a[g(t)]^k, \\ f(t,u), & \delta[g(t)]^k \leq u \leq a[g(t)]^k, \\ f(t, \delta[g(t)]^k), & u \leq \delta[g(t)]^k. \end{cases}$$

We see that, for each $y \in Y_k$,

$$(\mathcal{F}y)^{(k)}(t)$$
$$= \begin{cases} \displaystyle\int_t^\infty \frac{(r-t)^{n-k-1}}{(n-k-1)!}\overline{f}(r, \Psi[y](g(r)))dr, & t \geq T, \\ \displaystyle\int_T^\infty \frac{(r-T)^{n-k-1}}{(n-k-1)!}\overline{f}(r, \Psi[y](g(r)))dr, & t \in [T_* - k\tau, T], \end{cases}$$

so that $(\mathcal{F}y)^{(k)}(t)$ is nonincreasing on $[T-k\tau, \infty)$ and satisfies

$$0 \leq (\mathcal{F}y)^{(k)}(t) \leq \tau^{-k}\eta(t+k\tau), \quad t \geq T - k\tau, \tag{12.10}$$

and

$$(\mathcal{F}y)^{(k)}(t) = (\mathcal{F}y)^{(k)}(T), \quad t \in [T_* - k\tau, T].$$

From Lemma 12.4 it follows that $\mathcal{F}y \in Y_k$ for every $y \in Y_k$, and hence $\mathcal{F}$ maps Y_k into itself. It is easy to verify that $\mathcal{F}$ is continuous on Y_k and $\{(\mathcal{F}y)(t) : y \in Y_k\}$ is relatively compact. By the Schauder-Tychonoff fixed point theorem, there exists an element $\widetilde{y} \in Y_k$ such that $\widetilde{y} = \mathcal{F}\widetilde{y}$. Set $x(t) = \Psi[\widetilde{y}](t)$. The inequality (12.10) and the fact $\lim_{t\to\infty} \eta(t) = 0$ lead us $\lim_{t\to\infty} \widetilde{y}(t)/t^k = 0$. In view of Lemma 12.1, (12.3) and (12.8), we find that $x(t)$ satisfies (12.4) and $\delta[g(t)]^k \leq x(g(t)) \leq a[g(t)]^k$ for all large t, say $t \geq \widetilde{T}$. We have $\overline{f}(t, x(g(t))) = f(t, x(g(t)))$ for $t \geq \widetilde{T}$. From (12.9) it follows that

$$x(t) + h(t)x(t-\tau)$$
$$= c\,t^k$$
$$+ (-1)^{n-k-1}\sigma \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!}\overline{f}(r, x(g(r)))drds$$

for $t \geq \widetilde{T}$. Differentiation of the above equality yields

$$\frac{d^n}{dt^n}[x(t) + h(t)x(t-\tau)] = -\sigma \overline{f}(t, x(g(t))) = -\sigma f(t, x(g(t))), \quad t \geq \widetilde{T}.$$

This means that $x(t)$ is a solution of (12.1). The proof of is complete.

13. Proof of Proposition 11.1

Throughout this section, we assume that (11.2) holds.

For each $y \in Y$, we define the function $\Psi[y]$ by

$$\Psi[y](t) = \begin{cases} \displaystyle\sum_{i=1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t)), & t \geq \tau(T), \\ \Psi[y](\tau(T)), & t \in [T_*, \tau(T)], \end{cases}$$

where $H(t) = \max\{1, h(t)\}$. We note that $H(\tau(t)) = H(t)$ and $H(t) \geq 1$ for $t \geq t_0$.

Lemma 13.1. (i) *For each $y \in Y$, the series*

$$\sum_{i=1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))$$

converges uniformly on $[\tau(T), \infty)$, hence $\Psi[y]$ is well defined and is continuous on $[T_*, \infty)$;

(ii) *For each $y \in Y$, $\Psi[y]$ satisfies*

$$0 \leq \Psi[y](t) \leq \eta(\tau^{-1}(t)), \quad t \geq \tau(T), \tag{13.1}$$

and

$$\Psi[y](t) + H(t)\Psi[y](\tau(t)) = y(t), \quad t \geq T; \tag{13.2}$$

(iii) *Ψ is continuous on Y in the $C[T_*, \infty)$-topology.*

Proof. (i) Let $y \in Y$. We set

$$\Psi_m[y](t) = \sum_{i=1}^{m}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t)), \quad t \geq \tau(T), \quad m = 1, 2, \ldots.$$

Now we claim that

$$(13.3) \qquad 0 \le \Psi_m[y](t) \le \eta(\tau^{-1}(t)), \quad t \ge \tau(T),$$

for $m = 1, 2, \dots$. Since y is nonincreasing on $[T, \infty)$ and $H(t) \ge 1$, we have

$$(13.4) \qquad y(\tau^{-1}(t)) - [H(t)]^{-1} y(\tau^{-2}(t)) \ge 0, \quad t \ge \tau(T),$$

and

$$(13.5) \qquad [H(t)]^{-1} y(\tau^{-1}(t)) \le \eta(\tau^{-1}(t)), \quad t \ge \tau(T).$$

Hence, we easily see that (13.3) holds for the cases $m = 1$ and 2. If $m \ge 3$ is odd, we can rewrite $\Psi_m[y](t)$ as

$$\begin{aligned}\Psi_m[y](t) = &\sum_{j=1}^{(m-1)/2} [H(t)]^{-(2j-1)}[y(\tau^{-(2j-1)}(t)) - [H(t)]^{-1} y(\tau^{-2j}(t))] \\ &+ [H(t)]^{-m} y(\tau^{-m}(t))\end{aligned}$$

and

$$\begin{aligned}\Psi_m[y](t) &= [H(t)]^{-1} y(\tau^{-1}(t)) \\ &\quad - \sum_{j=1}^{(m-1)/2} [H(t)]^{-2j}[y(\tau^{-2j}(t)) - [H(t)]^{-1} y(\tau^{-(2j+1)}(t))].\end{aligned}$$

If $m \ge 4$ is even, we can rewrite $\Psi_m[y](t)$ as

$$\Psi_m[y](t) = \sum_{j=1}^{m/2} [H(t)]^{-(2j-1)}[y(\tau^{-(2j-1)}(t)) - [H(t)]^{-1} y(\tau^{-2j}(t))],$$

and

$$\begin{aligned}\Psi_m[y](t) &= [H(t)]^{-1} y(\tau^{-1}(t)) \\ &\quad - \sum_{j=1}^{(m/2)-1} [H(t)]^{-2j}[y(\tau^{-2j}(t)) - [H(t)]^{-1} y(\tau^{-(2j+1)}(t))] \\ &\quad - [H(t)]^{-m} y(\tau^{-m}(t)).\end{aligned}$$

From (13.4) and (13.5) we conclude that (13.3) holds for $m = 3, 4, \dots$.

Using (13.3), we find that if $m \geq p \geq 1$, then

$$(13.6) \qquad \left|\sum_{i=p}^{m}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))\right|$$
$$= \left|\sum_{i=1}^{m-p+1}(-1)^{(i+p-1)+1}[H(t)]^{-(i+p-1)}y(\tau^{-i}(\tau^{-p+1}(t)))\right|$$
$$= |(-1)^{(p-1)}[H(t)]^{-(p-1)}\Psi_{m-p+1}[y](\tau^{-p+1}(t))|$$
$$\leq \eta(\tau^{-p}(t)), \quad t \geq \tau(T).$$

Here, we have used the equality $H(t) = H(\tau^{-p+1}(t))$, $p \geq 1$. Since $\eta(\tau^{-p}(t)) \to 0$ as $p \to \infty$, the series $\sum_{i=1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))$ converges for each fixed $t \in [\tau(T), \infty)$. From (13.6) it follows that

$$\sup_{t\in[\tau(T),\infty)}\left|\sum_{i=p}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))\right|$$
$$\leq \sup_{t\in[\tau(T),\infty)} \eta(\tau^{-p}(t)) = \sup_{t\in[\tau^{-p+1}(T),\infty)} \eta(t) \to 0 \quad \text{as } p \to \infty,$$

which shows that the series $\sum_{i=1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))$ converges uniformly on $[\tau(T), \infty)$.

(ii) Letting $m \to \infty$ in (13.3), we have (13.1). It is easy to check that (13.2) holds.

(iii) Let $\varepsilon > 0$. There is an integer $p \geq 1$ such that

$$\sup_{t\in[\tau(T),\infty)} \eta(\tau^{-(p+1)}(t)) = \sup_{t\in[\tau^{-p}(T),\infty)} \eta(t) < \frac{\varepsilon}{3}.$$

Let $\{y_j\}_{j=1}^{\infty}$ be a sequence in Y converging to $y \in Y$ uniformly on every compact subinterval of $[T_*, \infty)$. Take an arbitrary compact subinterval I of $[\tau(T), \infty)$. There exists an integer $j_0 \geq 1$ such that

$$\sum_{i=1}^{p}|y_j(\tau^{-i}(t)) - y(\tau^{-i}(t))| < \frac{\varepsilon}{3}, \quad t \in I, \quad j \geq j_0.$$

It follows from (13.6) that

$$\begin{aligned}
&|\Psi[y_j](t) - \Psi[y](t)| \\
&\le \sum_{i=1}^{p}[H(t)]^{-i}|y_j(\tau^{-i}(t)) - y(\tau^{-i}(t))| \\
&\quad + \left|\sum_{i=p+1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y_j(\tau^{-i}(t))\right| + \left|\sum_{i=p+1}^{\infty}(-1)^{i+1}[H(t)]^{-i}y(\tau^{-i}(t))\right| \\
&\le \sum_{i=1}^{p}|y_j(\tau^{-i}(t)) - y(\tau^{-i}(t))| + 2\eta(\tau^{-(p+1)}(t)) < \varepsilon, \quad t \in I, \quad j \ge j_0,
\end{aligned}$$

which implies that $\Psi[y_j]$ converges $\Psi[y]$ uniformly on I. It is easy to see that $\Psi[y_j] \to \Psi[y]$ uniformly on $[T_*, \tau(T)]$. Consequently, we conclude that Ψ is continuous on Y. This completes the proof.

For each $y \in Y$, we assign the function $\varphi[y]$ as follows:

$$\varphi[y](t) = \begin{cases} \dfrac{y(T)}{1+h(T)} & \text{if } h(T) < 1, \\ \Psi[y](t) & \text{if } h(T) \ge 1, \end{cases} \qquad t \in [T_*, T].$$

Lemma 13.2. (i) *For each $y \in Y$, $\varphi[y]$ satisfies*

$$\varphi[y](T) + h(T)\varphi[y](\tau(T)) = y(T);$$

(ii) *Suppose that $\{y_j\}_{j=1}^{\infty}$ is a sequence in Y converging to $y \in Y$ uniformly on every compact subinterval of $[T_*, \infty)$. Then $\varphi[y_j]$ converges to $\varphi[y]$ uniformly on $[T_*, T]$.*

Proof. It is obvious that (i) and (ii) hold for the case $h(T) < 1$. For the case $h(T) \ge 1$, (i) and (ii) follow from (ii) and (iii) of Lemma 13.1.

For each $y \in Y$, we define the function $\Phi[y]$ as follows:

$$\Phi[y](t) = \begin{cases} \displaystyle\sum_{i=0}^{m}(-1)^i[h(t)]^i y(\tau^i(t)) + (-1)^{m+1}[h(t)]^{m+1}\varphi[y](\tau^{m+1}(t)), \\ \qquad\qquad t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)], \quad m = 0, 1, \dots, \\ \varphi[y](t), \qquad\qquad t \in [T_*, T]. \end{cases}$$

Lemma 13.3. *Let $y \in Y$.*

(i) $\Phi[y]$ *is continuous on* $[T_*, \infty)$;

(ii) $\Phi[y]$ *satisfies*

$$\Phi[y](t) + h(t)\Phi[y](\tau(t)) = y(t), \quad t \geq T;$$

(iii) *For* $t \in [\tau(T), \infty)$ *with* $h(t) \geq 1$,

$$\Phi[y](t) = \Psi[y](t);$$

(iv) Φ *is continuous on* Y *in the* $C[T_*, \infty)$*-topology.*

Proof. From Lemmas 2.1 and (i) of 13.2 it follows that (i) and (ii) hold. We prove (iii) and (iv).

(iii) If $h(T) < 1$, then there is no number $t \in [\tau(T), \infty)$ such that $h(t) \geq 1$ (recall the choice of T). Assume that $h(T) \geq 1$. Then

$$\Phi[y](t) = \varphi[y](t) = \Psi[y](t) \quad \text{for } t \in [\tau(T), T].$$

We suppose that there is an integer $m \geq 0$ such that $\Phi[y](t) = \Psi[y](t)$ for all $t \in [\tau^{-(m-1)}(T), \tau^{-m}(T)]$ with $h(t) \geq 1$. In view of (ii) of Lemma 13.3 and (13.2), we find that if $t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)]$ and if $h(t) \geq 1$, then

$$\Phi[y](t) = y(t) - h(t)\Phi[y](\tau(t)) = y(t) - H(t)\Psi[y](\tau(t)) = \Psi[y](t).$$

By induction, we conclude that $\Phi[y](t) = \Psi[y](t)$ for $t \in [\tau(T), \infty)$ with $h(t) \geq 1$.

(iv) Let $\{y_j\}_{j=1}^{\infty}$ be a sequence in Y converging to $y \in Y$ uniformly on every compact subinterval of $[T_*, \infty)$. Lemma 13.2 implies that $\Phi[y_j]$ converges to $\Phi[y]$ uniformly on $[T_*, T]$. It suffices to prove that $\Phi[y_j] \to \Phi[y]$ uniformly on $I_m \equiv [\tau^{-m}(T), \tau^{-(m+1)}(T)]$, $m = 0, 1, 2, \ldots$.

Since $|h(t)| \leq \lambda$ on $[t_0, \infty)$ for some $\lambda \geq 1$, we observe that

$$\begin{aligned}
&\sup_{t \in I_m} |\Phi[y_j](t) - \Phi[y](t)| \\
&\leq \sum_{i=0}^{m} \lambda^i \sup_{t \in I_m} |y_j(\tau^i(t)) - y(\tau^i(t))| \\
&\quad + \lambda^{m+1} \sup_{t \in I_m} |\varphi[y_j](\tau^{m+1}(t)) - \varphi[y](\tau^{m+1}(t))| \\
&\leq \lambda^m \sum_{i=0}^{m} \sup_{t \in I_{m-i}} |y_j(t) - y(t)| + \lambda^{m+1} \sup_{t \in [T_*, T]} |\varphi[y_j](t) - \varphi[y](t)|.
\end{aligned}$$

Then, $\sup_{t \in I_m} |\Phi[y_j](t) - \Phi[y](t)| \to 0$ as $j \to \infty$, so that $\Phi[y_j]$ converges to $\Phi[y]$ uniformly on I_m for $m = 0, 1, 2, \dots$.

Lemma 13.4. *Let $\{t_j\}_{j=0}^{\infty}$ be a sequence satisfying $\lim_{j \to \infty} t_j = \infty$ and $|h(t_j)| \leq \nu < 1$, $j = 1, 2, \dots$ for some $\nu > 0$. Then*

$$\lim_{j \to \infty} \Phi[y](t_j) = 0 \quad \text{for each } y \in Y.$$

Proof. Let $y \in Y$. Since $\lim_{t \to \infty} y(t) = 0$, for each $\varepsilon > 0$, there is an integer $p \geq 1$ such that

$$\frac{y(\tau^{-p}(T))}{1 - \nu} < \frac{\varepsilon}{3}.$$

There exists an integer $q \geq 1$ such that

$$\frac{y(T)\nu^{r-p+1}}{1 - \nu} < \frac{\varepsilon}{3} \quad \text{and} \quad \nu^{r+1} \sup_{t \in [T_*, T]} |\varphi[y](t)| < \frac{\varepsilon}{3} \quad \text{for all } r \geq p + q.$$

Let $m \geq p + q$. Then $\tau^{m-p}(t) \geq \tau^{-p}(T)$ for $t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)]$. In view of the monotonicity of y, we see that if $t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)]$

and $|h(t)| \le \nu$, then

$$\begin{aligned}
|\Phi[y](t)| &\le \sum_{i=0}^{m} \nu^i y(\tau^i(t)) + \nu^{m+1} |\varphi[y](\tau^{m+1}(t))| \\
&\le \sum_{i=0}^{m-p} \nu^i y(\tau^i(t)) + \sum_{i=m-p+1}^{m} \nu^i y(\tau^i(t)) + \frac{\varepsilon}{3} \\
&\le y(\tau^{m-p}(t)) \sum_{i=0}^{m-p} \nu^i + y(T)\nu^{m-p+1} \sum_{i=0}^{p-1} \nu^i + \frac{\varepsilon}{3} \\
&\le \frac{y(\tau^{-p}(T))}{1-\nu} + \frac{y(T)\nu^{m-p+1}}{1-\nu} + \frac{\varepsilon}{3} < \varepsilon.
\end{aligned}$$

This implies that $|\Phi[y](t)| < \varepsilon$ for $t \in [\tau^{-(p+q)}(T), \infty)$ with $|h(t)| \le \nu$ and hence the conclusion follows.

Lemma 13.5. *Let $m = 0, 1, 2, \ldots$. If t satisfies $t \ge \tau^{-m}(T)$ and $0 \le h(t) \le 1$, then*

$$\left| \sum_{i=0}^{m} (-1)^i [h(t)]^i y(\tau^i(t)) \right| \le 2y(\tau^m(t)), \quad y \in Y. \tag{13.7}$$

Proof. Let $t \ge \tau^{-m}(T)$ satisfying $0 \le h(t) \le 1$ and let $y \in Y$. Put

$$A(t) \equiv \sum_{i=0}^{m} (-1)^i [h(t)]^i y(\tau^i(t)).$$

It is easy to see that (13.7) holds for $m = 0$ and 1. If $m \ge 3$ is odd, we can rewrite $A(t)$ as

$$\begin{aligned}
A(t) = y(t) - &\sum_{j=1}^{(m-1)/2} [h(t)]^{2j-1} [y(\tau^{2j-1}(t)) - h(t) y(\tau^{2j}(t))] \\
&- [h(t)]^m y(\tau^m(t)),
\end{aligned}$$

and

$$A(t) = \sum_{j=0}^{(m-1)/2} [h(t)]^{2j} [y(\tau^{2j}(t)) - h(t) y(\tau^{2j+1}(t))].$$

If $m \ge 2$ is even, we can rewrite $A(t)$ as

$$A(t) = y(t) - \sum_{j=1}^{m/2} [h(t)]^{2j-1} [y(\tau^{2j-1}(t)) - h(t) y(\tau^{2j}(t))],$$

and

$$A(t) = \sum_{j=0}^{(m/2)-1} [h(t)]^{2j}[y(\tau^{2j}(t)) - h(t)y(\tau^{2j+1}(t))] + [h(t)]^m y(\tau^m(t)).$$

Since y is nonincreasing on $[T, \infty)$, we see that

$$y(t) - h(t)y(\tau(t)) \leq [1 - h(t)]y(t), \quad t \geq \tau^{-1}(T).$$

Hence, for the case where $m \geq 3$ is odd, we have

$$\begin{aligned} A(t) &\geq - \sum_{j=1}^{(m-1)/2} [h(t)]^{2j-1}[1 - h(t)]y(\tau^{2j-1}(t)) - [h(t)]^m y(\tau^m(t)) \\ &\geq - \sum_{j=1}^{(m-1)/2} [h(t)]^{2j-1}[1 - h(t)]y(\tau^m(t)) - [h(t)]^m y(\tau^m(t)) \\ &= y(\tau^m(t)) \sum_{i=1}^{m} (-1)^i [h(t)]^i \\ &= - y(\tau^m(t))h(t)\frac{1 - [-h(t)]^m}{1 + h(t)} \geq -2y(\tau^m(t)). \end{aligned}$$

In the same way, we can show that $A(t) \leq 2y(\tau^m(t))$ for the case where $m \geq 3$ is odd, and that $-2y(\tau^m(t)) \leq A(t) \leq 2y(\tau^m(t))$ for the case where $m \geq 2$ is even.

Lemma 13.6. *Let $y \in Y$. Then $\lim_{t\to\infty} \Phi[y](t) = 0$.*

Proof. Assume that $\lim_{t\to\infty} \Phi[y](t) = 0$ does not hold. Then we first claim that there is a sequence $\{t_j\}_{j=1}^{\infty}$ such that

(13.8)

$$\begin{cases} \lim\limits_{j\to\infty} t_j = \infty, \quad \lim\limits_{j\to\infty} \Phi[y](t_j) \quad \text{exists in } \mathbb{R} \cup \{\infty, -\infty\} \backslash \{0\}, \\ 0 < h(t_j) < 1 \quad \text{for } j \geq 1 \quad \text{and} \quad \lim\limits_{j\to\infty} h(t_j) = 1. \end{cases}$$

By assumption there is a sequence $\{s_j\}_{j=1}^{\infty}$ for which $s_j \to \infty$ and $\Phi[y](s_j) \to c \in \mathbb{R} \cup \{\infty, -\infty\} \backslash \{0\}$ as $j \to \infty$. Since $-1 < \mu \leq h(t) \leq \lambda$ for $t \geq t_0$, there is a subsequence $\{t_j\}_{j=1}^{\infty}$ of $\{s_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} h(t_j) = d \in [\mu, \lambda]$. Lemma 13.4 implies that $d \geq 1$. It can be shown that $h(t_j) < 1$, $j \geq j_0$ for some j_0. Otherwise, there exists a subsequence $\{\widetilde{t_j}\}_{j=1}^{\infty}$ of $\{t_j\}_{j=1}^{\infty}$ such that $h(\widetilde{t_j}) \geq 1$ for all j. From (iii) of Lemma 13.3 and (ii) of Lemma 13.1, it follows that

$$|c| = \left|\lim_{j\to\infty} \Phi[y](\widetilde{t_j})\right| = \left|\lim_{j\to\infty} \Psi[y](\widetilde{t_j})\right| \leq \lim_{j\to\infty} \eta(\tau^{-1}(\widetilde{t_j})) = 0,$$

which is a contradiction. Since $d \geq 1$, we see that $d = 1$, so that $0 < h(t_j) < 1$, $j \geq j_1$ for some $j_1 \geq j_0$. This proves the existence of $\{t_j\}_{j=1}^{\infty}$ satisfying (13.8).

Suppose that $\{t_j\}_{j=1}^{\infty}$ is a sequence satisfying (13.8). Let $\varepsilon > 0$ be arbitrary. There is an integer $p \geq 1$ such that

$$\eta(t) < \varepsilon, \quad t \geq \tau^{-p-1}(T).$$

There is a number $\delta > 0$ such that if $s_1, s_2 \in [\tau^{-p}(T), \tau^{-(p+1)}(T)]$ with $|s_1 - s_2| < \delta$, then

$$|\Phi[y](s_1) - \Phi[y](s_2)| < \varepsilon. \tag{13.9}$$

Consider the mapping $N : [\tau^{-p}(T), \infty) \longrightarrow \mathbb{N}\cup\{0\}$ such that

$$\tau^{N(t)}(t) \in [\tau^{-p}(T), \tau^{-(p+1)}(T)) \quad \text{for } t \geq \tau^{-p}(T).$$

We note that $\lim_{t\to\infty} N(t) = \infty$. It is easily verified that $\{t_j\}_{j=1}^{\infty}$ has a subsequence $\{u_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} \tau^{N(u_j)}(u_j) \quad \text{exists in } [\tau^{-p}(T), \tau^{-(p+1)}(T)].$$

Put $\overline{u} = \lim_{j\to\infty} \tau^{N(u_j)}(u_j)$. Then we find that

$$h(\overline{u}) = \lim_{j\to\infty} h(\tau^{N(u_j)}(u_j)) = \lim_{j\to\infty} h(u_j) = 1.$$

There exists an integer j_0 such that $u_j \geq \tau^{-p}(T)$ and $|\tau^{N(u_j)}(u_j) - \overline{u}| < \delta$ for $j \geq j_0$. From (ii) of Lemma 13.3, we observe that

$$\begin{aligned} \Phi[y](t) &= y(t) - h(t)\Phi[y](\tau(t)) \\ &= y(t) - h(t)y(\tau(t)) + [h(t)]^2\Phi[y](\tau^2(t)) \\ &= \sum_{i=0}^{m-1}(-1)^i[h(t)]^i y(\tau^i(t)) + (-1)^m[h(t)]^m\Phi[y](\tau^m(t)), \end{aligned} \tag{13.10}$$

for $t \geq \tau^{-m+1}(T)$. Since $h(\overline{u}) = 1$, we have

(13.11)

$$
\begin{aligned}
&|\Phi[y](u_j) - \Phi[y](\tau^{-N(u_j)}(\overline{u}))| \\
&\leq \left| \sum_{i=0}^{N(u_j)-1} (-1)^i [h(u_j)]^i y(\tau^i(u_j)) \right| + \left| \sum_{i=0}^{N(u_j)-1} (-1)^i y(\tau^i(\tau^{-N(u_j)}(\overline{u}))) \right| \\
&\quad + \left| [h(u_j)]^{N(u_j)} \Phi[y](\tau^{N(u_j)}(u_j)) - \Phi[y](\tau^{N(u_j)}(\tau^{-N(u_j)}(\overline{u}))) \right|.
\end{aligned}
$$

Lemma 13.5 implies that if $j \geq j_0$, then

$$
\begin{aligned}
(13.12) \qquad \left| \sum_{i=0}^{N(u_j)-1} (-1)^i [h(u_j)]^i y(\tau^i(u_j)) \right| &\leq 2y(\tau^{N(u_j)-1}(u_j)) \\
&\leq 2\eta(\tau^{N(u_j)-1}(u_j)) < 2\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
(13.13) \qquad \left| \sum_{i=0}^{N(u_j)-1} (-1)^i y(\tau^i(\tau^{-N(u_j)}(\overline{u}))) \right| &\leq 2y(\tau^{N(u_j)-1}(\tau^{-N(u_j)}(\overline{u}))) \\
&\leq 2\eta(\tau^{-1}(\overline{u})) < 2\varepsilon.
\end{aligned}
$$

From (iii) of Lemma 13.3, (ii) of Lemma 13.1 and the fact that $h(\overline{u}) = 1$, it follows that

$$|\Phi[y](\overline{u})| = |\Psi[y](\overline{u})| \leq \eta(\tau^{-1}(\overline{u})) < \varepsilon.$$

Then we observe that

$$
\begin{aligned}
(13.14) \qquad &|[h(u_j)]^{N(u_j)} \Phi[y](\tau^{N(u_j)}(u_j)) - \Phi[y](\tau^{N(u_j)}(\tau^{-N(u_j)}(\overline{u})))| \\
&\leq |[h(u_j)]^{N(u_j)}| |\Phi[y](\tau^{N(u_j)}(u_j)) - \Phi[y](\overline{u})| \\
&\qquad + |[h(u_j)]^{N(u_j)} - 1| |\Phi[y](\overline{u})| \\
&\leq |\Phi[y](\tau^{N(u_j)}(u_j)) - \Phi[y](\overline{u})| + 2|\Phi[y](\overline{u})| < 3\varepsilon, \quad j \geq j_0,
\end{aligned}
$$

because of (13.9). Combining (13.11)–(13.14), we obtain

$$|\Phi[y](u_j) - \Phi[y](\tau^{-N(u_j)}(\overline{u}))| < 7\varepsilon, \quad j \geq j_0.$$

This means that

$$\lim_{j\to\infty} |\Phi[y](u_j) - \Phi[y](\tau^{-N(u_j)}(\overline{u}))| = 0.$$

On the other hand, in view of (iii) of Lemma 13.3 and (ii) of Lemma 13.1, we see that

$$\lim_{j\to\infty} |\Phi[y](\tau^{-N(u_j)}(\overline{u}))| \le \lim_{j\to\infty} \eta(\tau^{-N(u_j)-1}(\overline{u})) = 0.$$

From (13.8) it follows that

$$\lim_{j\to\infty} |\Phi[y](u_j) - \Phi[y](\tau^{-N(u_j)}(\overline{u}))| \quad \text{exists and is not equal to } 0.$$

This is a contradiction. The proof is complete.

Proposition 11.1 follows from Lemmas 13.3 and 13.6.

14. Bibliographic notes

Sections 2–5 and 10 are based on [57]. Lemmas 2.1, 2.2 and 2.10 were obtained by Y. Naito [45]. The proofs of Theorems 4.1 and 4.2 are extended adaptations of the method introduced by Ruan [50]. Theorem 5.1 is a generalization of the results of Jaroš and Kusano [28], [29], [30] and [32], and Y. Naito [45]. Theorem 5.2 is an extension of the results of Chen, Yu and Wang [3], Chen [4] and Y. Naito [45]. Section 6 is taken from Y. Naito [47], and a related work can be found in [48]. Sections 7–9 are based on [58]. The existence of oscillatory solutions of neutral differential equations was first investigated by Jaroš and Kusano [33]. Corollary 7.1 is an improvement of the result in [33]. Corollary 7.3 was obtained by M. Naito [44]. Corollaries 8.1 and 8.2 were established by Kitamura and Kusano [35]. The proof of Theorem 8.3 is due to Zhang and Yang [66], see also Yang and Zhang [63] and Zhang and Yu [67]. Theorem 9.1 (i) extends the result in [33]. Theorem 9.1 (ii) was obtained by Kitamura and Kusano [35], see also Jaroš, Kitamura and Kusano [27], Kitamura, Kusano and Lalli [36], and Y. Naito [47]. Section 11 is due to [54] and [55]. Section 12 is based on [56].

References

[1] S. J. Bilchev, M. K. Grammatikopoulos and I. P. Stavroulakis, Oscillations of higher order neutral differential equations, J. Austral. Math. Soc. Ser. A **52** (1992), 261–284.

[2] R. K. Brayton, Nonlinear oscillations in a distributed network, Quart. Appl. Math. **24** (1967), 289–301.

[3] M. P. Chen, J. S. Yu and Z. C. Wang, Nonoscillatory solutions of neutral delay differential equations, Bull. Austral. Math. Soc. **48** (1993), 475–483.

[4] Y. Chen, Existence of nonoscillatory solutions of nth order neutral delay differential equations, Funkcial. Ekvac. **35** (1992), 557–570.

[5] Q. Chuanxi and G. Ladas, Oscillations of neutral differential equations with variable coefficients, Appl. Anal. **32** (1989), 215–228.

[6] Q. Chuanxi and G. Ladas, Oscillations of higher order neutral differential equations with variable coefficients, Math. Nacher. **150** (1991), 15–24.

[7] Q. Chuanxi, G. Ladas, B. G. Zhang and T. Zhao, Sufficient conditions for oscillation and existence of positive solutions, Appl. Anal. **35** (1990), 187–194.

[8] L. H. Erbe, Q. Kong and B. G. Zhang, Oscillation theory for functional-differential equations, Marcel Dekker, Inc., New York, 1995.

[9] L. H. Erbe and J. S. Yu, Linearized oscillations for neutral equations I: Odd order, Hiroshima Math. J. **26** (1996), 557–572.

[10] L. H. Erbe and J. S. Yu, Linearized oscillations for neutral equations II: Even order, Hiroshima Math. J. **26** (1996), 573–585.

[11] D. A. Georgiou and C. Qian, Oscillation criteria in neutral equations of n order with variable coefficients, Internat. J. Math. Math. Sci. **14** (1991), 689–696.

[12] K. Gopalsamy and B. G. Zhang, Oscillation and nonoscillation in first order neutral differential equations, J. Math. Anal. Appl. **151** (1990), 42–57.

[13] J. R. Graef, M. K. Grammatikopoulos and P. W. Spikes, Asymptotic properties of solutions of nonlinear neutral delay differential equations of the second order, Rad. Mat. **4** (1988), 133–149.

[14] J. R. Graef, M. K. Grammatikopoulos and P. W. Spikes, Asymptotic and oscillatory behavior of solutions of first order nonlinear neutral delay differential equations, J. Math. Anal. Appl. **155** (1991), 562–571.

[15] J. R. Graef, M. K. Grammatikopoulos and P. W. Spikes, On the asymptotic behavior of solutions of a second order nonlinear neutral delay differential equation, J. Math. Anal. Appl. **156** (1991), 23–39.

[16] J. R. Graef and P. W. Spikes, On the oscillation of an nth-order nonlinear neutral delay differential equation, J. Comput. Appl. Math. **41** (1992), 35–40.

[17] J. R. Graef, P. W. Spikes and M. K. Grammatikopoulos, On the behavior of solutions of a first order nonlinear neutral delay differential equation, Appl. Anal. **40** (1991), 111–121.

[18] M. K. Grammatikopoulos, E. A. Grove and G. Ladas, Oscillations of first-order neutral delay differential equations, J. Math. Anal. Appl. **120** (1986), 510–520.

[19] M. K. Grammatikopoulos, G. Ladas, and A. Meimaridou, Oscillations of second order neutral delay differential equations, Rad. Mat. **1** (1985), 267–274.

[20] M. K. Grammatikopoulos, G. Ladas, and A. Meimaridou, Oscillation and asymptotic behavior of second order neutral differential equations, Ann. Mat. Pura Appl. **148** (1987), 29–40.

[21] M. K. Grammatikopoulos, G. Ladas, and A. Meimaridou, Oscillation and asymptotic behavior of higher order neutral equations with variable coefficients, Chin. Ann. of Math. **9B** (1988), 322–338.

[22] M. K. Grammatikopoulos, G. Ladas and Y. G. Sficas, Oscillation and asymptotic behavior of neutral equations with variable coefficients, Rad. Mat. **2** (1986), 279–303.

[23] E. A. Grove, M. R. S. Kulenović and G. Ladas, Sufficient conditions for oscillation and nonoscillation of neutral equations, J. Differential Equations **68** (1987), 373–382.

[24] I. Györi and G. Ladas, Oscillation theory of delay differential equations, Oxford University Press, 1991.

[25] J. K. Hale, Theory of Functional Differential Equations, Springer Verlag, New York, 1977.

[26] J. Jaroš, An oscillation test for a class of linear neutral differential equations, J. Math. Anal. Appl. **159** (1991), 406–417.

[27] J. Jaroš, Y. Kitamura and T. Kusano, On a class of functional differential equations of neutral type, WSSIAA **1** (1992), 317–333.

[28] J. Jaroš and T. Kusano, Oscillation theory of higher order linear functional differential equations of neutral type, Hiroshima Math. J. **18** (1988), 509–531.

[29] J. Jaroš and T. Kusano, Asymptotic behavior of nonoscillatory solutions of nonlinear functional differential equations of neutral type, Funkcial. Ekvac. **32** (1989), 251–263.

[30] J. Jaroš and T. Kusano, On a class of first order nonlinear functional differential equations of neutral type, Czechoslovak Math. J. **40** (1990), 475–490.

[31] J. Jaroš and T. Kusano, On oscillation of linear neutral differential equations of higher order, Hiroshima Math. J. **20** (1990), 407–419.

[32] J. Jaroš and T. Kusano, Oscillation properties of first order nonlinear functional differential equations of neutral type, Differential Integral Equations **4** (1991), 425–436.

[33] J. Jaroš and T. Kusano, Existence of oscillatory solutions for functional differential equations of neutral type, Acta Math. Univ. Comenianae **60** (1991), 185–194.

[34] Y. Kitamura and T. Kusano, Oscillation and asymptotic behavior of solutions of first-order functional differential equations of neutral type, Funkcial. Ekvac. **33** (1990), 325–343.

[35] Y. Kitamura and T. Kusano, Existence theorems for a neutral functional differential equation whose leading part contains a difference operator of higher degree, Hiroshima Math. J. **25** (1995), 53–82.

[36] Y. Kitamura, T. Kusano and B. S. Lalli, Existence of oscillatory and nonoscillatory solutions for a class of neutral functional differential equations, Math. Bohemica **120** (1995), 57–69.

[37] G. Ladas and Y. G. Sficas, Oscillations of neutral delay differential equations, Canad. Math. Bull. **29** (1986), 438–445.

[38] G. Ladas, C. G. Philos and Y. G. Sficas, Oscillations is neutral equations with periodic coefficients, Proc. Amer. Math. Soc. **113** (1991), 123–134.

[39] H. J. Li, Oscillatory theorems for second-order neutral delay differential equations, Nonlinear Anal. **26** (1996), 1397–1409.

[40] O. Lopes, Forced oscillations in nonlinear neutral differential equations, SIAM J. Appl. Math. **29** (1975), 196–207.

[41] O. Lopes, Stability and forced oscillations, J. Math. Anal. Appl. **55** (1976), 686–698.

[42] W. D. Lu, Existence of nonoscillatory solutions of first order nonlinear neutral equations, J. Austral. Math. Soc. Ser. B **32** (1990), 180–192.

[43] W. D. Lu, Nonoscillation and oscillation of first order neutral equations with variable coefficients, J. Math. Anal. Appl. **181** (1994), 803–815.

[44] M. Naito, An asymptotic theorem for a class of nonlinear neutral differential equations, Czechoslovak Math. J. **48** (1998), 419–432.

[45] Y. Naito, Nonoscillatory solutions of neutral differential equations, Hiroshima Math. J. **20** (1990), 231–258.

[46] Y. Naito, Asymptotic behavior of decaying nonoscillatory solutions of neutral differential equations, Funkcial. Ekvac. **35** (1992), 95–110.

[47] Y. Naito, Existence and asymptotic behavior of positive solutions of neutral differential equations, J. Math. Anal. Appl. **188** (1994), 227–244.

[48] Y. Naito, A note on the existence of nonoscillatory solutions of neutral differential equations, Hiroshima Math. J. **25** (1995), 513–518.

[49] O. Quijada and A. Tineo, Oscillatory and nonoscillatory results for first order neutral differential equations, J. Math. Anal. Appl. **180** (1993), 37–42.

[50] J. Ruan, Type and criteria of nonoscillatory solutions for second order linear neutral differential equations, Chinese Ann. Math. Ser. A **8** (1987), 114–124. (Chinese)

[51] M. Slemrod, Nonexistence of oscillations in a nonlinear distributed network, J. Math. Anal. Appl. **36** (1971), 22–40.

[52] S. Tanaka, Oscillation properties of solutions of second order neutral differential equations with deviating arguments, Analysis **17** (1997) 99–111.

[53] S. Tanaka, Forced oscillations of first order nonlinear neutral differential equations, J. Appl. Anal. **3** (1997) 23–41.

[54] S. Tanaka, Existence of positive solutions for a class of first-order neutral functional differential equations, J. Math. Anal. Appl. **229** (1999), 501–518.

[55] S. Tanaka, Existence of positive solutions for a class of higher order neutral functional differential equations, preprint.

[56] S. Tanaka, Existence of positive solutions of higher order nonlinear neutral differential equations. (in preparation)

[57] S. Tanaka, Existence and asymptotic behavior of solutions of nonlinear neutral differential equations. (in preparation)

[58] S. Tanaka, Existence of oscillatory solutions and nonoscillatory solutions of neutral differential equations. (in preparation)

[59] S. Tanaka, Oscillation of solutions of first order neutral differential equations. (in preparation)

[60] S. Tanaka, Oscillation of solutions of higher order neutral differential equations. (in preparation)

[61] S. Tanaka and N. Yoshida, Oscillations of solutions to parabolic equations with deviating arguments, Tamkang J. Math. **28** (1997) 169–181.

[62] X. H. Tang and J. H. Shen, Oscillation and existence of positive solutions in a class of higher order neutral equations, J. Math. Anal. Appl. **213** (1997), 662–680.

[63] B. Yang and B. G. Zhang, Qualitative analysis of a class of neutral differential equations, Funkcial. Ekvac. **39** (1996), 347–362.

[64] J. S. Yu and M. P. Chen, Oscillation in neutral equations with an "integrally small" coefficient, Internat. J. Math. Math. Sci. **17** (1994), 361–368.

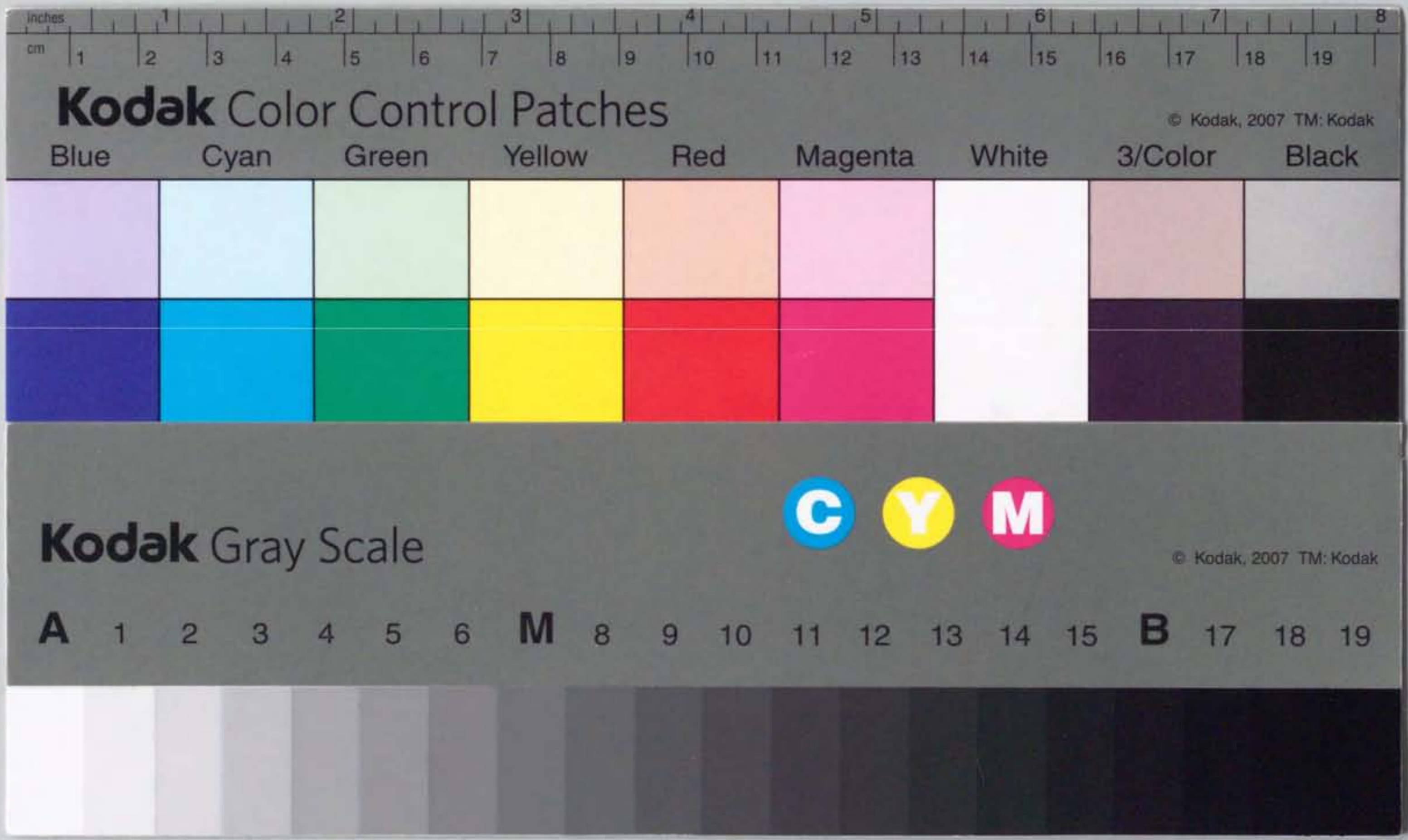
[65] J. S. Yu, Z. C. Wang and Q. Chuanxi, Oscillation of neutral delay differential equations, Bull. Austral. Math. Soc. **45** (1992), 195–200.

[66] B. G. Zhang and B. Yang, New approach of studying the oscillation of neutral differential equations, Funkcial. Ekvac. **41** (1998), 79–89.

[67] B. G. Zhang and J. S. Yu, Existence of positive solutions for neutral differential equations, Sci. China Ser. A **35** (1992), 1306–1313.

[68] B. G. Zhang and J. S. Yu, On the existence of asymptotically decaying positive solutions of second order neutral differential equations, J. Math. Anal. Appl. **166** (1992), 1–11.

[69] B. G. Zhang, J. S. Yu and Z. C. Wang, Oscillations of higher order neutral differential equations, Rocky Mountain J. Math. **25** (1995), 557–568.


inches
cm
Kodak Color Control Patches
© Kodak, 2007 TM: Kodak
Blue Cyan Green Yellow Red Magenta White 3/Color Black
C Y M
Kodak Gray Scale
© Kodak, 2007 TM: Kodak
A 1 2 3 4 5 6 M 8 9 10 11 12 13 14 15 B 17 18 19