Properties modelled on minimal almost periodicity, and small subgroup generating properties

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Chapter 1

Preliminaries

We assume that every topological space is Hausdorff. If X is any topological space, given $A \subseteq X$ we denote by $cl_X(A)$ the *closure* of A in X. $\mathbb{N}$ denotes the set of natural numbers, and $\mathbf{P}$ its subset of prime numbers. We use $\mathbb{N}^+$ to denote the set of strictly positive natural numbers.

1.1 Standard notations for groups

Symbols $\mathbb{Z}$ and $\mathbb{C}$ denote the groups of integer and complex numbers, respectively.

If $(G, \cdot)$ is a group and $X, Y \subseteq G$, then we let

$$X \cdot Y = \{x \cdot y : x \in X, y \in Y\}. \tag{1.1}$$

Given a group G and its subset $X \in G$ we denote by $\langle X \rangle$ *the smallest subgroup of* G *containing* X. In the event X is a singleton, i.e $X = \{x\}$ for some $x \in G$, we identify $\langle \{x\} \rangle = \langle x \rangle$ to simplify our notation.

Following [16], we define

$$\mathrm{Cyc}(A) = \{x \in G : \langle x \rangle \subseteq A\} \text{ for every } A \subseteq G. \tag{1.2}$$

Thus, we shall say that an element $x \in G$ *is a cyclic element of a subset* $X \subset G$ if and only if $x \in \mathrm{Cyc}(X)$. This notation shall be used in standard fashion throughout this text.

1.2 A small summary of algebraic ranks

Let $p \in \mathbb{N}$. For $m, n \in \mathbb{Z}$, we write that $m \equiv n \pmod{p}$ provided that $m - n = pk$ for some integer k.

Definition 1.2.1. Let G be an Abelian group and $p \in \mathbf{P} \cup \{0\}$.

(i) A subset X of G is said to be *p-independent* provided that for every $n \in \mathbb{N}$, each pairwise distinct elements $x_1, \ldots, x_n \in X$ and arbitrary integers $m_1, \ldots, m_n \in \mathbb{Z}$, the equation $\sum_{i=1}^{n} m_i x_i = 0$ implies that $m_i \equiv 0 \pmod{p}$ for all $i = 1, \ldots, n$.

(ii) The symbol $r_p(G)$ denotes the maximal cardinality of a p-independent subset of G (which exists by Zorn's lemma).

(iii) The cardinal $r_p(G)$ is called the *p-rank* of G.

(iv) The cardinal $r(G) = r_0(G) + \sum_{p \in \mathbf{P}} r_p(G)$ is called the *rank* of G.

The following fact is clear from these definitions.

Fact 1.2.2. If H is a subgroup of an Abelian group G, then $r_p(H) \leq r_p(G)$ for every $p \in \mathbf{P} \cup \{0\}$, and thus, $r(H) \leq r(G)$.

The following property of the 0-rank is well-known (see [17, Section 16] and [10, Corollary 2.5]):

Remark 1.2.3. The following inequality holds for every subgroup H of an Abelian group G:

$$r_0(G) \geq r_0(G/H). \tag{1.3}$$

Definition 1.2.4. [15, Definition 7.2] For an abelian group G, the cardinal

$$r_d(G) = \min\{r(nG) : n \in \mathbb{N}^+\} \tag{1.4}$$

is called the *divisible rank* of G.

The notion of the divisible rank was defined, under the name of *final rank*, by Szele [46] for p-groups. The following is easily seen:

Fact 1.2.5. The inequality $r_0(G) \leq r_d(G)$ holds for any group G.

The following is also straightforward:

Remark 1.2.6. An abelian group G satisfies $r_d(G) = 0$ if and only if G is a bounded torsion group; that is, if $nG = \{0\}$ for some $n \in \mathbb{N}^+$.

1.3 Topological groups

For a group G with an operation $\cdot_G$, there is a natural product mapping $m_G : G \times G \to G$ such that $m_G(x, y) = x \cdot_G y$ for all $x, y \in G$. Similarly, there is an *inversion* mapping $in_G : G \to G$ such that $in_G(x) = x^{-1}$ for all $x \in G$. In cases where we consider no confusion possible, we shall simply denote the product of $x, y \in G$ as $x \cdot y$. If this operation is commutative, the group G is said to be *Abelian*.

A topology τ defined on a group G is a *group topology* on G if the product mapping m_G and the inversion mapping i_G are continuous in the topological product $(G, \tau) \times (G, \tau)$ and in the topology τ for G respectively. The pair (G, τ) of a group with such a topology is called a *topological group*. In what follows, when we refer to a group G as being a topological group (without specifying τ), we assume the group G to be equipped with some Hausdorff group topology. A good overview of fundamentals and history of topological groups can be found in [8]. For even more detailed texts see [1, 9].

1.4 Minimally and maximally almost periodic groups

1.4.1 Historical background on minimal and maximal almost periodicity

For a topological space X, the set $B(X)$ denotes the family of all bounded complex-valued continuous functions on X equipped with the topology of uniform convergence. Given a topological group G, an element $g \in G$ and a complex-valued function f on G, we define the translation of f by g as the function $f_g : G \to \mathbb{C}$ satisfying $f_g(x) = f(xg)$ for all $x \in G$.

Definition 1.4.1. Let G be a topological group. A function $f \in B(G)$ is almost periodic if every sequence $\{f_{g_n} : n \in \mathbb{N}\}$ of translations of f by elements $g_n \in G$ $(n \in \mathbb{N})$ has a subsequence which is uniformly convergent in $B(G)$.

Real-valued almost periodic functions play a central role in the works of Bohr pertaining to harmonic analysis, and years later the same concept was considered by von Neumann in the context of complex-valued functions. The following concepts were introduced by von Neumann in [32]:

Definition 1.4.2. A topological group G is called:

(a) *maximally almost periodic* (MAP) if and only if the family of continuous homomorphisms to compact groups separate its points.

(b) *minimally almost periodic* (MinAP) if and only if it admits no non-trivial continuous homomorphism to a compact group.

These two properties are "orthogonal" in the following sense:

Remark 1.4.3. A maximally almost periodic group which is minimally almost periodic is trivial.

These concepts were motivated by [32, Theorem 36(i)], where von Neumann proves that the family of all almost periodic functions of a topological group G *separate its points* when G is either *compact or locally compact Abelian (and separable).* von Neumann notes in his monograph that verifying the existence of such almost periodic functions may be quite problematic, and from the desire of reducing this complexity he achieved the following result:

Theorem 1.4.4 ([32, Theorem 3](i))**.** *Let G be a topological group.*

*(i) The family of almost periodic functions of G separates its points if and only if the family of continuous homomorphisms to **unitary groups** separates its points.*

*(ii) The family of almost periodic functions of G is comprised of only the constant functions if and only if it admits no non-trivial continuous homomorphism to a **unitary group**.*

By the classical Peter-Weyl-van Kampen theorem, every compact group is isomorphic to a closed subgroup of a product of unitary groups. Therefore, one may replace the words "unitary group" by "compact group" in Theorem 1.4.4. This substitution (now presented in Definition 1.4.2) is the most commonly used definition for both MinAP and MAP groups in the modern literature. von Neumann's original results [32, Theorem 36(i)] show that compact groups and (separable) locally compact Abelian groups are contained in the class of MAP groups.

von Neumann and Wigner focused on minimally almost periodic groups in [33]. Here they construct a handful of examples of minimally almost periodic groups [33, Section 5] via linear transformations. And they note that constructing groups in this class is not a trivial effort. The class of minimally almost periodic groups gained a great deal of attention from experts in topological group theory thanks to two high-profile open problems which we shall describe in the next two subsections.

1.4.2 The connection of MinAP groups to extreme amenability

The first problem is related to the concept of *extremely amenable* groups.

Definition 1.4.5. A topological group is *extremely amenable* (or satisfies the *fixed point in compacta property*) if every continuous action of it on a compact space admits a fixed point.

Extremely amenable groups appeared in the context of Harmonic Analysis and Dynamical Systems (see [23, 35]). These groups are intimately connected to the class of MinAP groups by the following fact:

Fact 1.4.6. Every extremely amenable group is minimally almost periodic.

It is known that the converse implication does not hold in general. However, whether the converse implication holds or not in the realm of Abelian groups remains as a major open problem to this day:

Problem 1.4.7 (Pestov [35], 1998)**.** Is every Abelian MinAP topological group extremely amenable?

A topological group is *monothetic* if it contains a dense subgroup which is isomorphic to the group of integers. Every monothetic group is Abelian. The following particular version of Problem 1.4.7 was posed by Glasner as far back as 1988:

Problem 1.4.8 (Glasner [23], 1998)**.** Must every monothetic MinAP topological group be extremely amenable?

This particular version of Glasner has important implications in number theory. A negative answer to Problem 1.4.8 of Glasner would provide an answer to the following ancient problem (see [47]) of combinatoric number theory:

Problem 1.4.9. If S is a big set of the integers, is it true that the difference $S - S$ is a Bohr neighbourhood of 0?

Problems 1.4.7, 1.4.8 and 1.4.9 are still open.

1.4.3 Algebraic structure of MinAP groups

The difficulty in the construction of MinAP groups sparked a great deal of interest in regards to their algebraic structure. First examples of MinAP groups were the additive groups of some topological vector spaces [6], as explained in [28]. Nienhuys [34] constructed a connected monothetic group of cardinality at most continuum which is minimally almost periodic. This implies the existence of a MinAP group topology on the group $\mathbb{Z}$ of integers.

In 1984 Protasov posed the question of whether *every* Abelian group admits a minimally almost periodic group topology. In 1989 Remus provided an example of a bounded Abelian group which does not admit a MinAP group topology, so Comfort proposed the following modification of the original question of Protasov:

Problem 1.4.10 (Comfort, 1990 [3, Question 521])**.** Does every Abelian group which is not of bounded order admit a minimally almost periodic group topology?

The case of bounded torsion Abelian groups was resolved by Gabriyelyan by making use of their *Ulm-Kaplansky invariants* (see Fact 3.9.1):

Theorem 1.4.11 ([20, Corollary 3])**.** *A torsion bounded Abelian group admits a minimally almost periodic group topology if and only if all of its leading Ulm-Kaplansky invariants are infinite.*

The general case was resolved by Dikranjan and Shakhmatov [11] in 2014.

Theorem 1.4.12 (Dikranjan-Shakhmatov [11, Theorem 3.3])**.** *For an Abelian group G, the following conditions are equivalent:*

(i) G admits a minimally almost periodic group topology;

(ii) G is connected with respect to its Markov-Zariski group topology [13]*;*

(iii) for every $n \in \mathbb{N}$, the subgroup $nG = \{ng : g \in G\}$ of G is either trivial or infinite.

1.5 Properties related to small subgroups

1.5.1 The Hartman-Micyelski construction

In this section we describe the so-called Hartman-Mycielski construction [27].

Let G be an Abelian group and denote the unit interval $[0, 1]$ by I. We denote by G^I the set of all functions from I to G, which is a group under the coordinate-wise operations. Given $g \in G$ and $t \in (0, 1]$ we define the function $g^t \in G^I$ such that

$$g_t(x) = \begin{cases} g \text{ if } x < t, and \\ 0 \text{ if } x \geq t \end{cases}$$

where 0 is the zero element of G.

It is known that $G_t = \{g_t : g \in G\}$ is a subgroup of G^I that is isomorphic to G for every $t \in (0, 1]$. And furthermore the sum

$$\mathrm{HM}(G) = \bigoplus_{t \in (0,1]} G_t$$

is direct. If μ is the standard probability measure on I the *Hartman–Mycielski* topology on the group $\mathrm{HM}(G)$ is the topology generated by the family of all sets of the form

$$O(U, \epsilon) = \{g \in G^I : \mu(\{t \in I : g(t) \notin U\}) < \epsilon\} \tag{1.5}$$

where U is an open neighbourhood of 0 in G and $\epsilon > 0$ forms the base of the identity function of $\mathrm{HM}(G)$. This topology is known to be pathwise connected and locally pathwise connected [27].

1.5.2 The property DW of Dierolf and Warken

In 1978, Dierolf and Warken [7] proved that every topological group G can be embedded in a minimally almost periodic group. The group in question being the Hartman-Mycielski contruction $\mathrm{HM}(G)$ of G. To prove this, they showed that $\mathrm{HM}(G)$ has the following property:

Fact 1.5.1 (Dierolf and Warken [7, Theorem 1.1])**.** Let G be a topological group. For every neighbourhood U of the identity of $\mathrm{HM}(G)$ and every $g \in \mathrm{HM}(G)$ there exists a finite sequence of elements $g_1, \ldots, g_n \in U$ such that for every $i = 1, \ldots, n$ we have that $g_i \in \mathrm{Cyc}(U)$ and $g = \prod_{i=1}^{n} g_i$.

We can isolate the property above to obtain the following definition:

Definition 1.5.2 (Implicitly used in [7])**.** Let G be a topological group. We say that G *satisfies property* DW if for every open neighbourhood U of the identity of G, the equality $\langle \mathrm{Cyc}(U) \rangle = G$ holds.

In [7, Proof of Theorem 1.1], Dierolf and Warken essentially prove the following

Proposition 1.5.3. *A topological group with property* DW *is* MinAP*.*

The original theorem of Dierolf and Warken [7] can thus be stated as follows:

Theorem 1.5.4 ([7, Theorem 1.1])**.** *Every topological group G is topologically isomorphic to a closed subgroup of some topological group H_G (depending on G) which satisfies property* DW*. As a consequence, the group H_G is* MinAP*.*

1.5.3 The small subgroup generating properties of Gould

Gould [25] isolated Proposition 1.5.3 from the result of Dierolf and Warken and considered the following class of topological groups:

Definition 1.5.5 (Originally by Gould [25])**.** A topological group G has the *small subgroup generating property* (SSGP) if and only if for every open neighbourhood U of the identity of G, the set $\langle \mathrm{Cyc}(U) \rangle$ is **dense** in G.

The difference between the SSGP property of Gould and property DW used by Dierolf and Warken is subtle, but none the less non-trivial. Property DW is an algebraic expression for all elements of the group depending on the neighbourhoods of the identity. Meanwhile, in Definition 1.5.5, the requirement is that the set of elements which can be represented in the way proposed in property DW is topologically dense.

While it is easy to show that

$$\mathrm{DW} \to \mathrm{SSGP} \to \mathrm{MinAP}, \tag{1.6}$$

constructing group topologies with properties DW and SSGP is by no means simple. Particular evidence of this can be seen in the SSGP examples of Gould [4, 25, 26], which often require very careful manipulation of group metrics to be obtained. In the final paper [4] from this series, Comfort and Gould introduced a series of SSGP(n) properties, for every natural number n, defined by induction as follows:

Definition 1.5.6. Let G be a topological group. We say that G is:

(i) SSGP(0) when G is the trivial group, and

(ii) SSGP($n+1$) if the subgroup $K_U = cl_G(\langle \mathrm{Cyc}_G(U)\rangle)$ is normal in G, and the algebraic quotient $H_U = G/K_U$ has SSGP(n) for every neighbourhood U of the identity of G.

Notice that SSGP(1) coincides precisely with the original definition of SSGP. The following chain of implications is straightforward:

$$\mathrm{SSGP}(0) \to \mathrm{SSGP}(1) \to ... \to \mathrm{SSGP}(n) \to ... \to \mathrm{MinAP}. \tag{1.7}$$

Examples distinguishing all properties in (1.7) can be found in [4, Corollary 3.14, Theorem 4.6]. However, all properties in (1.7) coincide for bounded torsion abelian topological groups.

Theorem 1.5.7. *[4, Corollary 2.23] A bounded torsion abelian topological group has the* SSGP *if and only if it is* MinAP.

1.5.4 A family of `SSGP`(α) properties of Dikranjan and Shakhmatov

In 2016, Dikranjan and Shakhmatov [16] initiated the study of the SSGP-type properties of Gould by means of an operator-based approach with the goal of avoiding consecutive topological quotients as in Definition 1.5.6. They introduced an operator denoted by $\mathbf{S}_G$, which when used with carefully defined transfinite iterations (denoted by $\mathbf{S}_G^{(\alpha)}$ for every ordinal α) gives rise to the following definition:

Definition 1.5.8. For an ordinal α, a topological group G is said to be `SSGP`(α) if $\mathbf{S}_G^{(\alpha)}(U) = G$ for every neighbourhood U of the identity of G.

For a proper definition and more details regarding the $\mathbf{S}_G$ operator, see Section 2.6 of Chapter 2.

We highlight the difference in fonts between the properties SSGP(n) and $\texttt{SSGP}(n)$ for $n \in \mathbb{N}$, as their underlying definition is different. The connection between these properties is given in items (i), (iv) and (v) of the following theorem.

Theorem 1.5.9. *(i)* $\texttt{SSGP}(1)$ *coincides with* SSGP *(which is equivalent to* SSGP(1)*);*

(ii) $\texttt{SSGP}(\alpha) \to \texttt{SSGP}(\beta)$ *whenever* α, β *are ordinals satisfying* $\alpha < \beta$*;*

(iii) for each ordinal α*,* $\texttt{SSGP}(\alpha) \to$ MinAP*;*

(iv) SSGP(n) $\to \texttt{SSGP}(n)$ *for every* $n \in \mathbb{N}^+$*;*

(v) for every $n \in \mathbb{N}^+$*, an Abelian topological group has the property* SSGP(n) *if and only if it has the property* $\texttt{SSGP}(n)$*.*

Proof. Item (i) is [16, Lemma 2.2 and (7)], item (ii) is [16, Proposition 5.1], item (iii) is [16, Proposition 5.3 (iii)], item (iv) is [16, Corollary 6.3] and item (v) is [16, Theorem 6.4]. □

It is unknown if SSGP(n) and $\texttt{SSGP}(n)$ coincide for arbitrary topological groups [16, Question 13.4].

As a result of Theorem 1.5.9(v), there is absolutely no confusion between denoting SSGP(n) or $\texttt{SSGP}(n)$ while working with Abelian groups. From here, it follows that the properties $\texttt{SSGP}(\alpha)$ (where α is an ordinal) create a very natural extension of the implications detailed in (1.7):

$$\texttt{SSGP}(0) \to \cdots \to \texttt{SSGP}(n) \to \cdots \to \texttt{SSGP}(\alpha) \to \texttt{SSGP}(\alpha+1) \to \cdots \to \text{MinAP}.$$

1.5.5 The algebraic structure of SSGP groups

Comfort and Gould [4, Questions 5.2 and 5.3] asked the following:

Question 1.5.10. (a) What are the (Abelian) groups which admit an SSGP topology?

(b) Does every Abelian group which for some $n > 1$ admits an SSGP(n) topology also admit an SSGP topology?

As for Question 1.5.10, Dikranjan and Shakhmatov essentially split this problem into two parts: the case of groups of infinite divisible rank (see [15, Definition 7.2] for the definition of the divisible rank) and the case of groups of finite divisible rank.

In view of Remark 1.2.6, the next theorem coincides with [16, Corollary 1.7].

Theorem 1.5.11. *A non-trivial abelian group G satisfying $r_d(G) = 0$ admits an* SSGP *topology if and only if all leading Ulm-Kaplanski invariants of G are infinite.*

Theorem 1.5.11 resolves Question 1.5.10 (a) for Abelian groups of divisible rank zero, while the next theorem resolves it for Abelian groups of infinite divisible rank.

Theorem 1.5.12. *[16, Theorem 3.2] Every abelian group G satisfying $r_d(G) \geq \omega$ admits an* SSGP *topology.*

In [16], Dikranjan and Shakhmatov reduced the remaining case $0 < r_d(G) < \omega$ to a question regarding the existence of SSGP topologies on Abelian groups of a very particular type:

Question 1.5.13. Let $n \in \mathbb{N}^+$ and

$$G = G_0 \times \left(\bigoplus_{i=1}^{k} \mathbb{Z}(p_i^\infty)\right) \times F,$$

where F is a finite group, $k \in \mathbb{N}$, $p_1, p_2, \ldots, p_k$ are (not necessarily distinct) prime numbers, and G_0 is a subgroup of $\mathbb{Q}^n$ containing $\mathbb{Z}^n$ such that $G_0 \not\subseteq \mathbb{Q}_\pi^n$ for every finite set π of prime numbers. Is it true that G admits an SSGP topology?

For the precise definition of $\mathbb{Q}_\pi$ subgroups, see Definition 4.3.1. Assuming a positive answer to Question 1.5.13, they established in [16, Theorem 13.2] what is stated now as Theorem 4.1.1 in Chapter 4, thus giving a (*provisional*, at that moment) complete characterization of Abelian groups G admitting an SSGP topology in the remaining open case $0 < r_d(G) < \omega$.

1.6 Summary of results and content of each chapter

Chapters 1–3 are single-authored, while the contents of Chapters 4–6 constitute a joint work with Dmitri Shakhmatov.

We say that a topological group G is $\texttt{SSGP}(\infty)$ (or that it has the $\texttt{SSGP}(\infty)$ property) if G is $\texttt{SSGP}(\alpha)$ for some ordinal α (see Definition 2.3.1).

In Chapter 2 we introduce the following concept: Let $\mathcal{C}$ denote a class of topological groups. We say that a topological group is MinAP($\mathcal{C}$) (or satisfies the MinAP($\mathcal{C}$) property) if all its non-trivial homomorphisms to any group contained in the class $\mathcal{C}$ are discontinuous. Here the property MinAP(Compact) coincides with the classic MinAP property.

We investigate MinAP($\mathcal{C}$) for three standard classes $\mathcal{C}$: locally compact groups (LC), Lie groups, and groups with no small subgroups (NSS). This triad of properties is well-known for being part of the solution to Hilbert's 5th Problem by Gleason, Montgomery-Zippin and Yamabe: a topological group is Lie if and only if it is both locally compact and NSS.

The following diagram shows the relationships between the newly introduced properties (see Figure 2.1 and Chapter 2 for full details).

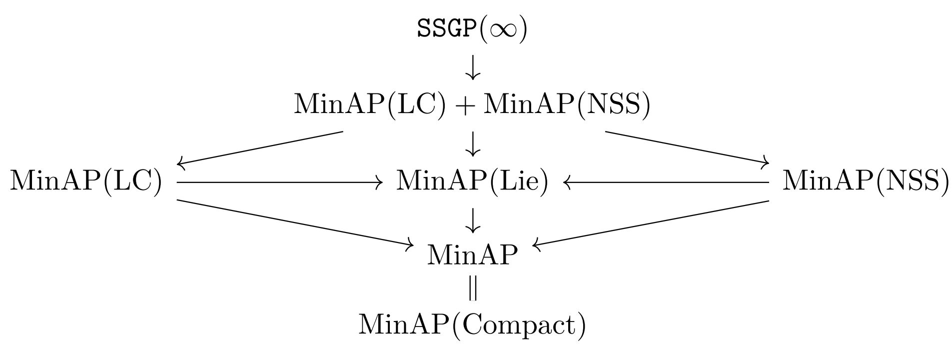

Figure 1.1: Diagram of implications. LC denotes Locally compact

We prove that all of the above implications are not reversible in general (see Theorem 2.3.5), yet some of them become reversible for Abelian topological groups:

Theorem 1.6.1 (Corollary 2.3.6)**.** *Properties* MinAP(*LC*)*,* MinAP(*Lie*) *and* MinAP *coincide for Abelian topological groups.*

Our main result of Chapter 2 connects the new MinAP(NSS) property with the hierarchy $\texttt{SSGP}(\infty)$ of Dikranjan and Shakhmatov (see Definition 2.3.1).

Theorem 1.6.2 (Corollary 2.3.8)**.** *The* $\texttt{SSGP}(\infty)$ *and* MinAP(NSS) *properties are equivalent for Abelian topological groups.*

As a consequence, we significantly simplify Figure 1.1 to the following in Abelian topological groups (see Figure 2.2):

$$\begin{array}{ccccc} & & \text{MinAP(NSS)} & \longleftrightarrow & \texttt{SSGP}(\infty) \\ & & \downarrow & & \downarrow \\ \text{MinAP} & \longleftrightarrow & \text{MinAP(Lie)} & \longleftrightarrow & \text{MinAP(Locally compact)} \end{array}$$

Figure 1.2: Simplified diagram of implications in Abelian topological groups

The group of integers $\mathbb{Z}$ is a well-known example of a group which can be equipped with a MinAP group topology, but never with an $\texttt{SSGP}(\infty)$ group topology (see [16, Corollary 3.9]). Therefore, the equivalent properties in the "upper-level" of the diagram do not imply any of the properties in the "lower-level" of the diagram.

In Chapter 3 we introduce the following concept inspired by the MinAP property. Let **P** be a property of topological groups. We say that a topological group G is MinAP *modulo* **P** (MinAP *mod* **P**) if for each continuous homomorphism $f : G \to K$ from G to a compact group K, the image $f[G]$ of G considered as a subgroup of K has property **P**.

When **P** is the property of being the *trivial group* then MinAP mod **P** coincides with classical minimal almost periodicity. Our interest in this modification of the MinAP property is to analyze potential changes to algebraic structure by weakening the restrictions on continuous homomorphic images to compact groups.

Naturally, when **P** and **Q** are properties of topological groups such that **P** implies **Q**, then

$$\text{MinAP mod } \mathbf{P} \to \text{MinAP mod } \mathbf{Q}.$$

We study five modifications of the MinAP property by considering a handful of properties **P**: one set-theoretic (finite), two algebraic (torsion and bounded torsion) and two topological (compact and connected). The properties relate as follows (see Figure 3.1):

$$\begin{array}{ccccc} \text{MinAP} & \longrightarrow & \text{MinAP mod finite} & \longrightarrow & \text{MinAP mod compact} \\ \downarrow & & \downarrow & & \\ \text{MinAP mod connected} & & \text{MinAP mod bounded} & \longrightarrow & \text{MinAP mod torsion.} \end{array}$$

Figure 1.3: Diagram of implications

We prove that none of the above implications are reversible (see Examples 3.3.2 and 3.6.8).

First, if **P** is an invariant of continuous homomorphisms, then we have a description of MinAP mod **P** groups in the following theorem:

Theorem 1.6.3 (Theorem 3.6.1 Corollary & 3.6.3)**.** *The following are equivalent for an Abelian group G and a property* **P** *invariant under continuous homomorphisms:*

(i) G is MinAP *modulo* **P***,*

(ii) The image $b_G(G)$ of G to its Bohr compactification has property **P***; and*

(iii) The quotent $G/\mathfrak{n}(G)$ of G with respect to its von Neumann kernel has property **P** *when equipped with its Bohr topology.*

Next, we apply our Theorem 1.6.3 along with Dikranjan-Shakhmatov's description of algebraic structure of Abelian MinAP topological groups (Theorem 1.4.12). This allows us to describe algebraic structure of Abelian groups corresponding to **all properties in Figure 1.3**:

Theorem 1.6.4 (Theorem 3.8.1 & Corollary 3.8.2)**.** *An Abelian group G admits a* MinAP *mod connected group topology if and only if it admits a* minimally almost periodic *group topology.*

Theorem 1.6.5 (Theorem 3.9.5 & Corollary 3.9.6)**.** *Every Abelian group G admits a group topology which is* MinAP *mod* **P** *for* **P** $=$ *{finite, bounded, compact, and torsion}.*

The former result shows that MinAP modulo connected imposes the same algebraic restrictions as the classical minimal almost periodicity. Meanwhile, the latter shows that MinAP mod **P** for **P** = {finite, bounded, compact, and torsion} is too lenient algebraically, as any Abelian group may be equipped with such a topology "easily".

Comfort and Gould previously asked for a characterization of Abelian groups which admit an SSGP group topology (see Question 1.5.10). Dikranjan and Shakhmatov essentially split this problem into two parts: the case of groups of infinite divisible rank and the case of groups of finite divisible rank. They obtained an almost complete description of the algebraic structure of SSGP *and* $\mathtt{SSGP}(\alpha)$ groups for any ordinal α, and they additionally reduced the sufficiency to a single remaining case (Question 1.5.13), which we resolve in Chapter 4.

The main result in Chapter 4 is the following theorem, which provides a solution to a more general statement than that of Question 1.5.13:

Theorem 1.6.6 (Theorem 4.7.4)**.** *Suppose that $m \in \mathbb{N}^+$ and G is a wide subgroup of $\mathbb{Q}^m$. Then for each at most countable abelian group H, the direct sum $K = G \oplus H$ admits a metric* SSGP *topology.*

The notion of a wide subgroup of $\mathbb{Q}^n$ is defined in Definition 4.3.1. We highlight the following:

Remark 1.6.7. By Corollary 6.5.5, the groups K considered in Theorem 1.6.6 *cannot be equipped with* DW *group topologies.*

To construct the topologies in Theorem 4.7.4 of Chapter 4 (as well as the one in Theorem 5.1.1 of Chapter 5) we devise a technique for extending a "finite neighbourhood system" on a group to some other "finite neighbourhood system" of a bigger group. With these extensions we define some canonical representations of elements of "extended neighbourhoods" by elements from "smaller neighbourhoods" and a fixed set which can be viewed as a base for such an extension. In this technique we introduce a partially ordered set $(\mathbb{P}, \leq)$ comprised of "finite neighbourhood systems" on our groups. Then, we construct a countable family $\mathscr{D}$ of dense subsets of the poset $(\mathbb{P}, \leq)$. We then select some linearly ordered subset $\mathbb{F}$ of $(\mathbb{P}, \leq)$ which intersects all members of the family $\mathscr{D}$ of dense subsets of $(\mathbb{P}, \leq)$; this can be done due to a folklore Lemma 4.7.3 (historically attributed to Rasiowa-Sikorski). Finally, we obtain a countable base of neighbourhoods of zero for a group topology $\mathscr{T}$, which is defined by means of elements of the linearly ordered set $\mathbb{F}$.

A reader who is familiar with Martin's Axiom undoubtedly notices that our technique makes use of a "ZFC version" of this axiom when the family of dense sets is at most countable. The choice of such an exposition was determined by the authors' desire to replace a direct construction of the topology $\mathscr{T}$ via an induction (which would be totally incomprehensible) by a "much smoother" forcing-type argument using a poset $(\mathbb{P}, \leq)$ and some dense subsets of it (which is much easier to follow than the direct inductive construction). We hope that, after discovering the technical complexity even of this "smooth" approach, the reader would fully agree with our judgment. We note that (other than the construction of SSGP-type group topologies) Shakhmatov and the author have succesfully applied this same technique for the construction of "coherent splitting-maps" in [39, Section 9].

Theorem 1.6.6 above resolves Question 1.5.13 of Dikranjan and Shakhmatov. And as a consequence we fully complete the following description for the algebraic structure of SSGP and $\texttt{SSGP}(\alpha)$

topological groups.

Theorem 1.6.8 (Theorem 4.1.1)**.** *The following are equivalent for an Abelian group G:*

(a) G admits an SSGP *group topology,*

(b) G admits an $\mathtt{SSGP}(\alpha)$ *group topology for some ordinal α, and*

(c) one of the two conditions is satisfied:

(i) G is of infinite divisible rank, or

(ii) the quotient $H = G/t(G)$ of G by its torsion part $t(G)$ has finite free rank $r_0(H)$ and $r(H/A) = \omega$ for some (equivalently, every) free subgroup A of H such that H/A is torsion.

In Chapter 5 we construct group topologies with the stronger DW property in *free groups of infinite rank.* This is our only non-Abelian case in this thesis. Our main results in this chapter are the following.

First, for free groups with *countably many generators*:

Theorem 1.6.9 (Theorem 5.1.1)**.** *The free group $F(X)$ over a countably infinite set X admits a metric* DW *group topology.*

In this case, we are able to make this topology metric. The construction of this topology is done by adapting our technique from Chapter 4 (explained above) to a substantially more complex *non-Abelian version.*

In case the set of generators is uncountable, we have an answer as well:

Theorem 1.6.10 (Theorem 5.1.2)**.** *Every free group with infinitely many generators admits an* DW *group topology.*

This result is achieved by finding an algebraically isomorphic copy of the desired free group (with uncountably many generators), which inherits a DW subgroup topology in some uncountable power of a free group obtained in Theorem 1.6.9. Whether the DW group topologies can be made metric in these cases remains open.

In Chapter 6 we provide some initial results on algebraic structure of Abelian groups which admit a group topology with property DW. The main result of this chapter consists of the following necessary conditions for Abelian groups of finite free rank:

Theorem 1.6.11 (Theorem 6.5.1)**.** *Let G be an Abelian group of finite free rank. If G admits a* DW *group topology, then either one of the following holds:*

(i) If $r_0(G) = 0$ (i.e. G is torsion) then every non-trivial p-component of G admits an DW *group topology.*

(ii) If $0 < r_0(G)$ then there exists $p \in \mathbf{P}$ such that the p-component of G has infinite divisible rank.

As a consequence this poses restrictions on certain common groups, such as finite powers of the group of rationals $\mathbb{Q}$:

Corollary 1.6.12 (Corollary 6.5.5)**.** *For every $n \in \mathbb{N}^+$ the following hold:*

(i) The only subgroup of $\mathbb{Q}^n$ which admits an DW *group topology is the trivial group.*

(ii) Every wide subgroup of $\mathbb{Q}^n$ admits an SSGP *group topology but not an* DW *one.*

As a consequence of Corollary 1.6.12(ii) the finite powers of $\mathbb{Q}$ are now shown to never admit DW group topologies even though they admit SSGP group topologies by our Theorem 1.6.6 above. This shows that the implication DW $\implies$ $\mathtt{SSGP}(\infty)$ from (1.6) cannot be reversed even by asking for "existence" of topologies.

Chapter 2

Strengthening minimal almost periodicity via the classical triad resolving Hilbert's Fifth Problem

2.1 Introduction

Inspired by Definition 1.4.2(b), for every class $\mathcal{C}$ of topological groups we propose corresponding MinAP-like and MAP-like classes of topological groups.

Definition 2.1.1. Let $\mathcal{C}$ be a non-empty class of topological groups and G be a topological group. We say that:

(i) G is MinAP($\mathcal{C}$) (or has the MinAP($\mathcal{C}$) *property*) if G admits no non-trivial continuous homomorphisms to groups contained in the class $\mathcal{C}$, and

(ii) G is MAP($\mathcal{C}$) (or has the MAP($\mathcal{C}$) *property*) if there exists a family of continuous homomorphisms of G to groups contained in the class $\mathcal{C}$ separating the points of G.

With a slight abuse of notation, we shall also denote by MinAP($\mathcal{C}$) and MinAP($\mathcal{C}$) the *class* of topological groups having the corresponding property.

Remark 2.1.2. In the terminology of Definition 2.1.1, the class of minimally almost periodic groups (MinAP) from Definition 1.4.2(b), is precisely the class of MinAP(Compact) groups. Analogously, the class of maximally almost periodic groups (MAP) from Definition 1.4.2(a), is precisely the class of MAP(Compact) groups.

In this chapter, our interest is in considering natural classes $\mathcal{C}$ of topological groups for which the class of MinAP($\mathcal{C}$) groups forms a proper subclass of MinAP groups. In particular, we investigate the classes of MinAP(Locally compact), MinAP(Lie) and MinAP(NSS) groups. Here NSS denotes the class of topological groups having No Small Subgroups.

Definition 2.1.3. A topological group G is said to have *no small subgroups* (commonly abbreviated to NSS) if there exists an open neighbourhood of the identity of G containing no non-trivial subgroups of G.

Locally compact groups and NSS groups play a fundamental role in the historical Hilbert Fifth Problem from the 1900s, concerning the characterization of Lie groups. The results which comprise the solution of this problem are due to Gleason [24] and Montgomery-Zippin [31]:

Theorem 2.1.4 (Gleason, Montgomery and Zippin)**.** *For a topological group G, the following are equivalent:*

(i) G is Locally compact NSS*, and*

(ii) G is a Lie group.

The following is an immediate consequence of this theorem.

Corollary 2.1.5. *The following are equivalent for a topological group G:*

(i) G is MinAP(*Locally compact* NSS)*, and*

(ii) G is MinAP(*Lie*)*.*

We prove that the Abelian groups in the classes MinAP(Locally compact) and MinAP(Lie) are precisely the minimally almost periodic groups (Corollary 2.3.6), while the Abelian MinAP(NSS) groups turn out to coincide with a certain class of Abelian groups considered by Dikranjan and Shakhmatov in [16]; see Corollary 2.3.8. The (more complex) relationships between these three classes of topological groups without the assumption of commutativity are summarized in Figure 2.1 found in Theorem 2.3.5.

2.2 Basic results about MinAP($\mathcal{C}$) and MAP($\mathcal{C}$) properties

The following three properties can be verified easily from Definition 2.1.1.

Proposition 2.2.1. *Let G be a topological group. If $\mathcal{C}$ is a subclass of $\mathcal{D}$, then the following holds:*

(a) If G is MinAP($\mathcal{D}$)*, then it is* MinAP($\mathcal{C}$)*;*

(b) If G is MAP($\mathcal{C}$)*, then it is* MAP($\mathcal{D}$)*;*

(c) If G is MinAP($\mathcal{D}$) *and* MAP($\mathcal{C}$)*, then it is the trivial group.*

Proposition 2.2.2. *Properties* MinAP(MAP($\mathcal{C}$)) *and* MinAP($\mathcal{C}$) *coincide for every class $\mathcal{C}$ of topological groups.*

Proof. First, observe that any group in the class $\mathcal{C}$ is MAP($\mathcal{C}$) by Definition 2.1.1(ii) (it suffices to take the identity homomorphism). This implies that $\mathcal{C}$ is a subclass of MAP($\mathcal{C}$). Proposition 2.2.1 (a) then shows that every MinAP(MAP($\mathcal{C}$)) group is MinAP($\mathcal{C}$).

Let us now show the converse. Assume G is MinAP($\mathcal{C}$) and let $f : G \to H$ be a continuous homomorphism from G to a MAP($\mathcal{C}$) group H. By contradiction, assume that f is non-trivial, so there exists some $x \in G$ such that $f(x) \neq e$. Since H is MAP($\mathcal{C}$), Definition 2.1.1(i) implies the existence of a continuous homomorphism $g : H \to K$ from H to a group $K \in \mathcal{C}$ which satisfies $g(f(x)) \neq e$. This implies that the composition $g \circ f : G \to K$ is a non-trivial continuous homomorphism from G to a group K in the class $\mathcal{C}$. By Definition 2.1.1(ii), this means that G is not MinAP($\mathcal{C}$), giving a contradiction with our assumption. Since the group $H \in$ MAP($\mathcal{C}$) and the continuous homomorphism $f : G \to H$ were arbitrary, we conclude that G is MinAP(MAP($\mathcal{C}$)) by Definition 2.1.1(i). □

As was mentioned in the introduction, our interest is in finding natural classes $\mathcal{C}$ of topological groups for which MinAP($\mathcal{C}$) groups form a proper subclass of MinAP groups. The next corollary shows that, in order for this to happen, $\mathcal{C}$ must contain at least one group which is *not* MAP.

Corollary 2.2.3. *If $\mathcal{C}$ is a subclass of* MAP *groups, then every* MinAP *group is* MinAP($\mathcal{C}$)*.*

Proof. Assume that G is a MinAP group. Then G is MinAP(Compact) by Remark 2.1.2, and so G is MinAP(MAP(Compact)) by Proposition 2.2.2. Note that the class MAP(Compact) coincides with the class of MAP groups by Remark 2.1.2, so G is MinAP(MAP). Since $\mathcal{C}$ is assumed to be the subclass of the class of MAP groups, we conclude that G is MinAP($\mathcal{C}$) by Proposition 2.2.1(a). $\square$

Corollary 2.2.4. *If $\mathcal{C}$ and $\mathcal{D}$ are classes of topological groups such that properties* MAP($\mathcal{C}$) *and* MAP($\mathcal{D}$) *coincide, then properties* MinAP($\mathcal{C}$) *and* MinAP($\mathcal{D}$) *coincide as well.*

Proof. It easily follows from the assumption of our corollary and Definition 2.1.1(i) that properties MinAP(MAP($\mathcal{C}$)) and MinAP(MAP($\mathcal{D}$)) coincide. By Proposition 2.2.2, the former property coincides with MinAP($\mathcal{C}$) while the latter one coincides with MinAP($\mathcal{D}$). $\square$

Corollary 2.2.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be classes of topological groups satisfying $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathrm{MAP}(\mathcal{C})$. Then properties* MinAP($\mathcal{C}$) *and* MinAP($\mathcal{D}$) *coincide.*

Proof. It follows from $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathrm{MAP}(\mathcal{C})$ and Proposition 2.2.1(b). that

$$\mathrm{MAP}(\mathcal{C}) \subseteq \mathrm{MAP}(\mathcal{D}) \subseteq \mathrm{MAP}(\mathrm{MAP}(\mathcal{C})) = \mathrm{MAP}(\mathcal{C}).$$

Therefore, $\mathrm{MAP}(\mathcal{C}) = \mathrm{MAP}(\mathcal{D})$, and the conclusion follows from Corollary 2.2.4. $\square$

Recall that a topological group is precompact if it is a subgroup of some compact group.

Corollary 2.2.6. *Properties* MinAP*,* MinAP(*Precompact*) *and* MinAP(MAP) *coincide.*

Proof. Indeed, let $\mathcal{C}$ be the class of compact groups. Since $\mathrm{MinAP}(\mathcal{C}) = \mathrm{MinAP}$ and $\mathrm{MAP}(\mathcal{C}) = \mathrm{MAP}$ by Remark 2.1.2, we have $\mathcal{C} \subseteq \{Precompact\} \subseteq \{\mathrm{MAP}\} = \mathrm{MAP}(\mathcal{C})$. The conclusion now follows from Corollary 2.2.5. $\square$

The following proposition is useful when dealing with Abelian topological groups.

Proposition 2.2.7. *Assume that $\mathcal{C}$ is a class of topological groups invariant under taking closed subgroups. If G is an Abelian topological group, then G is* MinAP($\mathcal{C}$) *if and only if G is* MinAP($\mathcal{C} \cap$ *Abelian*)*.*

Proof. Since $\mathcal{C} \cap$Abelian is a subclass of $\mathcal{C}$, it follows from Proposition 2.2.1(a) that every MinAP($\mathcal{C}$) group is MinAP($\mathcal{C} \cap$ Abelian). Therefore, it suffices to show that if G is MinAP($\mathcal{C} \cap$ Abelian), then it is MinAP($\mathcal{C}$). Let G be a MinAP($\mathcal{C} \cap$ Abelian) group, and let $f : G \to H$ be a continuous homomorphism where H is a group contained in the class $\mathcal{C}$. Since G is Abelian, the image $f[G]$ is an Abelian subgroup of H. If we take $K = cl_H(f[G])$ to be the closure of $f[G]$ in H, then K is also an Abelian group. Since K is closed in H, this implies that K is contained in $\mathcal{C}$ by our hypothesis. Since $f[G]$ is a subgroup of the Abelian group K from the class $\mathcal{C}$, the homomorphism f is trivial by the MinAP($\mathcal{C} \cap$ Abelian) property of G. To conclude, we note that the group H from $\mathcal{C}$ and the homomorphism $f : G \to H$ were arbitrary, so G is MinAP($\mathcal{C}$) by Definition 2.1.1(i). □

To close this section, let us recall the following classical facts in the theory of compact topological groups.

Remark 2.2.8. (i) It follows from the Peter-Weyl theorem that *every locally compact Abelian group is* MAP (see [8, Corollary 11.2.1]).

(ii) The Peter-Weyl-van Kampen theorem shows that *every compact group is* MAP($\mathbb{U}$)*, where* $\mathbb{U}$ *is the class of all unitary matrix groups* (see [8, Theorem 9.3.2]).

2.3 Results in this chapter

In [16, Proposition 5.3 (ii)], Dikranjan and Shakhmatov prove that, for every ordinal α, each SSGP(α) group is MinAP(NSS). To simplify this and upcoming statements regarding SSGP(α) groups, let us introduce the following notation.

Definition 2.3.1. We say that a topological group G is SSGP(∞) (or that it has the SSGP(∞) property) if G is SSGP(α) for some ordinal α.

We can now restate the result of Dikranjan and Shakhmatov as follows:

Proposition 2.3.2 ([16, Proposition 5.3 (ii)])**.** *Every* SSGP(∞) *group is* MinAP(NSS).

Our next result establishes a "sister implication":

Theorem 2.3.3. *Every* SSGP(∞) *group is* MinAP(*Locally compact*).

The proof of this theorem is postponed until Section 2.9.

Corollary 2.3.4. *Every* SSGP(∞) *group is both* MinAP(*Locally compact*) *and* MinAP(NSS).

Proof. This follows from Proposition 2.3.2 and Theorem 2.3.3. □

Our next theorem establishes connections between MinAP($\mathcal{C}$) properties for the triad of properties $\mathcal{C}$ appearing in Theorem 2.1.4, as well as the newly introduced SSGP(∞) property.

Theorem 2.3.5. *In the following diagram of implications, solid arrows hold for all topological groups, while dashed arrows hold (only) for Abelian topological groups.*

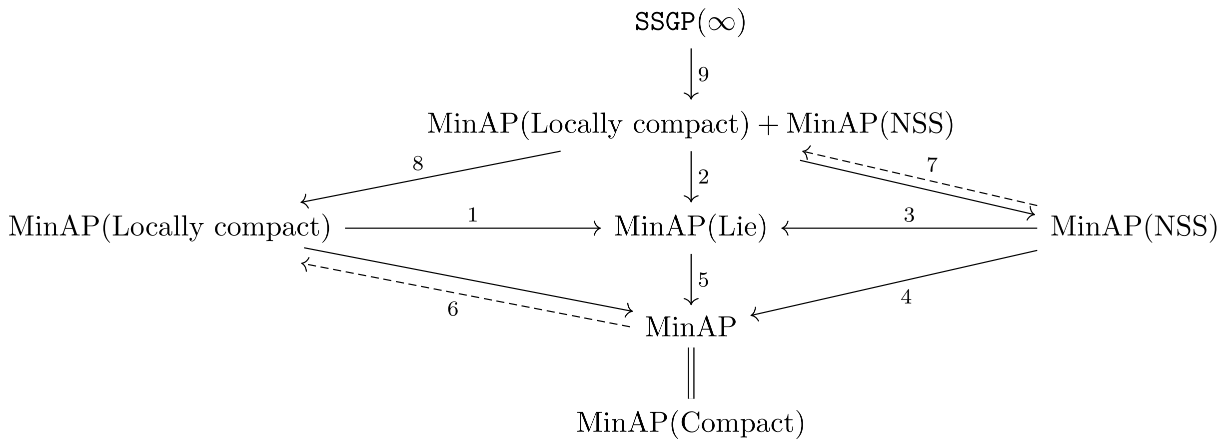

Figure 2.1: Diagram of implications

Proof. Since every unitary matrix group is a Lie group, Remark 2.2.8(ii) implies that the class of compact groups is a subclass of the broader class MAP(Lie). At the same time, the class of Lie groups is a very well-known subclass of NSS topological groups. From this, Propositions 2.2.1, 2.2.2 and Remark 2.1.2, we get

$$\text{MinAP(NSS)} \to \text{MinAP(Lie)} = \text{MinAP(MAP(Lie))} \to \text{MinAP(Compact)} = \text{MinAP}.$$

This proves the implications denoted by arrows 3 and 5 (and thus, also by arrow 4).

Each Lie group is locally compact, so arrow 1 follows from Proposition 2.2.1 (a). Arrows 2 and 8 are trivial. Arrow 9 is proved in Corollary 2.3.4. The remaining solid arrows in Figure 1 are either trivial or follow from the (solid) arrows whose validity is established above.

Now we turn our attention to dashed arrows. Observe that arrow 7 will follow as long as we show that arrow 6 does, so we focus on showing that arrow 6 holds for Abelian topological groups.

Let G be an Abelian MinAP group. Then G is MinAP(MAP) by Corollary 2.2.6. Let $f : G \to H$ be a continuous homomorphism from G to a locally compact Abelian group H. Since H is MAP by Remark 2.2.8(i), and G is MinAP(MAP), it follows that the homomorphism f is trivial. We have checked that every continuous homomorphism from G to a locally compact Abelian group is trivial. By Definition 2.1.1(i), this means that G is MinAP(Locally compact Abelian). Since G is Abelian and the class of locally compact groups is invariant under taking closed subgroups, G is MinAP(Locally compact) by Proposition 2.2.7. □

The triangle of arrows 1, 5 and 6 in Figure 1 establishes the following

Corollary 2.3.6. *For an Abelian topological group G the following are equivalent:*

(i) G *is* MinAP*,*

(ii) G *is* MinAP(*Locally compact*)*, and*

(iii) G *is* MinAP(*Lie*)*.*

The equivalence of items (ii) and (iii) of Corollary 2.3.6 shows that, in the realm of Abelian groups, Corollary 2.1.5 remains valid even without the word NSS in its item (i).

It is not immediately clear if the MinAP(NSS) property can be included in the list of equivalent conditions of Corollary 2.3.6, to fully encompass the solution triad of the fifth problem of Hilbert. Our nearest goal is to show that this cannot be done.

In the realm of Abelian groups, MinAP(NSS) topological groups are $\mathtt{SSGP}(\alpha)$ if one chooses a suitable ordinal α:

Theorem 2.3.7. *An Abelian* MinAP(NSS) *group G is* $\mathtt{SSGP}(|G|^+)$*.*

We leave the proof of this theorem for Section 2.8.

Our next result provides a complete characterization of Abelian MinAP(NSS) groups. Quite surprisingly, these are precisely $\mathtt{SSGP}(\infty)$ groups:

Corollary 2.3.8. *The following are equivalent for every Abelian topological group G:*

(i) G *is* $\mathtt{SSGP}(\infty)$*,*

(ii) G *is* MinAP(*Locally compact*) *and* MinAP(NSS)*.*

(iii) G *is* MinAP(NSS)*.*

Proof. (i) $\implies$ (ii) follows from Proposition 2.3.2 and Theorem 2.3.3.

(ii) $\implies$ (iii) clearly follows.

(ii) $\implies$ (i). By Theorem 2.3.7, G is $\mathtt{SSGP}(\alpha)$ for $\alpha = |G|^+$, so G is $\mathtt{SSGP}(\infty)$ by Definition 2.3.1. $\square$

This corollary shows that the implication of Proposition 2.3.2 becomes reversible for Abelian topological groups. Combining this with Corollaries 2.3.6 and 2.3.8, we obtain the "substantially collapsed" Figure 2.1 *in the Abelian case*:

$$\begin{array}{ccccc}
\text{MinAP(NSS)} & \stackrel{7}{\longleftrightarrow} & \mathtt{SSGP}(\infty) & \stackrel{9}{\longleftrightarrow} & \text{MinAP(Locally compact)} + \text{MinAP(NSS)} \\
\big\downarrow{\scriptstyle 3} & & & & \big\downarrow{\scriptstyle 8} \\
\text{MinAP(Lie)} & \longleftrightarrow & \text{MinAP} & \longleftrightarrow & \text{MinAP(Locally compact)}
\end{array}$$

Figure 2.2: Simplified diagram of implications in Abelian groups

In the above diagram, none of the equivalent "high-tier" properties imply any of the equivalent "lower-tier" properties. This shows that, in Abelian groups, the fine structure presented in Figure 2.1 essentially reduces to only two distinct properties, MinAP(NSS) and MinAP.

2.4 Reversibility of implications in Figure 2.1

In this section we turn our attention to the question of the reversibility of all arrows in Figure 2.1. Below is a summary of our examples:

(i) Arrow 1 is not reversible (Example 2.4.6),

(ii) arrows 2, 3, 4 and 8 are not reversible, even in Abelian groups (Example 2.4.3),

(iii) arrow 5 is not reversible (Example 2.4.1),

(iv) arrow 6 does not hold outside the class of Abelian groups (Example 2.4.1),

(v) arrow 7 does not hold outside the class of Abelian groups (Example 2.4.6),

(vi) arrow 9 is not reversible for all topological groups (Example 2.4.8).

Example 2.4.1. Each of the following three examples is *a non-trivial* MinAP *Lie group* G.

(i) Shakhmatov and Dikranjan point out in [16, Example 5.4 (f)] that the special linear group $\mathrm{SL}(2,\mathbb{R})$ is known to be minimally almost periodic as seen in [32].

(ii) It is shown in [2, Example 9.11] that the special linear group $\mathrm{SL}(2,\mathbb{C})$ is minimally almost periodic.

(iii) It is shown in [2, Example 9.11] that the special linear group $\mathrm{SL}(2,\mathbb{C})_d$ taken with the discrete topology is minimally almost periodic.

Since the identity map from G to itself is a non-trivial homomorphism, it follows that G is not MinAP(Lie) by Definition 2.1.1(i). Therefore, *arrow 5, and consequently, also arrows 4 and 6, are not reversible for general (non-Abelian) topological groups.* Since G is locally compact, a theorem of Veech [47, Theorem 2.2.1] implies that G *is not extremely amenable* as well.

For our next example, we shall need the following

Lemma 2.4.2. *Each* MinAP *group topology on the integers* $\mathbb{Z}$ *is* NSS*.*

Proof. Let τ be a MinAP group topology on $\mathbb{Z}$. Suppose that $(\mathbb{Z},\tau)$ is not NSS. By Definition 2.1.3, every τ-open neighbourhood of 0 contains a non-trivial subgroup. Since τ is Hausdorff, we can choose $U \in \tau$ such that $0 \in U$ and $cl_{(\mathbb{Z},\tau)}(U) \neq \mathbb{Z}$. Let N be a non-trivial subgroup of U. Then $K = cl_{(\mathbb{Z},\tau)}(N)$ is a τ-closed subgroup of $\mathbb{Z}$ contained in $cl_{(\mathbb{Z},\tau)}(U)$, so K is a proper subgroup of $\mathbb{Z}$ by our choice of U. Therefore, the quotient group $\mathbb{Z}/K$ is non-trivial. Since K is τ-closed, $\mathbb{Z}/K$ is Hausdorff. Let $q : (\mathbb{Z},\tau) \to \mathbb{Z}/K$ be the quotient homomorphism. Since N is non-trivial, so is K. Being a non-trivial subgroup of $\mathbb{Z}$, K has a finite index in $\mathbb{Z}$; that is, the quotient group $\mathbb{Z}/K$ is finite. In particular, $\mathbb{Z}/K$ is compact. We conclude that q is a continuous homomorphism of $(\mathbb{Z},\tau)$ onto a non-trivial compact group $\mathbb{Z}/K$. By Definition 1.4.2(b), this means that $(\mathbb{Z},\tau)$ is not MinAP, contradicting our assumption. This contradiction shows that $(\mathbb{Z},\tau)$ is NSS. □

Example 2.4.3. *There exists a* MinAP *group topology* τ *on the group of integers* $\mathbb{Z}$ *which is not* MinAP(NSS). Note that τ *is both* MinAP(*Lie*) *and* MinAP(*Locally compact*) by Corollary 2.3.6. Therefore, *arrows 2, 3, 4 and 8 of Figure 2.1 are not reversible, even for Abelian groups.* Indeed, by the result Nienhuys [34], $\mathbb{Z}$ is known to admit a minimally almost periodic group topology τ. The topology τ is NSS by Lemma 2.4.2. By Definition 2.1.1 (i), this implies that

$(\mathbb{Z}, \tau)$ is not MinAP(NSS), because the identity homomorphism of $(\mathbb{Z}, \tau)$ is a non-trivial continuous homomorphism onto itself, an NSS group.

Recall that a topological group G is said to be *topologically simple* if G does not contain non-trivial proper closed normal subgroups.

The following fact is part of folklore.

Lemma 2.4.4. *If $f : G \to H$ is a non-trivial continuous homomorphism from a topologically simple group G to a topological group H, then the kernel $ker(f) = \{x \in G : f(x) = e\}$ of f is trivial; that is, f is a monomorphism.*

Proof. Since f is non-trivial, the kernel $ker(f) = \{x \in G : f(x) = e\}$ of f is a proper subgroup of G. Since f is continuous, $ker(f)$ is a closed normal subgroup of G. Since G is topologically simple, we conclude that $ker(f) = \{e\}$. □

Lemma 2.4.5. *Let G be a topologically simple group which is not* NSS*. Then G is* MinAP(NSS)*.*

Proof. Assume that G is not MinAP(NSS). By Definition 2.1.1(i), there exists a non-trivial continuous homomorphism $f : G \to H$ from G to some NSS topological group H. Then $\ker(f) = \{e\}$ by Lemma 2.4.4. Since H is NSS, there exists an open neighbourhood U of the identity of H containing no non-trivial subgroups of H. Since $ker(f) = \{e\}$ and f is continuous, the preimage $f^{-1}[U]$ of U is an open neighbourhood of G containing no non-trivial subgroups of G. By Definition 2.1.3, this means that G is NSS, in contradiction with our assumption. □

The following is example makes use of a construction of Willis [48]:

Example 2.4.6. *There exists a locally compact group G which is* MinAP(NSS)*. As a consequence, the* MinAP(NSS) *property does not imply* MinAP(*Locally compact*)*.* In particular, *arrow 1 of Figure 2.1 is not reversible.*

Consider the topological group G constructed by Willis in [48, Section 3]. This group is non-discrete, locally compact, topologically simple and totally disconnected. Since G is locally compact and totally disconnected, a classical result of van Dantzig shows that its open subgroups form a neighbourhood basis of the identity. Since G is non-discrete, this implies that it is not NSS. Therefore, G satisfies the hypotheses of Lemma 2.4.5 from which we deduce that G is

MinAP(NSS). Since G is locally compact, the identity homomorphism is a non-trivial continuous homomorphism to itself, a locally compact group. By Definition 2.1.1(i), this shows that G is not MinAP(Locally compact).

Let X be the set. We denote by $S(X)$ the group of all bijections from X onto X with the composition of maps as the group operation. Recall that $S(X)$ is called the *symmetric group of* X. We endow $S(X)$ with the *pointwise convergence topology*, i.e. the topology $S(X)$ inherits from the Tychonoff product X^X when X is equipped with the discrete topology.

Corollary 2.4.7. *For every infinite set X, the symmetric group $S(X)$ is* MinAP(NSS)*.*

Proof. Indeed, $S(X)$ is known to be topologically simple [9, Proposition 7.2.1(b)]. Since X is infinite, every open neighbourhood of the identity of $S(X)$ contains a non-trivial subgroup. In particular, $S(X)$ is not NSS. Now the conclusion follows from Lemma 2.4.5. □

The following example is due to Shakhmatov.

Example 2.4.8 (Shakhmatov)**.** Let G be the symmetric group $S(\mathbb{N})$ of the natural numbers $\mathbb{N}$ equipped with the topology of pointwise convergence. Then *G is a Polish group which is both* MinAP(*Locally compact*) *and* MinAP(NSS)*, but G does not admit an* SSGP(∞) *group topology. Therefore, arrow 9 of Figure 2.1 is not reversible.* Indeed, according to [16, Example 5.4 (ii)], G does not admit an SSGP(∞) group topology. Furthermore, G is MinAP(NSS) by Corollary 2.4.7. So it remains only to show that G is MinAP(Locally compact). By contradiction, assume that G is not MinAP(Locally compact). By Definition 2.1.1(i), there exists a non-trivial continuous homomorphism $f : G \to H$ from G to some locally compact group H. Note that f is a monomorphism by Lemma 2.4.4. Since H is Hausdorff and the topology of pointwise convergence on G is the weakest Hausdorff group topology on G by [22, Theorem 2], it follows that f is a topological isomorphism between G and $f[G]$. Since G is complete, so is the subgroup $f[G]$ of H. Since a complete group is closed in every bigger (Hausdorff) group, $f[G]$ must be closed in H. Since H is locally compact, so is $f[G]$. Since f is a topological isomorphism between G and $f[G]$, we conclude that G must be locally compact as well. By a theorem of Gaughan [22, Theorem 3], a locally compact group topology on the symmetric group $S(\mathbb{N})$ must be discrete. Since G is non-discrete, this contradiction finishes the proof that G is MinAP(Locally compact).

2.5 The algebraic structure of Abelian MinAP(NSS) groups

In the next result we give a full description of the algebraic structure of Abelian MinAP(NSS) groups.

Corollary 2.5.1. *An Abelian group G admits a* MinAP(NSS) *group topology if and only if one of the following two conditions holds:*

(i) G is of infinite divisible rank, or

(ii) the quotient $H = G/t(G)$ of G by its torsion part $t(G)$ has finite free rank $r_0(H)$ and $r(H/A) = \omega$ for some (equivalently, every) free subgroup A of H such that H/A is torsion.

Proof. To prove the "if" part, assume that G satisfies either (i) or (ii). From the implication (c)→(b) of Theorem 4.1.1, G admits an $\texttt{SSGP}(\alpha)$ group topology for some ordinal α. This topology is $\texttt{SSGP}(\infty)$ by Definition 2.3.1, and so MinAP(NSS) by Corollary 2.3.8.

To prove the "only if" part, assume that G admits a MinAP(NSS) group topology. By Corollary 2.3.8 and Definition 2.3.1, this topology is $\texttt{SSGP}(\alpha)$ for some ordinal α. Applying the implication (b)→(c) of Theorem 4.1.1, we conclude that G satisfies either (i) or (ii). □

From the equivalence of items (a) and (c) of Theorem 4.1.1 and Corollary 2.5.1 we get the following

Corollary 2.5.2. *An Abelian group admits a* MinAP(NSS) *group topology if and only if it admits an* SSGP *group topology.*

This corollary can be used to find numerous examples of Abelian minimally almost periodic groups which cannot admit MinAP(NSS) group topologies. Indeed, by this result it is now enough to focus on MinAP groups which cannot admit SSGP group topologies. The author recommends the reader to check [4, 16, 25] for numerous examples. We highlight the following example, as it shows the existence of extremely amenable groups which do not admit a group topology with the MinAP(NSS) property:

Example 2.5.3. *No finite power of the integers $\mathbb{Z}$ admits a* MinAP(NSS) *group topology.* Indeed, by [4, Corollary 3.14] no finite power of the integers admits an SSGP group topology, so they may not admit a MinAP(NSS) group topology by Corollary 2.5.2. On the other hand, it is well-known that $\mathbb{Z}$ *can be equipped with an extremely amenable group topology* [23, Theorem 1.1].

We note that this example strengthens Example 2.4.3.

Remark 2.5.4. It follows from Corollary 2.3.6 that the algebraic structures of Abelian MinAP(Lie) groups and Abelian MinAP(Locally compact) groups are the same and coincide with the algebraic structure of Abelian MinAP groups. The latter is fully described in [11].

Remark 2.5.5. In 2019, Shakhmatov and the author [43] proved that a non-Abelian free group of infinite rank admits a group topology with the property of DW (see Chapter 5). To the knowledge of the author, this provides first examples of non-Abelian DW groups which are not obtained by means of Dierolf-Warken's construction.

2.6 NSS groups and the $\mathbf{S}_G$ operator

In this section we recall the concept of the $\mathbf{S}_G$ operator introduced in [16]. For a subgroup H of a group G we denote by $\mathrm{N}_G(H)$ the largest *normal* subgroup of G which is contained in H (this is often referred to as the *heart* of H in G). Observe that $\mathrm{N}_G(H)$ is closed whenever H is closed, as the closure of a normal subgroup of G is also normal in G. Finally, if $X \subseteq G$ is now a subset of a topological group G, then we denote by $\mathrm{Cs}_G(X)$ the *smallest closed subgroup* of G containing X. Since the closure of a subgroup of a given topological group is also a subgroup of it, we can describe Cs_G as

$$\mathrm{Cs}_G(X) = cl_G(\langle X \rangle) \tag{2.1}$$

for every subset $X \subseteq G$. We begin with the definition of the operator $\mathbf{S}_G$.

Definition 2.6.1 ([16, Section 2])**.** Let G be a topological group. We define the operator $\mathbf{S}_G : \mathcal{P}(G) \to \mathcal{P}(G)$ as the composition $\mathbf{S}_G = \mathrm{N}_G \circ \mathrm{Cs}_G \circ \mathrm{Cyc}_G$. That is, for every subset $X \subseteq G$ the operator $\mathbf{S}_G$ assigns the subset

$$\mathbf{S}_G(X) = \mathrm{N}_G \circ \mathrm{Cs}_G \circ \mathrm{Cyc}_G(X). \tag{2.2}$$

Observe that for every $X \subseteq G$ the result of the operator $\mathbf{S}_G(X)$ is always a closed normal subgroup of G. In the case of an *Abelian* group, however, every subgroup is normal. This implies that the operator N_G is simply the identity function when it is restricted to the set of all subgroups of G.

Remark 2.6.2. If G is an Abelian topological group, then the equality $\mathbf{S}_G = \mathrm{Cs}_G \circ \mathrm{Cyc}_G$ holds, so

$$\mathbf{S}_G(X) = cl_G(\langle \mathrm{Cyc}(X) \rangle) \text{ for every } X \subseteq G. \tag{2.3}$$

The relationship between NSS groups the $\mathbf{S}_G$ operator becomes clear from the following:

Proposition 2.6.3. *Let G be a topological group. Consider the following properties:*

(i) G is an NSS *group,*

(ii) $\mathrm{Cyc}_G(U) = \{e_G\}$ *for some open neighbourhood of the identity U of G, and*

(iii) $\mathbf{S}_G(U) = \{e_G\}$ *for some open neighbourhood of the identity U of G.*

Then (i) $\implies$ (ii) $\implies$ (iii). If we assume G to be Abelian, then all three properties are equivalent.

Proof. Implications (i) through (iii) were proven implicitly in [16, Proposition 5.3 (i)].

We now focus on proving the implication (iii) $\implies$ (i) for an Abelian group G. Let U be an open neighbourhood of 0 of an Abelian group G satisfying $\mathbf{S}_G(U) = \{0\}$.

Claim 1. If $h \in G$ and $\langle h \rangle \subseteq U$, then $h = 0$.

Proof. Note that $h \in \mathrm{Cyc}_G(U)$ by our assumption and (1.2). Since the group G is Abelian, we can use (2.3) to conclude that

$$h \in \mathrm{Cyc}_G(U) \subseteq cl_G(\langle \mathrm{Cyc}_G(U) \rangle) = \mathbf{S}_G(U). \tag{2.4}$$

Since $\mathbf{S}_G(U) = \{0\}$ holds by hypothesis, we deduce that $h = 0$. □

It easily follows from Claim 1 that U does not contain non-trivial subgroups of G, so G is NSS by Definition 2.1.3. □

Remark 2.6.4. As an observation, we point out that the hypothesis of G being Abelian is only used to perform the step outlined in (2.4). Indeed, if the group G is not Abelian, then the cyclic subgroup $\langle h \rangle$ may fail to be normal. If $\langle h \rangle$ is not normal, then it may not necessarily be a subset of $\mathbf{S}_G(U)$ as per (2.2) (the additional operation $\mathrm{N}_G(U)$ now plays a non-trivial role).

We shall need the following result about the relation of the $\mathbf{S}_G$ operator and topological quotients.

Lemma 2.6.5 ([16, Lemma 4.9](ii))**.** *Let H be a quotient of a topological group G. Let $q : G \to H$ be the corresponding quotient mapping. The equality*

$$\mathbf{S}_G(q^{-1}(Y)) = q^{-1}(\mathbf{S}_H(Y)) \tag{2.5}$$

holds for every subset Y of H satisfying $e_H \in Y$.

2.7 Transfinite iterations of the $\mathbf{S}_G$ operator

One important detail to observe is that the operator $\mathbf{S}_G$ is *idempotent*; that is, the equality $\mathbf{S}_G(\mathbf{S}_G(X)) = X$ holds for every $X \subseteq G$ [16, Lemma 4.1]. Therefore, if one wishes to iterate the operator $\mathbf{S}_G$, one needs to take an alternative approach described in [16, Section 2].

For every ordinal α, the α-th iteration $\mathbf{S}_G^{(\alpha)}$ of $\mathbf{S}_G$ is defined as follows. Let

$$\mathbf{S}_G^{(0)}(X) = \{e_G\} \text{ for every } X \subseteq G. \tag{2.6}$$

If $\alpha > 0$ is an ordinal and $\mathbf{S}_G^{(\beta)}$ has already been defined for all $\beta < \alpha$, then we let

$$\mathbf{S}_G^{(\alpha)}(X) = \mathbf{S}_G(X \cdot \bigcup_{\beta<\alpha} \mathbf{S}_G^{(\beta)}(X)) \text{ for every } X \subseteq G. \tag{2.7}$$

From (2.6) and (2.7) one can see that

$$\mathbf{S}_G^{(1)}(X) = \mathbf{S}_G(X) \text{ for every } X \subseteq G. \tag{2.8}$$

Here we make great emphasis on the use of the brackets (α) to indicate the α-th iteration of

$\mathbf{S}_G$. This is done deliberately to differentiate these operations from the usual function composition. As a helpful term we shall use the following.

Definition 2.7.1. Let G be a topological group. If $X \subseteq G$ is a subset of G and α is an ordinal, then we shall say that the operator $\mathbf{S}_G$ *stabilizes* X *in* α if the equality $\mathbf{S}_G^{(\alpha)}(X) = \mathbf{S}_G^{(\delta)}(X)$ holds for every $\delta > \alpha$.

One of the key properties held by the operator $\mathbf{S}_G^{(\alpha)}$ is its *monotonicity*:

Lemma 2.7.2 ([16, Lemma 4.8 (ii)])**.** *Let X be a subset of a topological group G. If β and α are ordinals satisfying $\beta < \alpha$, then the inclusion $\mathbf{S}_G^{(\beta)}(X) \subseteq \mathbf{S}_G^{(\alpha)}(X)$ holds.*

For a cardinal κ we denote by κ^+ its cardinal successor. We shall make use of the following observation from elementary Set Theory:

Lemma 2.7.3. *Let X be a set. Suppose that for every ordinal α we have a subset Y_α of X. If $Y_\beta \subseteq Y_\gamma$ holds whenever $\beta \leq \gamma$, then there exists some ordinal $\delta < |X|^+$ such that $Y_\beta = Y_\delta$ for all $\beta > \delta$.*

Lemma 2.7.4. *Let G be a topological group. For every subset X of G the operator $\mathbf{S}_G$ stabilizes X in $|G|^+$.*

Proof. Let $X \subseteq G$ be arbitrary. We use Lemma 2.7.2 to see that $\mathbf{S}_G^{(\beta)}(X) \subseteq \mathbf{S}_G^{(\alpha)}(X)$ holds whenever the ordinals β and α satisfy $\beta \leq \alpha$. We then use Lemma 2.7.3 (setting $Y_\alpha = \mathbf{S}_G^{(\alpha)}(X)$) to find some ordinal $\delta < |X|^+$ such that $\mathbf{S}_G^{(\beta)}(X) = \mathbf{S}_G^{(\delta)}(X)$ holds for every $\beta > \delta$. Finally, observe that the inequalities $\delta < |X|^+ \leq |G|^+$ hold. This implies that $\mathbf{S}_G$ stabilizes X in $|G|^+$ by Definition 2.7.1. Since the subset X of G was arbitrary, this concludes the proof of our lemma. □

Lemma 2.7.5. *Let G be a topological group. If H is an open subgroup of G then $\mathbf{S}_G^{(\alpha)}(H) \subseteq H$ for every ordinal α.*

Proof. We shall prove this by induction on α. Since H is an open subgroup of G, then it is also closed in G. Observe the following: if H is a closed subgroup of G then $\mathbf{S}_H(H) = \mathrm{N}_G(H) \subseteq H$ is satisfied by (2.2). We finally apply (2.6) and (2.7) to verify that

$$\mathbf{S}_G^{(0)}(H) = \{e_G\} \subseteq \mathbf{S}_G^{(1)}(H) = \mathbf{S}_H(H) \subseteq H. \tag{2.9}$$

As our induction hypothesis, let us assume that

$$\mathbf{S}_G^{(\beta)}(H) \subseteq H$$

holds for every ordinal $\beta < \alpha$. Our goal is to show that $\mathbf{S}_G^{(\alpha)}(H) \subseteq H$ holds.

Claim 2. The following inclusion is satisfied:

$$H \cdot \bigcup_{\beta<\alpha} \mathbf{S}_G^{(\beta)}(H) \subseteq H. \tag{2.10}$$

Proof. By our induction hypothesis, $\mathbf{S}_G^{(\beta)}(H) \subseteq H$ for every $\beta < \alpha$. This shows that

$$\bigcup_{\beta<\alpha} \mathbf{S}_G^{(\beta)}(H) \subseteq H.$$

Since H is a subgroup of G, the above equation and (1.1) imply our desired conclusion:

$$H \cdot \bigcup_{\beta<\alpha} \mathbf{S}_G^{(\beta)}(H) \subseteq H \cdot H \subseteq H.$$

□

Claim 3. The inclusion $\mathbf{S}_G^{(\alpha)}(H) \subseteq H$ holds.

Proof. If we apply Lemma 2.7.2(ii) (with respect to $\mathbf{S}_G^{(1)}$) to (2.10), then we have

$$\mathbf{S}_G^{(1)}(H \cdot \bigcup_{\beta<\alpha} \mathbf{S}_G^{(\beta)}(H)) \subseteq \mathbf{S}_G^{(1)}(H). \tag{2.11}$$

By (2.6), we can replace $\mathbf{S}_G^{(1)}$ with $\mathbf{S}_G$ in (2.11). We then use (2.7) and (2.9) to deduce that

$$\mathbf{S}_G^{(\alpha)}(H) = \mathbf{S}_G(H \cdot \bigcup_{\beta<\alpha} \mathbf{S}_G^{(\beta)}(H)) \subseteq \mathbf{S}_G(H) \subseteq H.$$

□

Claim 3 shows that the induction step is satisfied, concluding our induction. We have thus shown that the inclusion $\mathbf{S}_G^{(\alpha)}(H) \subseteq H$ holds for every ordinal α, as desired. □

2.8 Proof of theorem 2.3.7

Lemma 2.8.1. *Suppose that U is a subset of a topological group G containing its identity e_G, α is an ordinal, $H = G/\mathbf{S}_G^{(\alpha)}(U)$ is the quotient group of G and $q : G \to H$ is the corresponding quotient map. If the operator $\mathbf{S}_G$ stabilizes U in α, then $\mathbf{S}_H(V) = \{e_H\}$, where $V = q[U]$.*

Proof. Since V is a subset of H satisfying $e_H \in V$, we apply Lemma 2.6.5 to deduce the equality

$$\mathbf{S}_G(q^{-1}(V)) = q^{-1}(\mathbf{S}_H(V)). \tag{2.12}$$

Since q is a homomorphism with the kernel $\mathbf{S}_G^{(\alpha)}(U)$, we have

$$q^{-1}(V) = q^{-1}(q[U]) = U \cdot \mathbf{S}_G^{(\alpha)}(U),$$

so

$$\mathbf{S}_G(q^{-1}(V)) = \mathbf{S}_G(U \cdot \mathbf{S}_G^{(\alpha)}(U)). \tag{2.13}$$

We now apply Lemma 2.7.2 for every ordinal $\gamma < \alpha$ to obtain the equality

$$\mathbf{S}_G^{(\alpha)}(U) = \bigcup_{\gamma<\alpha+1} \mathbf{S}_G^{(\gamma)}(U). \tag{2.14}$$

We then apply (2.14) to the right-hand side of (2.13) to see that

$$\mathbf{S}_G(U \cdot \mathbf{S}_G^{(\alpha)}(U)) = \mathbf{S}_G\left(U \cdot \bigcup_{\gamma<\alpha+1} \mathbf{S}_G^{(\gamma)}(U)\right) = \mathbf{S}_G^{(\alpha+1)}(U) \tag{2.15}$$

holds by (2.7). Since the operator $\mathbf{S}_G$ stabilizes U in α by our hypothesis, Definition 2.7.1 implies that the equality

$$\mathbf{S}_G^{(\alpha)}(U) = \mathbf{S}_G^{(\alpha+1)}(U) \tag{2.16}$$

holds. We then combine (2.12),(2.13),(2.15) and (2.16) to obtain the equality

$$q^{-1}(\mathbf{S}_H(V)) = \mathbf{S}_G^{(\alpha)}(U).$$

Since the mapping q is surjective, we apply it to both sides of the above equation to get

$$\mathbf{S}_H(V) = q[q^{-1}(\mathbf{S}_H(V))] = q[\mathbf{S}_G^{(\alpha)}(U)] = \{e_H\}.$$

This finishes the proof. □

Corollary 2.8.2. *Suppose that U is an open neighbourhood of* 0 *of an Abelian topological group G such that the operator $\mathbf{S}_G$ stabilizes U in α for some ordinal α. Then the quotient group $H = G/\mathbf{S}_G^{(\alpha)}(U)$ is* NSS.

Proof. Denote by $q : G \to H$ the quotient mapping from G to H. Since q is open, the set $V = q[U]$ is an open neighbourhood of 0 of H. By Lemma 2.8.1, $\mathbf{S}_H(V) = \{0\}$ holds. Since the group H is Abelian, we can apply the implication (iii)→(i) of Proposition 2.6.3 to conclude that H is NSS. □

Proof of Theorem 2.3.7: Let G be an Abelian MinAP(NSS) group. Our goal is to prove that G is SSGP($|G|^+$). By Definition 1.5.8, to show this, it suffices to check the following

Claim 4. $\mathbf{S}_G^{(|G|^+)}(U) = G$ for every open neighbourhood U of 0 of G.

Proof. By Lemma 2.7.4, the operator $\mathbf{S}_G$ stabilizes U in $|G|^+$. Since the group G is Abelian, we can apply Corollary 2.8.2 to deduce that the quotient group $H = G/\mathbf{S}_G^{(|G|^+)}(U)$ of G is NSS. Let $q : G \to H$ be the (continuous) quotient homomorphism. Since G is MinAP(NSS) by hypothesis and H is an NSS group, q is trivial by Definition 2.1.1 (i). Therefore, $\mathbf{S}_G^{(|G|^+)}(U) = ker(q) = G$. □

2.9 Proof of Theorem 2.3.3

The following proposition is an improvement of [16, Example 5.4(b)] closely resembling a classical folklore fact that every Abelian MinAP topological group has no proper open subgroup. However, the strengthening of minimal almost periodicity to the SSGP(∞) property allows one to remove the hypothesis on the group being Abelian.

Proposition 2.9.1. *If H is an* $\mathtt{SSGP}(\infty)$ *subgroup of a topological group G, then H is contained in every open subgroup of G.*

Proof. Assume that G' is an open subgroup of G. Our goal is to show the inclusion $H \subseteq G'$. Since H is $\mathtt{SSGP}(\infty)$ by our assumption, we can use Definition 2.3.1 to fix some ordinal α such that H is $\mathtt{SSGP}(\alpha)$. Since G' is open in G, the intersection $H' = G' \cap H$ is an open subgroup of H. In particular, H' is an open neighbourhood of the identity of H. Since H is $\mathtt{SSGP}(\alpha)$, we deduce from Definition 1.5.8 that $\mathbf{S}_H^{(\alpha)}(H') = H$. Now $H = \mathbf{S}_H^{(\alpha)}(H') \subseteq H' = G' \cap H$ by Lemma 2.7.5. This establishes the inclusion $H \subseteq G'$. ☐

Proof of Theorem 2.3.3: Let G be an $\mathtt{SSGP}(\infty)$ group, and suppose that $f : G \to H$ is a continuous homomorphism from G to a locally compact group H. Note that G is MinAP(NSS) by Proposition 2.3.2, and in particular, G is MinAP by arrow 4 of Figure 2.1.

Claim 5. The image $f[G]$ of G is contained in every open subgroup H' of H.

Proof. Indeed, assume that H' is an open subgroup of H. Then $H' \cap f[G]$ is an open subgroup of $f[G]$. It follows from Definition 2.3.1 and [16, Theorem 5.2(i)] that every continuous homomorphic image of an $\mathtt{SSGP}(\infty)$ group is again an $\mathtt{SSGP}(\infty)$ group. As f is a continuous homomorphism, $f[G]$ is an $\mathtt{SSGP}(\infty)$ group. Since no $\mathtt{SSGP}(\infty)$ group has a proper open subgroup by Proposition 2.9.1, we conclude that $H' \cap f[G] = f[G]$ holds. This shows that $f[G] \subseteq H'$. ☐

Claim 6. There exists a compact subgroup N of H such that $f[G] \subseteq N$.

Proof. Recall that H is locally compact. By a theorem of Yamabe [49, Theorem 5'], there exists an open subgroup H' of H and a normal (with respect to H') compact subgroup N of H' such that the quotient H'/N is a Lie group. Since H' is open in H, Claim 5 implies that $f[G] \subseteq H'$. Therefore, if $q : H' \to H'/N$ denotes the quotient homomorphism, the composition $f \circ q : G \to H'/N$ is well-defined. Since G is MinAP(NSS) and H'/N is NSS (being a Lie group), the homomorphism $f \circ q$ must be trivial by Definition 2.1.1(i). This implies the inclusion $f[G] \subseteq ker(q) = N$. ☐

Claim 7. The homomorphism $f : G \to H$ is trivial.

Proof. By Claim 6, there exists a compact subgroup N of H such that $f[G] \subseteq N$. So f can be viewed as a continuous homomorphism from G to a compact group N. Since G is MinAP, the homomorphism f must be trivial by Definition 1.4.2(b). □

Since Claim 7 holds for every continuous homomorphism $f : G \to H$ from G to a locally compact group H, we conclude from Definition 2.1.1(i) that G is MinAP(Locally compact). □

2.10 Open questions

We finish this chapter with a list of open questions.

Question 2.10.1. Can one describe which (Abelian) MinAP(NSS) groups are also extremely amenable?

Question 2.10.2. Can one find a class $\mathcal{C}$ of topological groups such that an Abelian topological group is extremely amenable if and only if it is MinAP($\mathcal{C}$) for this class $\mathcal{C}$?

Question 2.10.3. Can one find a suitable class $\mathcal{C}$ of topological groups such that a (Abelian) topological group is DW (or SSGP) if and only if it is MinAP($\mathcal{C}$) for this class $\mathcal{C}$?

Question 2.10.4. For a "reasonable" class $\mathcal{C}$ of topological groups, can one describe the (Abelian) topological groups which belong to the class MinAP($\mathcal{C}$) (or MAP($\mathcal{C}$), respectively)?

Question 2.10.5. For a "reasonable" class $\mathcal{C}$ of topological groups, can one describe the algebraic structure of the (Abelian) groups which admit a MinAP($\mathcal{C}$) (or a MAP($\mathcal{C}$)) group topology?

The last question can be viewed as a natural extension of Problem 1.4.10.

Chapter 3

Precompact continuous homomorphic images of topological groups

3.1 Introduction

The MinAP topological groups (Definition 1.4.2(b)) were introduced by von Neumann and Wigner [33] with the idea to distinguish points via almost periodic functions. Obtaining examples of MinAP topological groups is difficult, as one would require to check all possible continuous homomorphisms to all possible compact groups. Due to this, providing a characterization of groups which can admit a minimally almost periodic group topology became a major open problem in the theory of topological groups.

In the realm of Abelian groups, the combination of results by Gabriyelyan [19] and Dikranjan-Shakhmatov [11] provides a full description of the algebraic structure of groups that admit a minimally almost periodic group topology. The former result focuses on the case of bounded groups, while the latter is a solution of the famous Protasov-Comfort problem regarding all remaining unbounded Abelian groups. These results were an important source of inspiration to the author, particularly for understanding the relationship between a property of topological groups, and the algebraic restrictions it demands for it to hold true.

The main motivation of the results in this chapter comes from attempting a very naïve modification of Definition 1.4.2(b) summarized in the following question:

Question 3.1.1. What are topological groups every continuous homomorphic image of which in a compact group is *finite*?

This question naturally splits into two parts:

Question 3.1.2. (i) Does this modification of the MinAP property result in a new class of topological groups?

(ii) If (i) has a positive answer, then how does this modification alter the algebraic restrictions imposed by the MinAP property in Abelian groups?

We completely resolve both items of the above question in this chapter. Furthermore, we introduce the following notion: A topological group is MinAP modulo a property **P** of topological groups if every continuous homomorphic image of it in a compact group has property **P**. Topological groups from our naïve Question 3.1.1 are precisely MinAP modulo finite groups in this new terminology. We compare simultaneously five similar naïve modifications of Definition 1.4.2(b): MinAP modulo finite, modulo compact, modulo bounded, modulo torsion and modulo connected.

The chapter has been organized as follows. In Section 3.2, for a given property **P** of topological groups, we introduce the notion of MinAP modulo **P** groups and study their basic properties.

In Section 3.3 we consider five concrete properties **P**: one set-theoretic (finite), two algebraic (torsion and bounded torsion) and two topological (compact and connected). We outline the relationships between these properties in Figure 3.1 and show that they are all distinct and differ from the classical minimal almost periodicity (Examples 3.3.2 and 3.6.8).

In Section 3.4 we give a quick summary of the Bohr compactification and the von Neumann kernel of a topological group, as well as the Bohr topology of a MAP group.

In Section 3.5 we interpret the categorical meaning of the Bohr compactification of a topological group (Proposition 3.5.3 and Corollary 3.5.4) and clarify the role of the von Neumann kernel when it is used in topological quotients. Specifically, we prove that the quotient group $G/\mathfrak{n}(G)$ of a topological group G with respect to its von Neumann kernel $\mathfrak{n}(G)$ is the reflection of G in the class of MAP groups, with the quotient map playing the role of the reflection homomorphism (Proposition 3.5.5). Furthermore, we describe the Bohr topology of the MAP quotient group $G/\mathfrak{n}(G)$ by showing that this group equipped with its Bohr topology (denoted by $(G/\mathfrak{n}(G))^+$) is topologically isomorphic to the image of G under the canonical map to the Bohr compactification of G (Corollary 3.5.8).

In Section 3.6 we introduce our main result which states that if a property $\mathbf{P}$ of topological groups is an invariant of surjective continuous homomorphisms, then a topological group G is MinAP modulo $\mathbf{P}$ if and only if the image of G under the canonical mapping to its Bohr compactification has property $\mathbf{P}$ (Theorem 3.6.1). Combining this with our results from Section 3.5, we obtain a new version of Theorem 3.6.1 which is significantly easier to verify (Corollary 3.6.2). As an application, we establish that a topological group G is MinAP modulo finite if and only if the von Neumann kernel $\mathfrak{n}(G)$ of G has finite index in G (Remark 3.6.6). This resolves item (i) of Question 3.1.2. We close this section by showing a couple of examples.

In Section 3.7 we find a necessary condition for the MinAP modulo connected property to coincide with standard minimal almost periodicity in Abelian groups. Specifically, we show that if the 0-rank of an Abelian group is less than the cardinality of the continuum, then MinAP modulo connected and MinAP properties coincide (see Theorem 3.7.5).

In Section 3.8 we prove that if an Abelian group G can be equipped with a MinAP modulo connected group topology, then for every positive integer $m \in \mathbb{N}^+$ the subgroup mG of G is either the trivial subgroup or has infinite cardinality. This condition becomes sufficient as well, as it is one of the conditions appearing in Dikranjan and Shakhmatov's description of Abelian groups admitting minimally almost periodic group topologies (see Theorem 1.4.12). These results provide some answers to Question in 3.10.1.

In Section 3.9 we deduce from the results of Gabriyelyan and Dikranjan-Shakhmatov on minimally almost periodic topologizations of Abelian groups that every Abelian group is a direct sum of a finite group and a group which admits a MinAP group topology (Theorem 3.9.5). It now easily follows that, for every property $\mathbf{P}$ satisfied by all finite groups, every Abelian group admits a MinAP modulo $\mathbf{P}$ group topology (Corollary 3.9.6). In particular, every Abelian group admits a MinAP modulo finite group topology (Corollary 3.9.7). This implies a slightly more general result: Every Abelian group G admits a group topology τ such that every continuous homomorphic image of (G, τ) in a precompact group is finite (Corollary 3.9.8).

Coming back to Question 3.1.1, we see that the modification of the definition of minimal almost periodicity proposed in this question becomes somewhat weak for Abelian groups, as it poses no restrictions on the algebraic structure. This resolves item (ii) of Question 3.1.2.

To finish this chapter, in Section 3.10 we propose some open questions.

3.2 MinAP modulo a property P of topological groups

We formulate a general concept inspired by Question 3.1.1.

Definition 3.2.1. Let **P** be a property of topological groups. We say that a topological group G is MinAP *modulo* **P** (MinAP *mod* **P**) if for each continuous homomorphism $f : G \to K$ from G to a compact group K, the image $f[G]$ of G considered as a subgroup of K has property **P**.

Remark 3.2.2. Clearly, when **P** is the property of being the trivial group, MinAP modulo **P** groups are precisely the classical minimally almost periodic groups.

Recall that a topological group is *precompact* if and only if it is topologically isomorphic to a subgroup of some compact group. The following remark is clear from this definition.

Remark 3.2.3. Let **P** be a property of topological groups.

(i) A topological group G is MinAP modulo **P** if and only if every continuous homomorphic image of G in a precompact group has property **P**.

(ii) If a topological group G is MinAP modulo **P** and precompact, then G has property **P**.

(iii) If **P** is an invariant of surjective continuous homomorphisms, then every topological group with property **P** is MinAP modulo **P**.

For our convenience, we shall use the following notation:

Definition 3.2.4. We shall say a property is **P** *upwards hereditary* provided that: for every Hausdorff space X and every pair of subspaces $A, B \subseteq X$, if the inclusion $A \subseteq B \subseteq cl_X(A)$ holds and A has property **P**, then B also has property **P** (here $cl_X(A)$ denotes the closure of A in X.)

Proposition 3.2.5. *Let* **P** *be a property of topological groups.*

(i) If $f : G \to H$ is a surjective continuous homomorphism from a MinAP *modulo* **P** *group G onto a topological group H, then H is* MinAP *modulo* **P**.

(ii) *Let H be a topological group having a dense* MinAP *modulo* **P** *subgroup. If* **P** *is upwards hereditary, then H is* MinAP *modulo* **P***.*

Proof. (i) Follows in fairly straightforward fashion.

(ii) Let $f : H \to K$ be a continuous homomorphism from H to a compact group K. Since G is a subgroup of H, the restriction mapping $f\restriction_G : G \to K$ is a continuous homomorphism from G to K. Since G is MinAP modulo **P**, the image $f[G] = f\restriction_G[G] \subseteq K$ has property **P** as a subgroup of K. By continuity of f, the following inclusion holds:

$$f[G] \subseteq f[H] = f[cl_H(G)] \subseteq cl_K(f[G]).$$

Since **P** is upwards hereditary by hypothesis, Definition 3.2.4 implies that $f[H]$ itself has property **P**. Since $f : H \to K$ was an arbitrary homomorphism from H to K, we conclude that H satisfies all conditions of Definition 3.2.1, and is therefore MinAP modulo **P**. □

Proposition 3.2.6. *Let* **P** *be a property of topological groups invariant under finite direct products and surjective continuous homomorphisms. Then every finite direct product (equivalently, direct sum) of* MinAP *modulo* **P** *groups is* MinAP *modulo* **P***.*

Proof. The conclusion of our proposition would easily follow by induction provided we can prove that it holds for two factors.

Let G and H be MinAP mod **P** topological groups. Let $f : G \times H \to K$ be an arbitrary continuous homomorphism from $G \times H$ to some compact group K. It suffices to show that the image $f[G \times H]$ has property **P**.

Let $\varphi : G \times H \to K \times K$ be the homomorphism defined by

$$\varphi(x, y) = (f(x, e), f(e, y)) \text{ for } (x, y) \in G \times H. \tag{3.1}$$

Since f is continuous, so is φ. Let us denote by $m : K \times K \to K$ the product mapping of K defined by $m(x, y) = x \cdot y$ for $(x, y) \in K \times K$.

Claim 8. The equality $m \circ \varphi = f$ holds.

Proof. For every $(x, y) \in G \times H$ we have

$$m(\varphi(x, y)) = m((f(x, e), f(e, y))) = f(x, e) \cdot f(e, y) = f(x, y),$$

as f is a homomorphism. This shows the equality $m \circ \varphi = f$. $\square$

Claim 9. The restriction $m \restriction_{\varphi[G\times H]} : \varphi[G \times H] \to K$ of m to $\varphi[G \times H]$ is a homomorphism.

Proof. This follows from Claim 8 and the fact that both φ and f are homomorphisms. $\square$

Claim 10. $\varphi[G \times H]$ has property **P**.

Proof. Note that G is topologically isomorphic to $M = G \times \{e\}$ and H is topologically isomorphic to $N = \{e\} \times H$. Since G and H are MinAP modulo **P**, so are M and N. Since f is a continuous homomorphism, both $f[M]$ and $f[N]$ have property **P** by Definition 3.2.1. Since **P** is assumed to be invariant under finite direct products, the product $f[M] \times f[N]$ has property **P**. Finally, $\varphi[G \times H] = f[M] \times f[N]$ by (3.1). $\square$

The mapping m is continuous, so Claim 9 implies $m \restriction_{\varphi[G\times H]}$ is a continuous homomorphism. Since $\varphi[G \times H]$ has property **P** by Claim 10, and **P** is an invariant of surjective continuous homomorphisms, $m \restriction_{\varphi[G\times H]} [\varphi[G \times H]]$ has property **P**. We now apply Claim 8 to deduce that

$$m \restriction_{\varphi[G\times H]} [\varphi[G \times H]] = m \circ \varphi[G \times H] = f[G \times H].$$

We have shown that $f[G \times H]$ has property **P**, as desired. $\square$

3.3 Five properties P and the corresponding MinAP modulo P groups

The following is clear from Definition 3.2.1.

Remark 3.3.1. If **P** and **Q** are properties of topological groups such that **P** implies **Q**, then

$$\text{MinAP mod } \mathbf{P} \to \text{MinAP mod } \mathbf{Q}.$$

In this chapter we shall focus on five properties **P**: finite, bounded torsion, torsion, compact and connected. The corresponding five new properties MinAP modulo **P** are weaker than MinAP. Indeed, from Remarks 3.2.2 and 3.3.1, we get the following diagram of implications:

$$\begin{array}{ccccc} \text{MinAP} & \xrightarrow{\;\;1\;\;} & \text{MinAP mod finite} & \xrightarrow{\;2\;} & \text{MinAP mod compact} \\ \big\downarrow{\scriptstyle 3} & & \big\downarrow{\scriptstyle 4} & & \\ \text{MinAP mod connected} & & \text{MinAP mod bounded} & \xrightarrow{\;5\;} & \text{MinAP mod torsion.} \end{array}$$

Figure 3.1: Diagram of implications

The following examples show that arrows 1,2,4 and 5 (in this order) are not reversible in general. We shall also show arrow 3 is not reversible in Example 3.6.8.

Example 3.3.2. (i) The group $\mathbb{Z}(2)$ consisting of two elements (with discrete topology) is a *finite (so* MinAP *mod finite by Remark 3.2.3(iii)) Abelian group which is not* MinAP *mod connected. Thus, arrow 1 is not reversible.* Indeed, since $\mathbb{Z}(2)$ is precompact and not connected, it is not MinAP mod connected by Remark 3.2.3(ii).

(ii) The circle group $\mathbb{T}$ is a *compact (so* MinAP *mod compact by Remark 3.2.3(iii)) metric Abelian group which is not* MinAP *mod torsion. Thus, arrow 2 is not reversible.* Indeed, since $\mathbb{T}$ is precompact and not torsion, it cannot be MinAP mod torsion by Remark 3.2.3(ii).

(iii) The Cantor cube $\mathbb{Z}(2)^\omega$ of countable weight is a *Boolean (thus, bounded and so* MinAP *mod bounded by Remark 3.2.3(iii)) compact metric group which is not* MinAP *mod finite. Thus, arrow 4 is not reversible.* Indeed, $\mathbb{Z}(2)^\omega$ is precompact and infinite, so it cannot be MinAP mod finite by Remark 3.2.3(ii).

(iv) For a prime number p, the Prüfer group $\mathbb{Z}(p^\infty)$ equipped with the topology inherited from the circle group $\mathbb{T}$ is a *countable precompact metric torsion (thus,* MinAP *mod torsion by Remark 3.2.3(iii)) Abelian group which is neither* MinAP *mod compact nor* MinAP *mod bounded. Thus, arrow 5 is not reversible.* Indeed, since the precompact group $\mathbb{Z}(p^\infty)$ is neither compact nor bounded, Remark 3.2.3(ii) implies that it is neither MinAP mod compact nor MinAP mod bounded.

Remark 3.3.3. It is worth noting that some of the arrows in Figure 1 are reversible under certain algebraic conditions:

(i) *Arrow 3 is reversible for torsion Abelian groups.* Indeed, let us take an Abelian torsion topological group G, and assume that G is MinAP mod connected. Suppose that G is not MinAP. By Definition 1.4.2(b), there exists a non-trivial homomorphism $f : G \to K$ from G to a compact group K. Since G is Abelian, so are both $f[G]$ and $N = cl_X(f[G])$. Fix $x \in G$ such that $f(x) \neq 0$. As N is a closed subgroup of a compact group K, it is compact as well. By the Peter-Weyl theorem, there exists a continuous character $\chi : N \to \mathbb{T}$ such that $\chi(f(x)) \neq 0$. Therefore, $h = \chi \circ f : G \to \mathbb{T}$ is a continuous homomorphism from G to the compact group $\mathbb{T}$. Since G is MinAP mod connected, $h[G]$ is a connected subgroup of $\mathbb{T}$ by Definition 3.2.1. Since G is torsion, so is $h[G]$. Since $\mathbb{T}$ does not contain non-trivial connected torsion subgroups, $h[G]$ must be trivial. This contradicts the fact that $0 \neq h(x) \in h[G]$.

(ii) *Arrow 5 is reversible for bounded groups.* Indeed, every homomorphic image of a bounded group is again a bounded group.

The next two propositions list basic properties of five new classes from Figure 1.

Proposition 3.3.4. *Let* **P** *be any of the five properties: finite, bounded torsion, torsion, compact and connected. Then:*

(i) the corresponding property MinAP *modulo* **P** *is preserved by taking finite direct products, and*

(ii) every topological group with property **P** *is* MinAP *modulo* **P***.*

Proof. Note that **P** is invariant under taking finite direct products and surjective continuous homomorphisms, so item (i) follows from Proposition 3.2.6 and item (ii) follows from Remark 3.2.3(iii). □

Proposition 3.3.5. *Let* **P** *be one of the four properties: finite, bounded torsion, compact and connected. Then the property* MinAP *modulo* **P** *is upwards hereditary.*

Proof. Indeed, the property **P** is upwards hereditary, so the conclusion follows from Proposition 3.2.5(ii). □

3.4 The Bohr compactification and the von Neumann kernel

All facts in this section (with the possible exception of Proposition 3.4.6), are either well known or part of folklore.

Definition 3.4.1. For every topological group G, there exists a compact group bG along with a continuous homomorphism $b_G : G \to bG$ having the following two properties:

(i) $b_G[G]$ is dense in bG, and

(ii) for each continuous homomorphism $f : G \to K$ from G to a compact group K, there exists a continuous homomorphism $f^+ : bG \to K$ satisfying the equality $f = f^+ \circ b_G$.

The group bG is called a *Bohr compactification of G with respect to b_G*.

The existence of a Bohr compactification is a well-known fact of topological group theory. The next remark asserts the uniqueness of such compactification, thereby allowing one to speak about *the* Bohr compactification.

Remark 3.4.2. The Bohr compactification of a topological group G is unique up to an isomorphism; that is, if bG and $b'G$ are Bohr compactifications of G with respect to b_G and b'_G, respectively, then there exists a topological isomorphism $\pi : bG \to b'G$ between bG and $b'G$ such that $b'_G = \pi \circ b_G$.

Definition 3.4.3. If bG is the Bohr compactification of a topological group G with respect to a homomorphism b_G, then $\mathfrak{n}(G) = ker(b_G)$ is called the *von Neumann kernel* of G.

It follows from Remark 3.4.2, Definition 3.4.3 and the existence of the Bohr compactification, that $\mathfrak{n}(G)$ is a well-defined normal subgroup of a topological group G.

Definition 3.4.4. For a topological group G, we denote by $q_G : G \to G/\mathfrak{n}(G)$ the quotient homomorphism from G to its quotient group $G/\mathfrak{n}(G)$.

Remark 3.4.5. Being a quotient homomorphism, the map q_G is continuous, open and surjective (see [1, Theorem 1.5.11]).

The von Neumann kernel has been an important object of study in the literature of MinAP and MAP topological groups. The following proposition shows that the quotient of a topological group with respect to its von Neumann kernel satisfies a universal property for MAP groups.

Proposition 3.4.6. *Let G be a topological group. For each continuous homomorphism $f : G \to H$ from G to a* MAP *group H there exists a continuous homomorphism $f^- : G/\mathfrak{n}(G) \to H$ satisfying the equality $f = f^- \circ q_G$.*

Proof. Let $f : G \to H$ be a continuous homomorphism from G to a MAP group H.

Claim 11. The inclusion $\mathfrak{n}(G) \subseteq ker(f)$ holds.

Proof. By contradiction, suppose that $\mathfrak{n}(G) \setminus ker(f) \neq \emptyset$. Let us fix some $x \in \mathfrak{n}(G) \setminus ker(f)$. Since H is a MAP group, Definition 1.4.2 implies that there exists a compact group K and a continuous homomorphism $g : H \to K$ such that $g(f(x)) \neq e$. The composition $g \circ f : G \to K$ is a continuous homomorphism from G to the compact group K. By condition (ii) of Definition 3.4.1 there exists a continuous homomorphism $(g \circ f)^+ : bG \to K$ satisfying $g \circ f = (g \circ f)^+ \circ b_G$. Since $x \in \mathfrak{n}(G)$, Definition 3.4.3 implies that

$$g \circ f(x) = ((g \circ f)^+ \circ b_G)(x) = (g \circ f)^+(e) = e.$$

This is in contradiction with $g(f(x)) \neq e$. This shows the inclusion $\mathfrak{n}(G) \subseteq ker(f)$. □

By Remark 3.4.5, q_G is an open surjective mapping onto the quotient $G/\mathfrak{n}(G)$. Therefore, Claim 11 and [1, Corollary 1.5.11] imply that there exists a continuous homomorphism $f^- : G/\mathfrak{n}(G) \to H$ from $G/\mathfrak{n}(G)$ to H such that $f = f^- \circ q_G$ holds, as desired. □

Recall that a homomorphism with the trivial kernel is called a *monomorphism*.

Corollary 3.4.7. *For every topological group G there exists a continuous monomorphism $\iota_G : G/\mathfrak{n}(G) \to bG$ such that*

$$b_G = \iota_G \circ q_G. \tag{3.2}$$

Proof. Since bG is a compact group, it is MAP. Applying Proposition 3.4.6 to $H = bG$ and $f = b_G$, we can find a homomorphism $\iota_G : G/\mathfrak{n}(G) \to bG$ satisfying condition (3.2). It follows from condition (3.2), Definition 3.4.3 and Definition 3.4.4 that $\ker(\iota_G) \subseteq q_G(\ker(b_G)) = q_G(\mathfrak{n}(G)) = \{e\}$. Therefore, q_G has the trivial kernel and thus is a monomorphism. □

Corollary 3.4.8. *$G/\mathfrak{n}(G)$ is* MAP *for a every topological group G.*

Proof. Let ι_G be a continuous monomorphism from the conclusion of Corollary 3.4.7. Clearly, ι_G separates the points of $G/\mathfrak{n}(G)$. Since bG is compact by Definition 3.4.1, $G/\mathfrak{n}(G)$ is MAP by Definition 1.4.2(a). □

In Section 3.5 we shall see that a monomorphism ι_G obtained in Corollary 3.4.7 plays a very important role in characterizing the Bohr topology of the quotient $G/\mathfrak{n}(G)$.

The following well-known proposition establishes the direct connection between the von Neumann kernel and the notions of MAP and MinAP.

Proposition 3.4.9. *A topological G group is:*

(i) maximally almost periodic (MAP) *if and only if* $\mathfrak{n}(G) = \{e\}$*;*

(ii) minimally almost periodic (MinAP) *if and only if* $\mathfrak{n}(G) = G$*.*

Proof. (i) If $\mathfrak{n}(G) = \{e\}$, then $G/\mathfrak{n}(G) \simeq G$ is MAP by Corollary 3.4.8.

For the converse, let us assume that G is MAP. The identity mapping $id_G : G \to G$ is a continuous homomorphism from G to a MAP group G. By Proposition 3.4.6 there exists a continuous homomorphism $id_G^- : G/\mathfrak{n}(G) \to G$ satisfying $id_G = id_G^- \circ q_G$. From this equality we deduce that q_G is injective, and therefore $\{e\} = ker(q_G) = \mathfrak{n}(G)$ holds by Definition 3.4.4.

(ii) Assume that $\mathfrak{n}(G) = G$. Let $f : G \to K$ be a continuous homomorphism from G to a compact group K. Since K is MAP, we may apply Proposition 3.4.6 to find a continuous homomorphism $f^- : G/\mathfrak{n}(G) \to K$ satisfying $f = f^- \circ q_G$. Since $ker(q_G) = \mathfrak{n}(G) = G$ by Definition 3.4.4, the previous equality implies that $G \subseteq ker(f)$. We deduce that f is the trivial homomorphism. Since the group K and the homomorphism $f : G \to K$ were arbitrary, it follows that G is MinAP by Definition 1.4.2(b).

For the converse, assume that G is MinAP. Since bG is compact, the continuous homomorphism $b_G : G \to bG$ is trivial. This shows $G = ker(b_G) = \mathfrak{n}(G)$ as desired. □

Since $\mathfrak{n}(G) = ker(b_G)$ by Definition 3.4.3, the above proposition has an equivalent form in terms of the universal mapping b_G to the Bohr compactification bG:

Remark 3.4.10. A topological G group is:

(i) maximally almost periodic (MAP) if and only if b_G is a monomorphism;

(ii) minimally almost periodic (MinAP) if and only if b_G is the trivial map.

When a topological group G is MAP, item (i) of the above remark implies that b_G is a monomorphism. As such, G is *algebraically* isomorphic to the subgroup $b_G[G]$ of its Bohr compactification bG. Therefore, one can identify G with the subgroup $b_G[G]$ of bG, thereby making possible the following definition (see [5]):

Definition 3.4.11. Let G be a MAP group. The group topology

$$\tau^+ = \{b_G^{-1}(U) : U \text{ is open in } bG\}$$

on G is known as the *Bohr topology* of G, and the pair (G, τ^+) is usually denoted by G^+.

It easily follows from Remark 3.4.2 that the Bohr topology of a MAP group G is well defined and the identity map $\mathrm{id}_G : G \to G^+$ is continuous.

The following is clear from the above definition.

Remark 3.4.12. For a MAP topological group G the group G^+ is isomorphic to the image $b_G[G]$ of G under the associated mapping $b_G : G \to bG$ to its Bohr compactification bG.

3.5 Universality through categorical reflection

Throughout Section 3.4 we encountered two distinct types of universal objects for topological groups. One of them is the Bohr compactification (Definition 3.4.1) which satisfies universality with respect to compact groups. The other is the quotient of a topological group with respect to its von Neumann kernel which is universal with respect to MAP groups (Proposition 3.4.6). In this section we give the appropriate categorical interpretation of the universality of these two objects. In order to do this, we need to recall the following folklore terminology cited from [38, Definition 9.1], with some simplifications.

Definition 3.5.1. (i) For a class $\mathcal{C}$ of topological groups we denote by $\bar{\mathcal{C}}$ the smallest (with respect to inclusion) class of topological groups containing $\mathcal{C}$ which is closed under taking arbitrary products and subgroups.

(ii) Given a topological group G and a class $\mathcal{C}$ of topological groups, a topological group $r(G) \in \bar{\mathcal{C}}$ is called a *reflection* of G in $\mathcal{C}$ provided that there exists a continuous homomorphism $r : G \to r(G)$ (called a *reflection homomorphism*) satisfying the following condition: For every $H \in \mathcal{C}$ and each continuous homomorphism $\phi : G \to H$ one can find a continuous homomorphism $\psi : r(G) \to H$ such that $\phi = \psi \circ r$.

Both the existence and uniqueness of the reflection is well known:

Proposition 3.5.2 ([38, Proposition 9.2])**.** *For every topological group G and each class $\mathcal{C}$ of topological groups, the reflection $r(G)$ of G in $\mathcal{C}$ exists and is unique up to a topological isomorphism.*

The following folklore proposition relates the Bohr compactification of a topological group to its precompact reflection:

Proposition 3.5.3. *For every topological group G, the image $b_G[G]$ under b_G is the reflection of G in the class of precompact groups, and the reflection homomorphism coincides with the associated map $b_G : G \to b_G[G] \subseteq bG$ from G to its Bohr compactification bG.*

Proof. The group $b_G[G]$ is precompact, by virtue of being a subgroup of the compact group bG.

Let $f : G \to H$ be a continuous homomorphism from G to a precompact group H. Let K be a compact topological group containing H as a subgroup. By universality of the Bohr compactification of G (condition (ii) of Definition 3.4.1), there exists a continuous homomorphism $f^+ : bG \to K$ from bG to K satisfying $f = f^+ \circ b_G$. The previous equality implies that the range of f^+ is contained in H. Consider the mapping $f' = f^+ \restriction_{b_G[G]} : b_G[G] \to H$. By construction

$$f = f^+ \circ b_G = f^+ \restriction_{b_G[G]} \circ b_G = f' \circ b_G.$$

By condition (ii) of Definition 3.5.1, $b_G[G]$ is the reflection of G in the class of precompact groups, with its associated reflection mapping $b_G : G \to b_G[G]$. □

Recall that a topological group G is *complete* (in the sense of Weil) if every left Cauchy net converges in G. Basic results about completeness can be consulted in [1, Section 3.6] and [8, Sections 6.2 and 7].

Corollary 3.5.4. *For every topological group G, the Bohr compactification of G coincides with the Weil completion of its reflection in the class of precompact groups. The associated universal mapping coincides with the reflection mapping.*

Proof. Denote by H the reflection of G in the class of precompact groups and denote by $r : G \to H$ the associated reflection mapping. It suffices to prove that the Weil completion cH of H is the Bohr compactification of G with r as its associated mapping.

Claim 12. The group cH is compact.

Proof. Since the group H is precompact, we may assume that it is a subgroup of some compact group K. Every compact group is Weil complete [8, Lemma 7.2.6], so the Weil completion cH of H is contained in K. Since every Weil complete subgroup of K is closed (see [8, Proposition 6.2.7]), this shows that cH is a compact group. □

Claim 13. The reflection map $r : G \to H \subseteq cH$ satisfies all conditions of Definition 3.4.1.

Proof. Since $r(G) = H$ is dense in its completion cH, item (i) of Definition 3.4.1 holds.

Let us check item (ii). Let $f : G \to K$ be a continuous homomorphism to a compact group K. Since K is precompact and $H = r(G)$ is the precompact reflection of G, it follows from condition (ii) of Definition 3.5.1 that there exists a continuous homomorphism $\hat{f} : H \to K$ satisfying $f = \hat{f} \circ r$. Since K is compact, it is Weil complete [8, Lemma 7.2.6]. Therefore, [8, Theorem 6.2.4] allows us to find a continuous homomorphism $\varphi : cH \to cK = K$ between the completions of H and K respectively, satisfying $\hat{f} = \varphi \restriction_H$. This implies that $\varphi \circ r = \varphi \restriction_H \circ r = \hat{f} \circ r = f$. This equality shows that r satisfies condition (ii) of Definition 3.4.1 for the group cH. □

By Claims 12 and 13 we conclude that cH together with the mapping $r : G \to cH$ satisfies all conditions of Definition 3.4.1, as desired. □

In Proposition 3.4.6 we established that, for any topological group G, the quotient $G/\mathfrak{n}(G)$ features some universality with respect to the class of MAP topological groups. The following proposition interprets this universality in categorical terms.

Proposition 3.5.5. *For every topological group G, its quotient group $G/\mathfrak{n}(G)$ is the reflection of G in the class of* MAP *topological groups, with the quotient map $q_G : G \to G/\mathfrak{n}(G)$ playing the role of the reflection homomorphism.*

Proof. The reflection of G in the class of MAP groups is unique up to isomorphism by Proposition 3.5.2. It then suffices to verify the conditions of Definition 3.5.1 for the group $G/\mathfrak{n}(G)$ and the mapping q_G. The group $G/\mathfrak{n}(G)$ is MAP by Corollary 3.4.8. Finally, Proposition 3.4.6 shows that the map q_G satisfies condition (ii) of Definition 3.5.1 with the group $G/\mathfrak{n}(G)$, thereby acting as the reflection map for G in the class of MAP groups. □

We are now ready to state the main result of this section.

Theorem 3.5.6. *For a topological group G, the subgroup $b_G[G]$ of its Bohr compactification bG is the precompact reflection of the quotient $G/\mathfrak{n}(G)$, with a monomorphism $\iota_G : G/\mathfrak{n}(G) \to bG$ from Corollary 3.4.7 as the associated reflection homomorphism.*

Proof. Since a map ι_G satisfies condition (3.2), we have $\iota_G[G/\mathfrak{n}(G)] = b_G[G]$. The group $b_G[G]$ is precompact, so it now suffices to show that ι_G serves as the associated reflection mapping to the class of precompact groups. Let $f : G/\mathfrak{n}(G) \to H$ be a continuous homomorphism to a precompact group H. Since H is precompact, it is a subgroup of some compact group K. Since $f \circ q_G : G \to K$ is a continuous homomorphism from G to K, by condition (ii) of Definition 3.4.1 we can find a homomorphism $(f \circ q_G)^+ : bG \to K$ satisfying $f \circ q_G = (f \circ q_G)^+ \circ b_G$. From condition (3.2) we have

$$f \circ q_G = (f \circ q_G)^+ \circ b_G = (f \circ q_G)^+ \circ \iota_G \circ q_G.$$

Since q_G is surjective, we obtain that

$$f = (f \circ q_G)^+ \circ \iota_G.$$

This shows that $b_G[G]$ together with the mapping ι_G satisfies condition (ii) of Definition 3.5.1 for the class of precompact groups. Our conclusion now follows from the uniqueness of the precompact reflection of G (Proposition 3.5.2). □

Corollary 3.5.7. *For every topological group G, the group bG is the Bohr compactification of $G/\mathfrak{n}(G)$ with a monomorphism ι_G from Corollary 3.4.7 as the associated mapping.*

Proof. Apply Theorem 3.5.6 to deduce that the group $b_G[G]$ is the precompact reflection of $G/\mathfrak{n}(G)$ with ι_G as its associated reflection mapping. Next, apply Corollary 3.5.4 to deduce that the Weil completion bG of $b_G[G]$ is the Bohr compactification of $G/\mathfrak{n}(G)$ with ι_G as the associated mapping. □

The next corollary describes precisely the Bohr topology of the quotient $G/\mathfrak{n}(G)$:

Corollary 3.5.8. *For every topological group G, the subgroup $b_G[G]$ of bG is topologically isomorphic to $(G/\mathfrak{n}(G))^+$.*

Proof. Indeed, by Corollary 3.5.7 the subgroup $\iota_G[G/\mathfrak{n}(G)]$ of bG is the image of $G/\mathfrak{n}(G)$ under the associated mapping to its Bohr compactification. Remark 3.4.12 implies that $\iota_G[G/\mathfrak{n}(G)]$ is isomorphic to $(G/\mathfrak{n}(G))^+$. It suffices to note condition (3.2) implies that $\iota_G[G/\mathfrak{n}(G)] = b_G[G]$. This shows that $b_G[G]$ is isomorphic to $(G/\mathfrak{n}(G))^+$. □

3.6 When is a topological group MinAP modulo a property P?

For any property **P** of topological groups which is invariant under surjective continuous homomorphisms, the following result reduces the difficulty of checking if a topological group G is MinAP modulo **P** to the verification of property **P** for the image of G in its Bohr compactification bG.

Theorem 3.6.1. *Let G be a topological group and let $b_G : G \to bG$ be a canonical mapping to its Bohr compactification. Let* **P** *be a property of topological groups invariant under surjective continuous homomorphisms. Then G is* MinAP *modulo* **P** *if and only if its image $b_G[G]$ in the Bohr compactification bG has property* **P***.*

Proof. Assume that G is MinAP modulo **P**. Since the homomorphism $b_G : G \to bG$ is continuous, and the group bG is compact, the image $b_G[G]$ has property **P** by Definition 3.2.1.

For the converse, let us assume that the image $b_G[G]$ has property **P**. Let K be a compact topological group, and assume that $f : G \to K$ is a continuous homomorphism from G to K. By the universality of the Bohr compactification (condition (ii) of Definition 3.4.1), there exists a continuous homomorphism $f^+ : bG \to K$ from bG to K such that the equality $f = f^+ \circ b_G$ holds. This implies that

$$f[G] = f^+[b_G[G]].$$

Recall that $\mathbf{P}$ is an invariant of surjective continuous homomorphisms. Since $b_G[G]$ has property $\mathbf{P}$, its continuous image $f^+[b_G[G]]$ also has property $\mathbf{P}$. This concludes our proof. □

From Corollary 3.5.8 and Theorem 3.6.1, we obtain the following

Corollary 3.6.2. *Let* $\mathbf{P}$ *be a property of topological groups invariant under surjective continuous homomorphisms. Then a topological group* G *is* MinAP *modulo* $\mathbf{P}$ *if and only if* $(G/\mathfrak{n}(G))^+$ *has property* $\mathbf{P}$*.*

When the topological group G in question is assumed to be MAP, we can obtain the following simplification of the above corollary:

Corollary 3.6.3. *Let* $\mathbf{P}$ *be a property of topological groups invariant under surjective continuous homomorphisms. A* MAP *group* G *is* MinAP *modulo* $\mathbf{P}$ *if and only if* G^+ *has property* $\mathbf{P}$*.*

Proof. Since G is MAP, we have $\mathfrak{n}(G) = \{e\}$ by Proposition 3.4.9(i), so the quotient $G/\mathfrak{n}(G)$ is topologically isomorphic to G. We then apply Corollary 3.6.2 to deduce that G is MinAP modulo $\mathbf{P}$ if and only if G^+ has property $\mathbf{P}$. □

In case our group is not MAP, we can use a simpler version of Corollary 3.6.2 to verify the MinAP modulo $\mathbf{P}$ property for some carefully selected properties $\mathbf{P}$:

Corollary 3.6.4. *Let* G *be a topological group and* $\mathbf{P}$ *be a property of topological groups invariant under surjective continuous homomorphisms. If the quotient group* $G/\mathfrak{n}(G)$ *has property* $\mathbf{P}$*, then the group* G *is* MinAP *modulo* $\mathbf{P}$*.*

Proof. Assume that the quotient $H = G/\mathfrak{n}(G)$ has property $\mathbf{P}$. The mapping $b_H : H \to bH$ is a continuous homomorphism from H to its Bohr compactification bH. Since H has property $\mathbf{P}$ and $\mathbf{P}$ is invariant under surjective continuous homomorphisms, the image $b_H[H]$ has property $\mathbf{P}$. Finally, by Remark 3.4.12 the image $b_H[H]$ is topologically isomorphic to H^+. Since $H^+ = (G/\mathfrak{n}(G))^+$ has property $\mathbf{P}$, we conclude that G is MinAP modulo $\mathbf{P}$ by Corollary 3.6.2. □

It is worth restating Theorem 3.6.1 for the concrete classes **P** discussed in Figure 1. The nicest reformulation is obtained for the algebraic properties:

Corollary 3.6.5. *A topological group G is* MinAP *modulo finite (bounded, torsion) if and only if the quotient $G/\mathfrak{n}(G)$ of G with respect to its von Neumann kernel $\mathfrak{n}(G)$ is finite (bounded, torsion, respectively).*

Proof. Indeed, let **P** be one of the three algebraic properties: finite, bounded, or torsion. Each of the three properties **P** are invariant under surjective (continuous) homomorphisms, and the topological group $(G/\mathfrak{n}(G))^+$ has property **P** if and only if the (abstract) group $G/\mathfrak{n}(G)$ has property **P**. Now the conclusion follows from Corollary 3.6.2. □

Remark 3.6.6. It follows from Corollary 3.6.5 that a topological group G is MinAP modulo finite if and only if its von Neumann kernel $\mathfrak{n}(G)$ has finite index in G.

Corollary 3.6.7. *A topological group G is* MinAP *modulo compact (connected) if and only if the topological group $(G/\mathfrak{n}(G))^+$ is compact (connected, respectively).*

Proof. Both compactness and connectedness are invariant under (surjective) continuous mappings, so the conclusion follows from Corollary 3.6.2. □

We finish this section with two examples.

Example 3.6.8. *The real line $\mathbb{R}$ is connected (so* MinAP *mod connected by Proposition 3.3.4(ii)), yet it is not* MinAP *mod compact. This shows that arrow 3 of Figure 1 is not reversible.* To check this, we shall need to recall the folklore fact that $\mathbb{R}^+$ is not compact. Indeed, if $\mathbb{R}^+$ was a compact group, then the identity mapping $id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}^+$ would be a continuous surjective homomorphism between locally compact groups. Since $\mathbb{R}$ is a countable union of compact subsets (σ-compact), the open mapping theorem ([8, Theorem 7.3.1]) would imply that the identity mapping $id_{\mathbb{R}}$ above is open. Therefore, $\mathbb{R}$ and $\mathbb{R}^+$ would be topologically isomorphic, and therefore $\mathbb{R}$ itself would be compact, which is a contradiction.

Being locally compact, $\mathbb{R}$ is a MAP group by the Peter-Weyl theorem. Since $\mathbb{R}^+$ is not compact, $\mathbb{R}$ is not MinAP mod compact by Corollary 3.6.3.

The infinite cyclic group $\mathbb{Z}$ behaves in a rather "extreme" manner.

Example 3.6.9. *Every group topology on the integers* $\mathbb{Z}$ *is either* MAP *or* MinAP *modulo finite.* Indeed, recall that every non-trivial subgroup of $\mathbb{Z}$ has finite index. This implies that if τ is a group topology on the integers with a non-trivial von Neumann kernel, then the topological quotient $\mathbb{Z}/\mathfrak{n}(\mathbb{Z}, \tau)$ is a finite group. We then apply Corollary 3.6.4 to see that $(\mathbb{Z}, \tau)$ is a MinAP mod finite group. If we now assume that τ is a group topology on $\mathbb{Z}$ with a trivial von Neumann kernel, then $(\mathbb{Z}, \tau)$ is now a MAP group by Proposition 3.4.9(i).

The above argument shows, in particular, that every non-MAP group topology of the integers is MinAP modulo finite. As we shall see in the following section, *any* Abelian topological group can be equipped with a MinAP modulo finite group topology by carefully using some well-known results.

3.7 Abelian groups where MinAP modulo connected coincides with MinAP

Let us recall the following terminology.

Definition 3.7.1 ([15])**.** An Abelian group is a *Markov group* if for every positive integer $m \in \mathbb{N}$ the subgroup mG of G is either trivial or has cardinality $\mathfrak{c}$.

The Markov groups play a key role in Dikranjan and Shakhmatov's solution [15] of Markov's ancient problem on Abelian connected group topologies (see and [29, 30]):

Theorem 3.7.2 ([15, Theorem 1.9, Corollary 1.10])**.** *For an Abelian group* G*, the following conditions are equivalent:*

(i) G *is a Markov group,*

(ii) G *admits a connected group topology.*

We shall use the following straightforward lemma which can be extracted from [12, Proposition 2.2]:

Lemma 3.7.3. *Let* G *be an Abelian group. If* G *satisfies the inequality* $r_0(G) \geq \mathfrak{c}$*, then* G *is a Markov group.*

In Remark 3.3.3 we established that the MinAP modulo connected property coincides with MinAP for Abelian torsion groups. This motivates us to find which algebraic restrictions on an Abelian group happen to make this equivalence hold (see Question 3.10.2).

Remark 3.3.3 shows that torsion groups belong to the class of groups asked for in Question 3.10.2. The following strengthening of [11, Lemma 6.3] shows that we are able to add a number of additional group types to this list, not all of which are necessarily torsion.

For our following theorem, we shall take advantage of working in the Abelian realm. It is well known that in Abelian groups one may substantially simplify Definition 1.4.2(b) thanks to the classical Peter-Weyl theorem:

Proposition 3.7.4 ([8, Section 9])**.** *If G is an Abelian topological group, then*

(a) G is MAP *if and only if the family of continuous characters of G separates its points; and*

(b) G is MinAP *if and only if the only continuous character of G is trivial.*

Theorem 3.7.5. *If an Abelian group G satisfies the inequality $r_0(G) < \mathfrak{c}$, then every* MinAP *modulo connected group topology on G is* MinAP*.*

Proof. We proceed by contradiction, so let us assume that G is equipped with a MinAP modulo connected group topology which is not MinAP. By Proposition 3.7.4(b) there exists some non-trivial continuous character $\chi : G \to \mathbb{T}$ of G. Since $\mathbb{T}$ is compact and G is MinAP modulo connected, the image $\chi(G) \subseteq \mathbb{T}$ is a connected subgroup of the circle group. Since χ is a non-trivial character of G, the image $\chi(G)$ is a non-trivial connected subgroup of $\mathbb{T}$. The circle group contains no connected subgroups other than itself and $\{0\}$, and so we obtain that $\chi(G) = \mathbb{T}$. This proves that the algebraic quotient $G/ker(\chi)$ is algebraically isomorphic to the circle group $\mathbb{T}$. It follows from (1.3) that

$$\mathfrak{c} = r_0(\mathbb{T}) = r_0(G/ker(\chi)) \leq r_0(G).$$

This contradicts our hypothesis of G satisfying $r_0(G) < \mathfrak{c}$. We conclude that G is MinAP. This shows that every MinAP modulo connected group topology on G is MinAP, as desired. □

The following particular cases are worth stating for the cases where they are easier to verify.

Corollary 3.7.6. *If an Abelian group G satisfies either of the following properties, then every* MinAP *modulo connected group topology on G is* MinAP*:*

(i) G is torsion, or

(ii) There exists some positive integer $m \in \mathbb{N}^+$ such that $0 < |mG| < \mathfrak{c}$.

Proof. (i) If the group is torsion, then it contains no torsion-free elements. This implies that $r_0(G) = 0$, and so the conclusion follows from Theorem 3.7.5.

(ii) By the contrapositive of Lemma 3.7.3, the group G satisfies the inequality $r_0(G) < \mathfrak{c}$. The conclusion follows from Theorem 3.7.5 as well. □

Theorem 3.7.5 goes in hand with the example presented in [51] which shows that arrow 3 of Figure 3.1 is not reversible:

Example 3.7.7 ([51, Example 6.8])**.** The Euclidean topology on the real line $\mathbb{R}$ is MinAP modulo connected, but not minimally almost periodic.

This example shows the limitations of Theorem 3.7.5. Indeed, since $\mathbb{R}$ is a divisible group, the subgroup $m\mathbb{R} = \mathbb{R}$ has cardinality $\mathfrak{c}$ for all positive integers $m \in \mathbb{N}^+$. So we may not improve Theorem 3.7.5 by asking for bigger cardinality.

Finally, we can give a different interpretation of Corollary 3.7.6(ii) for finding counter-examples. Indeed, item (ii) states that any topological group G which is counter-example to the reversibility of arrow 3 in Figure 3.1 is a Markov group.

Combining this with our interpretation of Corollary 3.7.6(ii) as well as Theorem 3.7.5, we deduce that every possible counter-example to the reversibility of arrow 3 *is a group which admits a connected group topology and with $\mathfrak{c}$ size* 0*-rank.*

3.8 Algebraic structure of Abelian MinAP mod connected groups

In the following theorem we describe the algebraic structure of Abelian groups which admit a MinAP modulo connected group topology. We begin with the following necessary condition:

Theorem 3.8.1. *Let G be an Abelian group G. If G admits a* MinAP *mod connected group topology, then for all $m \in \mathbb{N}$ the subgroup $mG = \{mg : g \in G\}$ is either trivial or infinite.*

Proof. If G is MinAP mod connected, Theorem 3.6.1(iii) implies that $(G/\mathfrak{n}(G))^+$ is connected. We have two cases, depending whether the group $(G/\mathfrak{n}(G))^+$ is trivial or not.

Claim 14. The result holds if $(G/\mathfrak{n}(G))^+$ is trivial.

Proof. If $(G/\mathfrak{n}(G))^+$ is trivial, then the group $G/\mathfrak{n}(G)$ is trivial as well. This implies that $G = \mathfrak{n}(G)$, and so G is minimally almost periodic. If there exists some $m \in \mathbb{N}$ such that mG is a finite group, then mG is a compact and continuous surjective image of G (under the mapping which sends each $g \in G$ to mg). Since G is minimally almost periodic and the group mG is compact, then it must be trivial by Definition 1.4.2(b). This shows that for each $m \in \mathbb{N}$ the group mG is either the trivial group or infinite. □

Claim 15. The result holds if $(G/\mathfrak{n}(G))^+$ is non-trivial.

Proof. The group $(G/\mathfrak{n}(G))^+$ is connected by Theorem 3.6.1(iii) (and therefore MinAP mod connected), precompact and non-trivial. Every precompact group is MAP, and so $(G/\mathfrak{n}(G))^+$ is not minimally almost periodic. By the contrapositive of Theorem 3.7.5 we deduce that $r_0((G/\mathfrak{n}(G))^+) \geq \mathfrak{c}$. However, $(G/\mathfrak{n}(G))^+$ is algebraically isomorphic to the quotient $G/\mathfrak{n}(G)$, and therefore $r_0(G/\mathfrak{n}(G)) \geq \mathfrak{c}$ also holds. It follows from (1.3) that

$$r_0(G) > r_0(G/\mathfrak{n}(G)) \geq \mathfrak{c}.$$

By Lemma 3.7.3 we conclude that G is a Markov group. Observe that by Definition 3.7.1 every Markov group G satisfies that mG is either the trivial group or infinite for all $m \in \mathbb{N}$. □

Our desired conclusion now follows from Claims 14 and 15. □

The necessary condition found in Theorem 3.8.1 is very similar to that of a Markov group, where the main difference is precisely the size of the non-trivial subgroups mG. For a Markov group, these have size continuum, while in this condition we are asking for them to be infinite. Groups with this property were key for Dikranjan and Shakhmatov's characterization of Abelian groups which admit a minimally almost periodic group topology (Theorem 1.4.12).

Thus, if we combine Theorem 1.4.12 and 3.8.1, we obtain the following:

Corollary 3.8.2. *An Abelian topological group admits a* MinAP *modulo connected group topology if and only if it admits a minimally almost periodic group topology.*

Proof. This follows from the fact that every MinAP topology is MinAP modulo connected (arrow 3 of Figure 3.1), and Theorems 3.8.1 and 1.4.12.

□

To close this section, we would like to highlight condition (ii) of Theorem 1.4.12, which very surprisingly links the classical MinAP property with that of connectedness. In our case, we obtain that connectedness with respect to the Markov-Zariski topology is also a necessary and sufficient condition to equip a group with a MinAP modulo connected group topology. We note that the Markov-Zariski topology is **not** a group topology, and it is a quite complicated topology to work with in general (even more so if our group happens to be non-Abelian). Details about this topology (along with some history) can be consulted in a series of papers by Dikranjan and Shakhmatov (see [13, 15, 11]).

3.9 Abelian groups which admit a MinAP modulo finite group topology

In this section we shall be applying some well-known results about the theory of Abelian minimally almost periodic groups. Our goal is to show that, in fact, every Abelian group admits a group topology which is MinAP modulo finite. The following well-known fact will be used to show this.

Fact 3.9.1. If G is a bounded Abelian group, then it is a direct sum of cyclic groups

$$G = \bigoplus_{p\in\pi(G)} \bigoplus_{i=1}^{m_p} \mathbb{Z}(p^i)^{(\alpha_{p,i})},$$

where $m_p \in \mathbb{N}$ for all $p \in \pi(G)$ and $\pi(G)$ is a non-empty finite set of primes. The cardinals $\alpha_{p,i}$ are known as the Ulm-Kaplansky invariants of G. Moreover, while some of them may be equal to zero, the cardinals α_{p,m_p} are positive; these cardinals are known as the *leading Ulm-Kaplansky invariants* of G.

The following is an easy corollary of this fact:

Corollary 3.9.2. *Every bounded Abelian group G admits a decomposition $G = H \oplus F$, where H is a group whose leading Ulm-Kaplansky invariants are infinite and F is a finite group.*

The following result of Gabriyelyan [19] is a characterization of bounded groups which admit a MinAP group topology:

Theorem 3.9.3. *A bounded Abelian group admits a minimally almost periodic group topology if and only if all of its leading Ulm-Kaplansky invariants are infinite.*

Finally, we recall the following particular case of Dikranjan and Shakhmatov's characterization of Abelian groups which admit a MinAP group topology (see [11]):

Theorem 3.9.4 ([11, Theorem 3.1])**.** *Every unbounded Abelian group admits a minimally almost periodic group topology.*

With these results in hand, we are ready to obtain the main results of this section.

Theorem 3.9.5. *Every Abelian group G has a decomposition $G = H \oplus F$, where H is a group which admits a minimally almost periodic group topology and F is a finite group.*

Proof. If G is unbounded, then letting $H = G$ and $F = \{0\}$ produces the desired decomposition by Theorem 3.9.4. Suppose now that G is bounded, and let $G = H \oplus F$ be a decomposition from Corollary 3.9.2. It remains only to note that H admits a minimally almost periodic group topology by Theorem 3.9.3. □

Corollary 3.9.6. *Assume that* **P** *is a property of topological groups satisfied by all finite groups. Then every Abelian group G admits a* MinAP *mod* **P** *group topology.*

Proof. Let $G = H \oplus F$ be a decomposition from Theorem 3.9.5. We equip H with a minimally almost periodic group topology, F with the discrete topology and $G = H \oplus F \cong H \times F$ with the product topology.

Let $f : G \to K$ be a continuous homomorphism from G to a compact group K. Then $f[H]$ is the trivial subgroup of K by Definition 1.4.2(b), so $f[G] = f[F]$ is a finite subgroup of K. Being a finite group, $f[G]$ has property **P** by our assumption. □

Applying Corollary 3.9.6 to the property **P** defined as "being a finite group", we obtain the following

Corollary 3.9.7. *Every Abelian group admits a* MinAP *mod finite group topology.*

Form Remark 3.2.3(i) and Corollary 3.9.7 we can deduce the following

Corollary 3.9.8. *Every Abelian group G admits a group topology τ such that every continuous homomorphic image of (G, τ) in a precompact group is finite.*

3.10 Open questions

The first question is related to the algebraic structure of MinAP modulo **P** groups:

Question 3.10.1. Can one characterize all properties **P** of topological groups such that every (Abelian) topological group admits a MinAP modulo **P** group topology if and only if it admits a MinAP group topology?

By Corollary 3.8.2, we have shown that **P** = *connected* is a property which satisfies the above equivalence in the *Abelian case*. So a characterization of all properties **P** required in Question 3.10.1 must include the topological property of being connected.

Let us now recall that all finite (and therefore, compact) MinAP groups are trivial. It follows from Remark 3.2.3(ii) that a necessary condition for a positive answer to Question 3.10.1 is that the trivial group is the only finite group satisfying property **P**.

We motivated the following question in Section 3.7:

Question 3.10.2. For which (Abelian) groups do the MinAP and MinAP modulo connected properties become equivalent?

We provide a partial answer to this question in the Abelian case in our Theorem 3.7.5. In this result we prove that if a group has 0-rank less than the continuum $\mathfrak{c}$, then the above question is *affirmative*. The case of Abelian groups with more than continuum 0-rank is open, as well as all non-Abelian variants.

Question 3.10.3. Consider any of the new properties introduced in Figure 3.1. Can one show that every (Abelian) group admits a (Hausdorff) group topology in this class?

This question was answered positiviely in the realm of Abelian groups for MinAP modulo finite, torsion, bounded and compact in Corollary 3.9.6, as these four properties are possessed by all finite groups. The only property from Figure 3.1 which answers this question negatively for Abelian groups is connectedness, as proved in Corollary 3.8.2.

Question 3.10.4. For a "reasonable" property **P** of topological groups, can one describe the algebraic structure of (Abelian) groups which admit a MinAP mod **P** group topology?

This question is answered partially in Corollary 3.9.6. The solution we present lies in the realm of Abelian groups, and it holds for every property **P** of topological groups which is satisfied by all finite groups. As a consequence, the above question remains open (in the Abelian case) for any property **P** not satisfied by all finite groups. The question is open in general in the non-Abelian case.

Chapter 4

SSGP group topologies on Abelian groups of positive finite divisible rank

4.1 Introduction

The main goal of the chapter is to provide a positive answer to (a more general version of) Question 1.5.13 given in Theorem 4.7.4. It follows from this theorem that [16, Theorem 13.2] holds *unconditionally*; that is, the next theorem holds:

Theorem 4.1.1. *For an abelian group G satisfying $1 \leq r_d(G) < \omega$, the following conditions are equivalent:*

(i) G admits an SSGP *topology;*

(ii) G admits an $\mathtt{SSGP}(\alpha)$ *topology for some ordinal α;*

(iii) the quotient $H = G/t(G)$ of G with respect to its torsion part $t(G)$ has finite rank $r_0(H)$ and $r(H/A) = \omega$ for some (equivalently, every) free subgroup A of H such that H/A is torsion.

A combination of Theorems 1.5.11, 1.5.12 and 4.1.1 provides to Question 1.5.10(a) for Abelian groups. Similarly, the implication (iii)→(i) of the next corollary provides a complete solution to Question 1.5.10(b) for Abelian groups. Both items of Question 1.5.10 remain widely open for non-commutative groups.

Corollary 4.1.2. *For an abelian group G, the following conditions are equivalent:*

(i) G admits an SSGP *topology;*

(ii) G admits an $\mathtt{SSGP}(\alpha)$ *topology for some ordinal α;*

(iii) G admits an $\mathrm{SSGP}(n)$ *topology for some integer $n > 1$.*

Proof. The implication (i)→(iii) follows from (1.7), while the implication (iii)→(ii) follows from Theorem 1.5.9(iv).

(ii)→(i) Let G be an abelian $\mathtt{SSGP}(\alpha)$ group for some ordinal α. We consider three cases.

Case 1. $r_d(G) = 0$. By Remark 1.2.6, G is a bounded torsion group. Since G is an $\mathtt{SSGP}(\alpha)$ group, it is minimally almost periodic by Theorem 1.5.9(iii), so G is SSGP by Theorem 1.5.7.

Case 2. $1 \leq r_d(G) < \omega$. In this case G admits an SSGP topology by the implication (ii)→(i) of Theorem 4.1.1.

Case 3. $r_d(G) \geq \omega$. In this case G admits an SSGP topology by Theorem 1.5.12. □

The rest of the chapter is organized as follows. In Section 4.2 we obtain a convenient reformulation of the SSGP property. This reformulation will be used later for our main proof. In Section 4.3 we prove two lemmas about the algebraic structure of a particular type of subgroups of finite powers $\mathbb{Q}^m$ of $\mathbb{Q}$. In Section 4.4 we introduce the notion of a wide subgroup of $\mathbb{Q}^m$ and we establish two auxiliary lemmas about wide subgroups. The main result of this chapter is Theorem 4.7.4 which states that the direct sum $G \oplus H$ of a wide subgroup G of $\mathbb{Q}^m$ and an at most countable abelian group H admits a metric SSGP topology $\mathscr{T}$.

We employ the strategy described in Section 1.6. First, we introduce a partially ordered set $(\mathbb{P}, \leq)$ in Section 4.5. Next, we define in (4.32) a countable family $\mathscr{D}$ of subsets of $(\mathbb{P}, \leq)$. Section 4.6 collects lemmas establishing the density in $(\mathbb{P}, \leq)$ of various sets participating in the family $\mathscr{D}$. The "technical heart" of this section is Lemma 4.6.6 which is responsible for the SSGP property of $\mathscr{T}$. In Section 4.7, we select a linearly ordered subset $\mathbb{F}$ of $(\mathbb{P}, \leq)$ intersecting all members of the family $\mathscr{D}$ of dense subsets of $(\mathbb{P}, \leq)$.We produce a countable base of neighbourhoods of zero for $\mathscr{T}$ is defined via elements of $\mathbb{F}$.

4.2 An auxiliary reformulation of the SSGP property

Definition 4.2.1. For a subset A of group G and $k \in \mathbb{N}^+$, we let

$$\langle A \rangle_k = \left\{ \prod_{i=1}^{j} a_i : j \leq k, a_1, \ldots, a_j \in A \right\}. \tag{4.1}$$

Remark 4.2.2. If A is a subset of a group G such that $A = A^{-1}$, then $\langle A \rangle = \bigcup_{k \in \mathbb{N}^+} \langle A \rangle_k$.

Using the notation from (4.1), one can get a convenient reformulation of the SSGP property.

Proposition 4.2.3. *For a topological group G, the following conditions are equivalent:*

(i) *G has the small subgroup generating property* SSGP*;*

(ii) *$G = \bigcup_{k \in \mathbb{N}^+} W \cdot \langle \mathrm{Cyc}(W) \rangle_k$ for every neighbourhood W of the identity of G;*

(iii) *for each $g \in G$ and every neighbourhood W of the identity of G, there exist $k \in \mathbb{N}^+$ (depending on g and W), $g_0 \in W$ and $g_1, \ldots, g_k \in G$ such that $g = \prod_{i=0}^{k} g_i$ and $\langle g_i \rangle \subseteq W$ for $i \in \{1, \ldots, k\}$.*

Proof. (i)→(ii) Suppose that G has the small subgroup generating property. Let W be a neighbourhood of the identity e of G. Since $\mathrm{Cyc}(W) = (\mathrm{Cyc}(W))^{-1}$, we have $\langle \mathrm{Cyc}(W) \rangle = \bigcup_{k \in \mathbb{N}^+} \langle \mathrm{Cyc}(W) \rangle_k$ by Remark 4.2.2. Furthermore, $\langle \mathrm{Cyc}(W) \rangle$ is dense in G by Definition 1.5.5, so

$$G = W \cdot \langle \mathrm{Cyc}(W) \rangle = W \cdot \bigcup_{k \in \mathbb{N}^+} \langle \mathrm{Cyc}(W) \rangle_k = \bigcup_{k \in \mathbb{N}^+} W \cdot \langle \mathrm{Cyc}(W) \rangle_k.$$

(ii)→(i) Suppose that G has the property from (ii). Fix a neighbourhood U of e in G. By Definition 1.5.5, to check that G has the small subgroup generating property, it suffices to prove that $\langle \mathrm{Cyc}(U) \rangle$ is dense in G. This is equivalent to establishing that $G = V \cdot \langle \mathrm{Cyc}(U) \rangle$ for every neighbourhood V of e in G. Let V be such a neighbourhood. Then $W = U \cap V$ is also a neighbourhood of e in G, so we can apply (ii) to this W to obtain

$$G = \bigcup_{k \in \mathbb{N}^+} W \cdot \langle \mathrm{Cyc}(W) \rangle_k = W \cdot \bigcup_{k \in \mathbb{N}^+} \langle \mathrm{Cyc}(W) \rangle_k \subseteq V \cdot \bigcup_{k \in \mathbb{N}^+} \langle \mathrm{Cyc}(U) \rangle_k \subseteq V \cdot \langle \mathrm{Cyc}(U) \rangle.$$

The converse inclusion $V \cdot \langle \mathrm{Cyc}(U) \rangle \subseteq G$ is clear.

(ii)↔(iii) follows from (1.2) and (4.1). □

4.3 The algebraic structure of subgroups $\mathbb{Q}_\pi^m$ of $\mathbb{Q}^m$

Definition 4.3.1. For a non-empty set π of of prime numbers, we use $\mathbb{Q}_\pi$ to denote the set of all rational numbers q whose irreducible representation $q = z/n$ with $z \in \mathbb{Z}$ and $n \in \mathbb{N}^+$ is such that all prime divisors of n belong to π. For convenience, we let $\mathbb{Q}_\emptyset = \mathbb{Z}$.

Definition 4.3.2. Given an integer $s \in \mathbb{Z}$, we shall denote by $\lceil s \rceil$ the subgroup $s\mathbb{Z}$ of $\mathbb{Q}$.

A straightforward proof of the following lemma is left to the reader.

Lemma 4.3.3. *Let $\pi \subseteq \mathbf{P}$ and $m \in \mathbb{N}^+$. Then:*

(i) $\mathbb{Q}_\pi$ is a subgroup of $\mathbb{Q}$, so $\mathbb{Q}_\pi^m$ is a subgroup of $\mathbb{Q}^m$.

(ii) $\mathbb{Z} \subseteq \mathbb{Q}_\pi$, and so $\lceil s \rceil^m \subseteq \mathbb{Q}_\pi^m$ for each $s \in \mathbb{N}^+$.

(iii) If $\pi \subseteq \pi' \subseteq \mathbf{P}$, then $\mathbb{Q}_\pi \subseteq \mathbb{Q}_{\pi'}$, and so $\mathbb{Q}_\pi^m \subseteq \mathbb{Q}_{\pi'}^m$.

Our next lemma clarifies the algebraic structure of subgroups $\mathbb{Q}_\pi^m$ of $\mathbb{Q}^m$.

Lemma 4.3.4. (A) *Suppose that $s \in \mathbb{Z}$, $k \in \mathbb{N}^+$, $g_1, \dots, g_k \in \mathbb{Q}^m$, $\pi_0, \pi_1, \dots, \pi_k \in [\mathbf{P}]^{<\omega}$,*

$$\pi_0 \subseteq \pi_1 \subseteq \pi_2 \subseteq \cdots \subseteq \pi_k, \tag{4.2}$$

and the following conditions hold for every $j \in \{1, \dots, k\}$:

(a_j) $g_j \in \mathbb{Q}_{\pi_j}^m$,

(b_j) $\langle g_j \rangle \cap \mathbb{Q}_{\pi_{j-1}}^m \subseteq \lceil s \rceil^m$.

Then:

(i) $\langle \{g_1, \dots, g_i\} \rangle + \mathbb{Q}_{\pi_0}^m \subseteq \mathbb{Q}_{\pi_i}^m$ for every $i \in \{1, \dots, k\}$.

(ii) $\langle \{g_i, \dots, g_k\} \rangle \cap \mathbb{Q}_{\pi_{i-1}}^m \subseteq \lceil s \rceil^m$ for every $i \in \{1, \dots, k\}$.

(B) *In addition to the assumptions of(A), suppose that for every $j \in \{1, \dots, k\}$,*

(c_j) $lg_j \notin \mathbb{Q}^m_{\pi_{j-1}}$ for every $l \in \mathbb{Z} \setminus \{0\}$ satisfying $|l| \le k$.

Finally, assume that $g \in \mathbb{Q}^m_{\pi_0}$, J is a proper subset of $\{1, \dots, k\}$, $l \in \mathbb{Z}$, $|l| \le k$ and

$$lg_0 \in \langle \{g_j : j \in J\} \rangle + \mathbb{Q}^m_{\pi_0}, \tag{4.3}$$

where

$$g_0 = g - \sum_{j=1}^{k} g_j. \tag{4.4}$$

Then $l = 0$.

Proof. First, we check item (A).

(i) Let $i \in \{1, \dots, k\}$. Fix $j \in \{1, \dots, i\}$. Then $\pi_j \subseteq \pi_i$ by (4.2). Applying Lemma 4.3.3 (iii), we get $\mathbb{Q}^m_{\pi_j} \subseteq \mathbb{Q}^m_{\pi_i}$. Since $g_j \in \mathbb{Q}^m_{\pi_j}$ by (a_j), we obtain $g_j \in \mathbb{Q}^m_{\pi_i}$. Since this holds for all $j \in \{1, \dots, i\}$, we conclude that $\{g_1, \dots, g_i\} \subseteq \mathbb{Q}^m_{\pi_i}$. Since $\pi_0 \subseteq \pi_i$ by (4.2), applying Lemma 4.3.3 (iii) once again, we obtain $\mathbb{Q}^m_{\pi_0} \subseteq \mathbb{Q}^m_{\pi_i}$. Since $\mathbb{Q}^m_{\pi_i}$ is a subgroup of $\mathbb{Q}^m$ by Lemma 4.3.3 (i), this implies $\langle \{g_1, \dots, g_i\} \rangle + \mathbb{Q}^m_{\pi_0} \subseteq \mathbb{Q}^m_{\pi_i}$.

(ii) We use induction on k.

Basis of induction. If $k = 1$, then the conclusion of (ii) holds by (b_1).

Inductive step. Let $k \ge 2$ and suppose that (ii) has already been proved for $k - 1$.

Let $i \in \{1, \dots, k\}$. Fix

$$h \in \langle \{g_i, \dots, g_k\} \rangle \cap \mathbb{Q}^m_{\pi_{i-1}}. \tag{4.5}$$

Then there exist $x \in \langle \{g_i, \dots, g_{k-1}\} \rangle$ and $y \in \langle g_k \rangle$ such that $h = x + y$. In particular, $x \in \mathbb{Q}^m_{\pi_{k-1}}$ by item (i), as $0 \in \mathbb{Q}^m_{\pi_0}$. As $i - 1 \le k - 1$, we have $\pi_{i-1} \subseteq \pi_{k-1}$ by (4.2), which implies $\mathbb{Q}^m_{\pi_{i-1}} \subseteq \mathbb{Q}^m_{\pi_{k-1}}$ by Lemma 4.3.3 (iii). From this and (4.5), we obtain $h \in \mathbb{Q}^m_{\pi_{k-1}}$. Therefore, $y = h - x \in \mathbb{Q}^m_{\pi_{k-1}} - \mathbb{Q}^m_{\pi_{k-1}} = \mathbb{Q}^m_{\pi_{k-1}}$, as $\mathbb{Q}^m_{\pi_{k-1}}$ is a subgroup of $\mathbb{Q}^m$ by Lemma 4.3.3(i). Now we consider two cases.

If $i < k$, then $i \in \{1, \dots, k-1\}$, so $\langle \{g_i, \dots, g_{k-1}\} \rangle \cap \mathbb{Q}^m_{\pi_{i-1}} \subseteq \lceil s \rceil^m$ by our inductive assumption, which implies $x \in \lceil s \rceil^m$. Since $y \in \lceil s \rceil^m$ as well, $h = x + y \in \lceil s \rceil^m + \lceil s \rceil^m = \lceil s \rceil^m$.

Suppose now that $i = k$. Then $h \in \langle g_k \rangle \cap \mathbb{Q}^m_{\pi_{k-1}}$ by (4.5), so $h \in \lceil s \rceil^m$ by (b_k).

Next, we check item (B). From (4.3) and (4.4), we get

$$lg - \sum_{j=1}^{k} lg_j \in \langle\{g_j : j \in J\}\rangle + \mathbb{Q}^m_{\pi_0}.$$

Since $\mathbb{Q}^m_{\pi_0}$ is a subgroup of $\mathbb{Q}^m$ by Lemma 4.3.3 (i) and $g \in \mathbb{Q}^m_{\pi_0}$ by our assumption, it follows that

$$-\sum_{j=1}^{k} lg_j \in \langle\{g_j : j \in J\}\rangle + \mathbb{Q}^m_{\pi_0}. \tag{4.6}$$

Since J is a proper subset of $\{1, \dots, k\}$, we can fix $i \in \{1, \dots, k\} \setminus J$. From this and (4.6), we can find

$$x \in \langle\{g_1, \dots, g_{i-1}\}\rangle \ \text{ and } \ y \in \langle\{g_{i+1}, \dots, g_k\}\rangle \tag{4.7}$$

such that

$$lg_i \in x + y + \mathbb{Q}^m_{\pi_0}. \tag{4.8}$$

(If $i = 1$, we define $\langle\{g_1, \dots, g_{i-1}\}\rangle = \{0\}$, and if $i = k$, we define $\langle\{g_{i+1}, \dots, g_k\}\rangle = \{0\}$.) Then

$$y \in lg_i - x + \mathbb{Q}^m_{\pi_0} \in \langle g_i\rangle - \langle\{g_1, \dots, g_{i-1}\}\rangle + \mathbb{Q}^m_{\pi_0} = \langle\{g_1, \dots, g_i\}\rangle + \mathbb{Q}^m_{\pi_0} \subseteq \mathbb{Q}^m_{\pi_i}$$

by (A)(i) and our special definition of $\langle\{g_1, \dots, g_{i-1}\}\rangle$ in case $i = 1$. Recalling the second inclusion in (4.7), we get $y \in \langle\{g_{i+1}, \dots, g_k\}\rangle \cap \mathbb{Q}^m_{\pi_i}$. If $i < k$, then applying (A)(ii), we conclude that $y \in \lceil s\rceil^m$. Combining this with Lemma 4.3.3(ii), we get $y \in \mathbb{Q}^m_{\pi_{i-1}}$. If $i = k$, then $y = 0$ by (4.7) and our special definition of $\langle\{g_{i+1}, \dots, g_k\}\rangle$ in case $i = k$ given after (4.8). Since $\mathbb{Q}^m_{\pi_{i-1}}$ is a subgroup of $\mathbb{Q}^m$ by Lemma 4.3.3(i), we get $y \in \mathbb{Q}^m_{\pi_{i-1}}$.

From the first inclusion in (4.7) and (A)(i), we get $x \in \mathbb{Q}^m_{\pi_{i-1}}$. Note that $\mathbb{Q}^m_{\pi_0} \subseteq \mathbb{Q}^m_{\pi_{i-1}}$ by (4.2), as $i \in \{1, \dots, k\}$. Since $\mathbb{Q}^m_{\pi_{i-1}}$ is a subgroup of $\mathbb{Q}^m$, from $x, y \in \mathbb{Q}^m_{\pi_{i-1}}$, the inclusion $\mathbb{Q}^m_{\pi_0} \subseteq \mathbb{Q}^m_{\pi_{i-1}}$ and (4.8), one obtains $lg_i \in \mathbb{Q}^m_{\pi_{i-1}}$. Since $l \in \mathbb{Z}$ and $|l| \leq k$, applying (c_i), we get $l = 0$. □

4.4 Wide subgroups of $\mathbb{Q}^m$

Our next definition gives a name to subgroups of $\mathbb{Q}^m$ having the property from [16, Question 13.1].

Definition 4.4.1. Let $m \in \mathbb{N}^+$. We shall call a subgroup G of $\mathbb{Q}^m$ *wide* if $\mathbb{Z}^m \subseteq G$ and $G \setminus \mathbb{Q}^m_\pi \neq \emptyset$ for every $\pi \in [\mathbf{P}]^{<\omega}$.

Lemma 4.4.2. *Let $m \in \mathbb{N}^+$, G be a wide subgroup of $\mathbb{Q}^m$, $\pi \in [\mathbf{P}]^{<\omega}$, $k \in \mathbb{N}^+$ and $s \in \mathbb{Z} \setminus \{0\}$. Then there exists $g \in G$ such that:*

(i) $\langle g \rangle \cap \mathbb{Q}^m_\pi \subseteq \lceil s \rceil^m$,

(ii) $lg \notin \mathbb{Q}^m_\pi$ *for every* $l \in \mathbb{Z} \setminus \{0\}$ *satisfying* $|l| \leq k$.

Proof. Let ϖ be the set of all prime numbers not exceeding $\max\{k, s\}$. Then $\pi' = \pi \cup \varpi \in [\mathbf{P}]^{<\omega}$. Since G is wide, we can find $h \in G \setminus \mathbb{Q}^m_{\pi'}$. Let $h = (h_1, \dots, h_m)$, where $h_1, \dots, h_m \in \mathbb{Q}$. For every $i \in \{1, \dots, m\}$, let $h_i = a_i/b_i$ be the irreducible fraction with $a_i \in \mathbb{Z}$ and $b_i \in \mathbb{N}^+$.

Since $h \notin \mathbb{Q}^m_{\pi'}$, we can fix $t \in \{1, \dots, m\}$ such that $h_t \notin \mathbb{Q}_{\pi'}$. Since a_t/b_t is an irreducible representation of $h_t \in \mathbb{Q} \setminus \mathbb{Q}_{\pi'}$, we can fix $p \in \mathbf{P} \setminus \pi'$ dividing b_t. Since $\varpi \subseteq \pi'$ and $p \in \mathbf{P} \setminus \pi'$, we have $p \in \mathbf{P} \setminus \varpi$. Since ϖ includes all prime numbers not exceeding $\max\{k, s\}$, we conclude that

$$p > \max\{k, s\}. \tag{4.9}$$

For every $i \in \{1, \dots, m\}$, let

$$n_i = \max\{n \in \mathbb{N} : p^n \text{ divides } b_i\}. \tag{4.10}$$

Then

$$c_i = b_i/p^{n_i} \in \mathbb{N} \text{ is not divisible by } p. \tag{4.11}$$

Note that

$$n_t \geq 1, \tag{4.12}$$

as p divides b_t by our choice of p. It follows from $s \in \mathbb{N}$ and (4.11) that

$$m_0 = sc_1 \cdots c_m \in \mathbb{N}. \tag{4.13}$$

Since G is a group and $h \in G$, it follows that $g = m_0 h \in G$. Note that

$$g = (m_0 h_1, \dots, m_0 h_m). \tag{4.14}$$

We claim that g is the required element of G, that is, conditions (i) and (ii) are satisfied.

Fix $i \in \{1, \dots, m\}$. It follows from (4.11) and (4.13) that

$$m_0 h_i = s c_1 c_2 \cdots c_m \frac{a_i}{b_i} = s \left(\prod_{j=1, j \neq i}^{m} c_j \right) \frac{b_i}{p^{n_i}} \frac{a_i}{b_i} = s \left(\prod_{j=1, j \neq i}^{m} c_j \right) \frac{a_i}{p^{n_i}} = \frac{s d_i}{p^{n_i}}, \tag{4.15}$$

where

$$d_i = c_1 \cdots c_{i-1} a_i c_{i+1} \cdots c_m \in \mathbb{Z}. \tag{4.16}$$

Claim 16. The right-hand side of (4.15) is the irreducible representation of $m_0 h_i$.

Proof. If $n_i = 0$, then $p^{n_i} = 1$, so $s d_i / 1$ is the irreducible representation of $m_0 h_i$. Suppose now that $n_i \geq 1$. In this case, it suffices to check that neither s nor d_i is divisible by p. The first statement follows from (4.9). Since $n_i \geq 1$, (4.10) implies that p divides b_i. Since a_i / b_i is an irreducible fraction, p does not divide a_i. By (4.11), no c_j is divisible by p. By (4.16), this means that p does not divide d_i. $\square$

(i) By (4.14), in order to check (i), we need to show that $\langle m_0 h_i \rangle \cap \mathbb{Q}_\pi \subseteq \lceil s \rceil$ for every $i \in \{1, \dots, m\}$. Fix such an i. Let $x \in \langle m_0 h_i \rangle \cap \mathbb{Q}_\pi$. Then $x = l m_0 h_i$ for some $l \in \mathbb{Z}$. Since $p \notin \pi$, it follows from $x \in \mathbb{Q}_\pi$, Definition 4.3.1 and Claim 16 that p^{n_i} must divide l, so $l = p^{n_i} l'$ for some $l' \in \mathbb{Z}$. Therefore, $x = l m_0 h_i = l' s d_i$ by (4.15). Since $l' \in \mathbb{Z}$ and $d_i \in \mathbb{Z}$ by (4.16), we conclude that $x \in \lceil s \rceil$. This implies $\langle m_0 h_i \rangle \cap \mathbb{Q}_\pi \subseteq \lceil s \rceil$.

(ii) By (4.14), in order to check (ii), it suffices to show that $l m_0 h_t \notin \mathbb{Q}_\pi^m$ for every $l \in \mathbb{Z} \setminus \{0\}$ satisfying $|l| \leq k$. Fix such an l. Since $|l| \leq k < p$ by (4.9), p does not divide l. Recalling Claim 16 and (4.15), we conclude that $l s d_t / p^{n_t}$ is the irreducible representation of $l m_0 h_t$. Since $p \notin \pi$ and $n_t \geq 1$ by (4.12), from this and Definition 4.3.1, we obtain that $l m_0 h_t \notin \mathbb{Q}_\pi$. $\square$

Lemma 4.4.3. *Let $m \in \mathbb{N}^+$, G be a wide subgroup of $\mathbb{Q}^m$, $\pi_0 \in [\mathbf{P}]^{<\omega}$, $k \in \mathbb{N}^+$ and $s \in \mathbb{Z} \setminus \{0\}$. Then there exist an increasing sequence $\pi_0 \subseteq \pi_1 \subseteq \pi_2 \subseteq \cdots \subseteq \pi_k$ of finite subsets of $\mathbf{P}$ and elements*

$g_1, \ldots, g_k \in G$ *such that conditions* (a_j), (b_j), (c_j) *of Lemma 4.3.4 hold for every* $j \in \{1, \ldots, k\}$.

Proof. Use finite induction on $k \in \mathbb{N}^+$ based on Lemma 4.4.2. □

4.5 The poset $(\mathbb{P}, \leq)$

From now on we shall follow the standard set-theoretic practice of using the inclusion $m \in n$ for $n \in \mathbb{N}$ as an abbreviation for "$m \in \mathbb{N}$ and $m < n$".

In this section we define a (technically rather involved) poset $(\mathbb{P}, \leq)$ which will be used to construct a metric SSGP topology $\mathscr{T}$ on the direct sum $G \oplus H$ of a wide subgroup of $\mathbb{Q}^m$ for $m \in \mathbb{N}^+$ and an arbitrary at most countable abelian group H. The definition of the poset itself does not require the group G to be wide and the countability restriction on H is not essential, so we impose neither of these two conditions in the next definition.

Definition 4.5.1. Let H be an abelian group. For a fixed $m \in \mathbb{N}^+$, consider the direct sum $\mathbb{Q}^m \oplus H$. Furthermore, let G be a subgroup of $\mathbb{Q}^m$ containing $\mathbb{Z}^m$. Then the sum $G + H = G \oplus H$ is direct.

(a) Let $\mathbb{P}$ be the set of all structures $p = \langle\langle \pi^p, n^p, \{U_i^p : i \in n^p + 1\}, \{s_i^p : i \in n^p + 1\} \rangle\rangle$ satisfying:

- (1_p) $\pi^p \in [\mathbf{P}]^{<\omega}$,
- (2_p) $n^p \in \mathbb{N}$,
- (3_p) $s_i^p \in \mathbb{N}^+$ for every $i \in n^p + 1$,
- (4_p) $0 \in U_i^p \subseteq (G \cap \mathbb{Q}^m_{\pi^p}) + H$ for every $i \in n^p + 1$,
- (5_p) $-U_i^p = U_i^p$ for every $i \in n^p + 1$,
- (6_p) $U_i^p + \lceil s_i^p \rceil^m = U_i^p$ for every $i \in n^p + 1$,
- (7_p) $U_{i+1}^p + U_{i+1}^p \subseteq U_i^p$ for every $i \in n^p$,
- (8_p) s_i^p divides s_{i+1}^p for every $i \in n^p$.

(b) Given structures

$$p = \langle\langle \pi^p, n^p, \{U_i^p : i \in n^p + 1\}, \{s_i^p : i \in n^p + 1\} \rangle\rangle \in \mathbb{P}$$

and

$$q = \langle\langle \pi^q, n^q, \{U_i^q : i \in n^q + 1\}, \{s_i^q : i \in n^q + 1\}\rangle\rangle \in \mathbb{P},$$

we define $q \leq p$ if and only if:

(i_q^p) $\pi^p \subseteq \pi^q$,

(ii_q^p) $n^p \leq n^q$,

(iii_q^p) $U_i^q \cap (\mathbb{Q}_{\pi^p}^m + H) = U_i^p$ for every $i \in n^p + 1$,

(iv_q^p) $s_i^q = s_i^p$ for every $i \in n^p + 1$.

Remark 4.5.2. Let $p = \langle\langle \pi^p, n^p, \{U_i^p : i \in n^p + 1\}, \{s_i^p : i \in n^p + 1\}\rangle\rangle \in \mathbb{P}$. Then:

(i) $U_{i+1}^p \subseteq U_i^p$ *for every* $i \in n^p$. Indeed, $0 \in U_{i+1}^p$ by (4_p), so $U_{i+1}^p = 0 + U_{i+1}^p \subseteq U_{i+1}^p + U_{i+1}^p \subseteq U_i^p$ by (7_p).

(ii) $\lceil s_i^p \rceil^m \subseteq U_i^p$ *for every* $i \in n^p + 1$. Indeed, $U_i^p + \lceil s_i^p \rceil^m = U_i^p$ by (6_p). Since $0 \in U_i^p$ by (4_p), this implies $\lceil s_i^p \rceil^m \subseteq U_i^p$.

A straightforward proof of the following lemma is left to the reader.

Lemma 4.5.3. $(\mathbb{P}, \leq)$ *is a poset.*

Lemma 4.5.4. $\mathbb{P} \neq \emptyset$*.*

Proof. Define $\pi^p = \emptyset$, $n^p = 0$, $s_0^p = 1$, $U_0^p = \mathbb{Z}^m$, and

$$p = \langle\langle \pi^p, n^p, \{U_i^p : i \in 1\}, \{s_i^p : i \in 1\}\rangle\rangle = \langle\langle \emptyset, 0, \{U_0^p\}, \{s_0^p\}\rangle\rangle.$$

To show that $p \in \mathbb{P}$, we need to check the conditions (1_p)–(8_p) of Definition 4.5.1 (a). Conditions (1_p)–(3_p) are clear, and conditions (7_p) and (8_p) are vacuous. Conditions (5_p) and (6_p) are satisfied, as $U_0^p = \mathbb{Z}^m$ is a subgroup of $\mathbb{Q}^m$ and $\lceil 1 \rceil^m = \mathbb{Z}^m$. Finally condition ($4_p$) is satisfied because $U_0^p = \mathbb{Z}^m \subseteq G \cap \mathbb{Q}_{\pi^p} \subseteq (G \cap \mathbb{Q}_{\pi^p}) + H$ by our assumption on G in Definition 4.5.1, Lemma 4.3.3(ii) and $0 \in H$. □

Lemma 4.5.5. *Given* $p \in \mathbb{P}$, $g \in G \cap \mathbb{Q}_{\pi^p}^m$ *and* $h \in H$ *with* $g + h \neq 0$, *one can find* $q \in \mathbb{P}$ *such that* $q \leq p$, $n^q = n^p + 1$ *and* $g + h \notin U_{n^q}$.

Proof. Let

- $\pi^q = \pi^p$,
- $n^q = n^p + 1$,
- $U_i^q = U_i^p$ and $s_i^q = s_i^p$ for every $i \in n^p + 1$.

It remains only to define $U_{n^q}^q$ and $s_{n^q}^q$.

Since $\bigcap_{k \in \mathbb{N}^+} \lceil k s_{n^p}^p \rceil^m = \{0\}$ and $g + h \neq 0$, there exists $k \in \mathbb{N}^+$ such that $g + h \notin \lceil k s_{n^p}^p \rceil^m$. Define $s_{n^q}^q = k s_{n^p}^p$. Since $s_{n^p}^p \in \mathbb{N}^+$ by (3_p) and $k \in \mathbb{N}^+$, we see that $s_{n^q}^q \in \mathbb{N}^+$ and

$$s_{n^p}^p \ \text{ divides } \ s_{n^q}^q. \tag{4.17}$$

By our choice of $s_{n^q}^q$, we have

$$g + h \notin \lceil s_{n^q}^q \rceil^m. \tag{4.18}$$

Finally, we define

$$U_{n^q}^q = \lceil s_{n^q}^q \rceil^m. \tag{4.19}$$

Claim 17. $q = \langle\!\langle \pi^q, n^q, \{U_i^q : i \in n^q + 1\}, \{s_i^q : i \in n^q + 1\} \rangle\!\rangle \in \mathbb{P}$.

Proof. According to Definition 4.5.1 (a), we have to check that the structure q satisfies conditions (1_q)–(8_q). By our construction, (1_q), (2_q), (3_q) and (8_q) hold.

Let us check conditions (4_q)–(7_q). Since conditions (4_p)–(7_p) hold and $U_i^q = U_i^p$, $s_i^q = s_i^p$ for every $i \in n^p + 1$, it follows that (4_q)–(6_q) hold for each $i \in n^p + 1 = n^q$ and condition (7_q) holds for each $i \in n^p$. Therefore, it remains only to check the following four conditions:

(a) $0 \in U_{n^q}^q \subseteq (G \cap \mathbb{Q}_{\pi^q}^m) + H$,

(b) $-U_{n^q}^q = U_{n^q}^q$,

(c) $U_{n^q}^q + \lceil s_{n^q}^q \rceil^m = U_{n^q}^q$,

(d) $U_{n^q}^q + U_{n^q}^q \subseteq U_{n^p}^q$.

Conditions (b) and (c) are immediate from (4.19) and the fact that $\lceil s_{n^q}^q \rceil^m$ is a subgroup of $\mathbb{Q}^m$.

Clearly, $0 \in \lceil s^q_{n^q} \rceil^m \subseteq \mathbb{Z}^m \subseteq G$ by our assumption on G made in Definition 4.5.1. Furthermore, $\lceil s^q_{n^q} \rceil^m \subseteq \mathbb{Q}^q_{\pi^q}$ by Lemma 4.3.3 (ii). Combining this with (4.19), we get $0 \in U^q_{n^q} \subseteq G \cap \mathbb{Q}^m_{\pi^q} \subseteq (G \cap \mathbb{Q}^m_{\pi^q}) + H$. Thus, (a) holds.

From (4.19), we get $U^q_{n^q} + U^q_{n^q} = \lceil s^q_{n^q} \rceil^m + \lceil s^q_{n^q} \rceil^m = \lceil s^q_{n^q} \rceil^m$. Since $s^p_{n^p}$ divides $s^q_{n^q}$ by (4.17), we have the inclusion $\lceil s^q_{n^q} \rceil^m \subseteq \lceil s^p_{n^p} \rceil^m$. Finally, $\lceil s^p_{n^p} \rceil^m \subseteq U^q_{n_p}$ by Remark 4.5.2 (ii). This finishes the check of (d). □

The inequality $q \leq p$ is clear from our construction of q and Definition 4.5.1 (b). Finally, from (4.18) and (4.19), we get $g + h \notin U_{n^q}$. □

4.6 Density lemmas

Definition 4.6.1. Let $(\mathbb{P}, \leq)$ be a poset. Recall that a set $D \subseteq \mathbb{P}$ is called:

(i) *dense in* $(\mathbb{P}, \leq)$ provided that for every $p \in \mathbb{P}$ there exists $q \in D$ such that $q \leq p$;

(ii) *downward-closed in* $(\mathbb{P}, \leq)$ if for every $p \in D$ and $q \in \mathbb{P}$ the inequality $q \leq p$ implies that $q \in D$.

The relation between these two notions is made apparent by the following straightforward lemma.

Lemma 4.6.2. *If $A, B \subseteq \mathbb{P}$ are dense subsets of a poset $(\mathbb{P}, \leq)$ and A is downward-closed, then $A \cap B$ is dense in $(\mathbb{P}, \leq)$.*

Lemma 4.6.3. *For every $n \in \mathbb{N}$, the set $A_n = \{q \in \mathbb{P} : n \leq n^q\}$ is dense and downward-closed in $(\mathbb{P}, \leq)$.*

Proof. Let $n \in \mathbb{N}$ and $p \in \mathbb{P}$. If $n \leq n^p$, then $p \in A_n$, so we shall assume from now on that $n^p < n$. Then $k = n - n^p \geq 1$.

Note that $\mathbb{Z}^m \subseteq G$ by our assumption on G and $\mathbb{Z}^m \subseteq \mathbb{Q}^m_{\pi^p}$ by Lemma 4.3.3 (ii), so $\mathbb{Z}^m \subseteq G \cap \mathbb{Q}^m_{\pi^p}$. This allows us to fix $g \in G \cap \mathbb{Q}^m_{\pi^p}$ with $g \neq 0$. Let $h = 0$. Then $g + h = g \neq 0$.

Let $q_0 = p$. By finite induction on $i \in \{1, \dots, k\}$, we can use Lemma 4.5.5 to find $q_i \in \mathbb{P}$ such that $q_i \leq q_{i-1}$ and $n^{q_i} = n^{q_{i-1}} + 1 = n^p + i$. (Note that g and h play a "dummy role" in this argument; their sole purpose here is to make the assumptions of Lemma 4.5.5 satisfied.) Now

$q_k \leq q_{k-1} \leq \cdots \leq q_1 \leq q_0 = p$ and $n^{q_k} = n^p + k = n$, so $q_k \in A_n$. This shows that A_n is dense in $(\mathbb{P}, \leq)$.

Finally, given $p \in A_n$ and $q \in \mathbb{P}$ such that $q \leq p$, we have that $n \leq n^p \leq n^q$, thus showing that $q \in A_n$ and therefore that A_n is downward-closed. □

Lemma 4.6.4. *For every $\pi \in [\mathbf{P}]^{<\omega}$, the set $B_\pi = \{q \in \mathbb{P} : \pi \subseteq \pi^q\}$ is dense in $(\mathbb{P}, \leq)$.*

Proof. Let $\pi \in [\mathbf{P}]^{<\omega}$ and $p \in \mathbb{P}$. Define

- $\pi^q = \pi^p \cup \pi$,
- $n^q = n^p$,
- $s_i^q = s_i^p$ for every $i \in n^p + 1$,
- $U_i^q = U_i^p$ for every $i \in n^p + 1$.

A straightforward check using Definition 4.5.1 (a) shows that

$$q = \langle\!\langle \pi^q, n^q, \{U_i^q : i \in n^q + 1\}, \{s_i^q : i \in n^q + 1\} \rangle\!\rangle \in \mathbb{P}.$$

From our definition of q and Definition 4.5.1 (b), one easily concludes that $q \leq p$. Clearly, $q \in B_\pi$. This shows that B_π is dense in $(\mathbb{P}, \leq)$. □

Lemma 4.6.5. *The set $C_{g+h} = \{q \in \mathbb{P} : g + h \in (\mathbb{Q}^m_{\pi^q} + H) \setminus U^q_{n^q}\}$ is dense in $(\mathbb{P}, \leq)$ whenever $g \in G$, $h \in H$ and $g + h \neq 0$.*

Proof. Fix $g \in G$ and $h \in H$ such that $g + h \neq 0$. Let $r \in \mathbb{P}$ be arbitrary. Since $g \in G \subseteq \mathbb{Q}^m$, there exists $\pi \in [\mathbf{P}]^{<\omega}$ such that $g \in \mathbb{Q}^m_\pi$. Since B_π is dense in $(\mathbb{P}, \leq)$ by Lemma 4.6.4, there exists $p \in B_\pi$ such that $p \leq r$. Now $\pi \subseteq \pi^p$ by definition of B_π, so $\mathbb{Q}^m_\pi \subseteq \mathbb{Q}^m_{\pi^p}$ by Lemma 4.3.3 (iii). Since $g \in \mathbb{Q}^m_\pi$, we also have $g \in \mathbb{Q}^m_{\pi^p}$. We have checked that $g \in G \cap \mathbb{Q}^m_{\pi^p}$. Applying Lemma 4.5.5, we can find $q \in \mathbb{P}$ such that $q \leq p \leq r$ and $g + h \notin U_{n^q}$. Since $q \leq p$, we have $\pi^p \subseteq \pi^q$ by (i^p_q), so $\mathbb{Q}^m_{\pi^p} \subseteq \mathbb{Q}^m_{\pi^q}$ by Lemma 4.3.3 (iii). Since $g \in \mathbb{Q}^m_{\pi^p}$, we also have $g \in \mathbb{Q}^m_{\pi^q}$. Therefore, $g + h \in \mathbb{Q}^m_{\pi^q} + H$. We have proved that $g + h \in (\mathbb{Q}^m_{\pi^q} + H) \setminus U^q_{n^q}$, so $q \in C_{g+h}$. Since $q \leq r$, this shows that C_{g+h} is dense in $(\mathbb{P}, \leq)$. □

Lemma 4.6.6. *If G is a wide subgroup of $\mathbb{Q}^m$, then the set*

$$D_{g+h} = \left\{ q \in \mathbb{P} : g + h \in \bigcup_{k \in \mathbb{N}^+} U^q_{n^q} + \langle \mathrm{Cyc}(U^q_{n^q}) \rangle_k \right\} \tag{4.20}$$

is dense in $(\mathbb{P}, \leq)$ for all $g \in G$ and $h \in H$.

Proof. Let $g \in G$, $h \in H$ and $p \in \mathbb{P}$ be arbitrary. Arguing as at the beginning of the proof of Lemma 4.6.5, we may assume, without loss of generality, that $g \in \mathbb{Q}^m_{\pi^p}$. Define

$$k = 2^{n^p} + 1. \tag{4.21}$$

Applying Lemma 4.4.3 to $\pi_0 = \pi^p$ and $s = s^p_{n^p}$, we can find an increasing sequence

$$\pi^p = \pi_0 \subseteq \pi_1 \subseteq \pi_2 \subseteq \cdots \subseteq \pi_k = \pi^q \tag{4.22}$$

of finite subsets of $\mathbf{P}$ and elements $g_1, \ldots, g_k \in G$ such that conditions (a$_j$), (b$_j$), (c$_j$) of Lemma 4.3.4 (in which we let $s = s^p_{n^p}$) hold for every $j \in \{1, \ldots, k\}$.

Define g_0 as in (4.4). Since G is a subgroup of $\mathbb{Q}^m$, $g \in \mathbb{Q}^m_{\pi_0}$ and (a$_j$) holds for every $j \in \{1, \ldots, k\}$, it follows from (4.2) and (4.4) that

$$\{g_j : j \in k + 1\} \subseteq \mathbb{Q}^m_{\pi^q}. \tag{4.23}$$

Let

$$U^q_{n^p} = U^p_{n^p} \cup \left(\left(\{g_0 + h\} \cup \{-(g_0 + h)\} \cup \bigcup_{j=1}^{k} \langle g_j \rangle \right) + \lceil s^p_{n^p} \rceil^m \right). \tag{4.24}$$

By finite reverse induction, we define

$$U^q_i = U^p_i \cup (U^q_{i+1} + U^q_{i+1} + \lceil s^p_i \rceil^m) \tag{4.25}$$

for every $i = n^p - 1, \ldots, 0$.

Let $n^q = n^p$ and $s^q_i = s^p_i$ for every $i \in n^q + 1 = n^p + 1$.

Claim 18. $q = \langle\!\langle \pi^q, n^q, \{U^q_i : i \in n^q + 1\}, \{s^q_i : i \in n^q + 1\} \rangle\!\rangle \in \mathbb{P}$.

Proof. We need to check conditions (1_q)–(8_q).

Conditions (1_q)–(3_q) are trivial.

(4_q) By (4_p) and (4.25), $0 \in U_i^p \subseteq U_i^q$ for every $i \in n^q+1 = n^p+1$. By (4_p), $U_i^p \subseteq (G \cap \mathbb{Q}^m_{\pi^p}) + H$ for every $i \in n^q+1 = n^p+1$. Since G is a subgroup of $\mathbb{Q}^m$, from this, $g_1, \dots, g_k \in G$, (4.2) and (4.24), it follows that $U^q_{n^q} = U^q_{n^p} \subseteq G+H$. Furthermore, (4.23) and (4.24) imply that $U^q_{n^q} = U^q_{n^p} \subseteq \mathbb{Q}^m_{\pi_q} + H$. Since $(G \cap \mathbb{Q}^m_{\pi^q}) + H$ is a subgroup of $\mathbb{Q}^m + H$, from this, (4.25) and finite reverse induction on $i = n^q, n^q-1, \dots, 0$, one concludes that $U_i^q \subseteq (G \cap \mathbb{Q}^m_{\pi^q}) + H$ for every $i \in n^q + 1$.

(5_q) From (4.24) and (5_p), we get $-U_{n^p} = U_{n^p}$. Starting with this and using (4.25), (5_p) and finite reverse induction on $i = n^q, n^q-1, \dots, 0$, one concludes that $-U_i^q = U_i^q$ for every $i = n^q+1$.

(6_q) The equation $U^q_{n^p} + \lceil s^p_{n^p} \rceil^m = U^q_{n^p}$ follows from (4.24) and (6_p). Starting with this and using (4.25), (6_p) and finite reverse induction on $i = n^p, n^p-1, \dots, 0$, one concludes that $U_i^q + \lceil s_i^p \rceil^m = U_i^q$ for every $i \in n^p + 1$. Since $n^q = n^p$ and $s_i^q = s_i^p$ for every $i \in n^q+1$, this establishes (6_q).

(7_q) follows from (4.25).

(8_q) follows from (8_p), as $s_i^q = s_i^p$ for every $i \in n^q + 1$. □

Claim 19. $q \in D_{g+h}$.

Proof. It follows from (4.24) and $n^q = n^p$ that $g_0 + h \in U^q_{n^q}$ and $g_1, \dots, g_k \in \mathrm{Cyc}(U^q_{n^q})$. Furthermore, $g = g_0 + \sum_{i=1}^k g_i$ by (4.4). Thus,

$$g + h = g_0 + h + \sum_{i=1}^{k} g_i \in U^q_{n^q} + \langle \mathrm{Cyc}(U^q_{n^q}) \rangle_k$$

by Definition 4.2.1, so $q \in D_{g+h}$ by (4.20). □

Our final goal is to prove the inequality $q \leq p$. Before establishing this, we need to check two auxiliary facts.

Claim 20. The inclusion

$$U_i^q \subseteq \bigcup \left\{ U_i^p + l(g_0 + h) + \sum_{j \in J} \langle g_j \rangle + \lceil s_i^p \rceil^m : l \in \mathbb{Z}, |l| \leq 2^{n^q - i}, J \in \{1, \dots, k\}^{\leq 2^{n^q-i}} \right\} \tag{4.26}$$

holds for every $i \in n^p + 1$. (Here $|l|$ denotes the absolute value of l.)

Proof. We use reverse induction on $i = n^p, n^p - 1, \dots, 0$. For $i = n^p$ this follows from (4.24), as $n^q = n^p$.

For the inductive step, suppose that $i \in n^p$ and we have already proved the inclusion (4.26) for $i+1$; that is, we have shown that

$$U^q_{i+1} \subseteq \bigcup \left\{ U^p_{i+1} + l(g_0 + h) + \sum_{j \in J} \langle g_j \rangle + \lceil s^p_{i+1} \rceil^m : l \in \mathbb{Z}, |l| \leq 2^{n^q - i - 1}, J \in \{1, \dots, k\}^{\leq 2^{n^q - i - 1}} \right\} \tag{4.27}$$

holds. We are going to show that the inclusion (4.26) holds as well.

Let $x \in U^q_i$. By equation (4.25), either $x \in U^p_i$ or $x \in U^q_{i+1} + U^q_{i+1} + \lceil s^p_i \rceil^m$. In the former case, $x = x + 0(g_0 + h) + 0 + 0$ is the decomposition witnessing that x belongs to the right-hand side of (4.26). In the latter case, $x = y_0 + y_1 + s^p_i w$, where $w \in \mathbb{Z}^m$ and $y_t \in U^q_{i+1}$ for $t = 0, 1$. Applying (4.27), we can find $u_t \in U^p_{i+1}$, $l_t \in \mathbb{Z}$, $J_t \in \{1, 2, \dots, k\}^{\leq 2^{n^q - i - 1}}$, $b_t \in \sum_{j \in J_t} \langle g_j \rangle$ and $z_t \in \mathbb{Z}^m$ for $t = 0, 1$ such that $|l_t| \leq 2^{n^q - i - 1}$ and $y_t = u_t + l_t(g_0 + h) + b_t + s^p_{i+1} z_t$. Since $x = y_0 + y_1 + s^p_i w$, it follows that $x = u + l(g_0 + h) + b + s^p_{i+1} z + s^p_i w$, where $u = u_0 + u_1$, $l = l_0 + l_1$, $b = b_0 + b_1$ and $z = z_0 + z_1$. Since s^p_i divides s^p_{i+1} by (8_p), $s^p_{i+1} = s^p_i k_0$ for some $k_0 \in \mathbb{Z}$, so

$$x = u + l(g_0 + h) + b + s^p_i (k_0 z + w). \tag{4.28}$$

Since $u_t \in U^p_{i+1}$ for $t = 0, 1$, $u = u_0 + u_1 \in U^p_{i+1} + U^p_{i+1} \subseteq U^p_i$ by (7_p). Since $l_t \in \mathbb{Z}$ and $|l_t| \leq 2^{n^q - i - 1}$ for $t = 0, 1$, we have $l = l_0 + l_1 \in \mathbb{Z}$ and

$$|l| \leq |l_0| + |l_1| \leq 2^{n^q - i - 1} + 2^{n^q - i - 1} = 2 \cdot 2^{n^q - i - 1} = 2^{n^q - i}.$$

Since $J_t \in \{1, 2, \dots, k\}^{\leq 2^{n^q - i - 1}}$ for $t = 0, 1$, the set $J = J_0 \cup J_1$ satisfies

$$|J| \leq |J_1| + |J_2| \leq 2^{n^q - i - 1} + 2^{n^q - i - 1} = 2 \cdot 2^{n^q - i - 1} = 2^{n^q - i},$$

so $J \in \{1, 2, \dots, k\}^{\leq 2^{n^q - i}}$. Since $b_t \in \sum_{j \in J_t} \langle g_j \rangle$ for $t = 0, 1$, we have $b = b_0 + b_1 \in \sum_{j \in J} \langle g_j \rangle$. Since $z_t \in \mathbb{Z}^m$ for $t = 0, 1$, we have $z = z_0 + z_1 \in \mathbb{Z}^m$. Finally, since $k_0 \in \mathbb{Z}$, we get $k_0 z + w \in \mathbb{Z}^m$ and thus $s_i^p(k_0 z + w) \in \lceil s_i^p \rceil^m$. Combining all this with (4.28), we conclude that

$$x \in U_i^p + l(g_0 + h) + \sum_{j \in J} \langle g_j \rangle + \lceil s_i^p \rceil^m$$

with $l \in \mathbb{Z}$, $J \in \{1, \dots, k\}^{\leq 2^{n^q - i}}$ and $|l| \leq 2^{n^q - i}$. This finishes the proof of the inclusion (4.26). $\square$

Claim 21. $U_i^q \cap (\mathbb{Q}_{\pi^p}^m + H) = U_i^p$ for every $i \in n^p + 1$.

Proof. It follows from (4.25) that $U_i^p \subseteq U_i^q$. Since $U_i^p \subseteq \mathbb{Q}_{\pi^p}^m + H$ by (4_p), this establishes the inclusion $U_i^p \subseteq U_i^q \cap (\mathbb{Q}_{\pi^p}^m + H)$.

To prove the reverse inclusion, we fix an arbitrary $x \in U_i^q \cap (\mathbb{Q}_{\pi^p}^m + H)$ and we are going to show that $x \in U_i^p$. Since $x \in U_i^q$, we can apply Claim 20 to fix $y \in U_i^p$, $l \in \mathbb{Z}$, a finite set $J \subseteq \{1, 2, \dots, k\}$, a family $\{m_j : j \in J\} \subseteq \mathbb{Z}$ and $z \in \mathbb{Z}^m$ such that $|l| \leq 2^{n^q - i}$, $|J| \leq 2^{n^q - i}$ and

$$x = y + l(g_0 + h) + \sum_{j \in J} m_j g_j + s_i^p z. \tag{4.29}$$

Recall that $x \in \mathbb{Q}_{\pi^p}^m + H$. Furthermore, $y \in U_i^p \subseteq \mathbb{Q}_{\pi^p}^m + H$ by (4_p). Finally, $s_i^p z \in \mathbb{Z}^m \subseteq \mathbb{Q}_{\pi^p}^m$ by Lemma 4.3.3(ii). Since $\pi_0 = \pi^p$ by (4.22) and $\mathbb{Q}_{\pi_0}^m$ is a group, from (4.29) we get

$$l(g_0 + h) + \sum_{j \in J} m_j g_j = x - y - s_i^p z \in \mathbb{Q}_{\pi_0}^m + H, \tag{4.30}$$

so

$$l g_0 \in -lh - \sum_{j \in J} m_j g_j + \mathbb{Q}_{\pi_0}^m + H \in \langle \{g_j : j \in J\} \rangle + \mathbb{Q}_{\pi_0}^m + H. \tag{4.31}$$

Since $\{g_0\} \cup \{g_j : j \in J\} \subseteq \mathbb{Q}^m$, $\mathbb{Q}_{\pi_0}^m \subseteq \mathbb{Q}^m$, $\mathbb{Q}^m$ is a group and the sum $\mathbb{Q}^m + H = \mathbb{Q}^m \oplus H$ is direct, from the inclusion (4.31) we obtain the stricter inclusion $l g_0 \in \langle \{g_j : j \in J\} \rangle + \mathbb{Q}_{\pi_0}^m$.

As $|J| \leq 2^{n^q - i} \leq 2^{n^q} = 2^{n^p} < k$ by (4.21), J is a proper subset of $\{1, \dots, k\}$. Furthermore, $|l| \leq 2^{n^q - i} < k$. Now all the assumptions of Lemma 4.3.4 are satisfied (with $s = s_{n^p}^p$), so from item (B) of this lemma we conclude that $l = 0$. From this and (4.30), we obtain $\sum_{j \in J} m_j g_j \in \mathbb{Q}_{\pi_0}^m + H$, so we can fix $a \in \mathbb{Q}_{\pi_0}^m$ and $b \in H$ satisfying $\sum_{j \in J} m_j g_j = a + b$. Since $\mathbb{Q}_{\pi^q}^m$ is a subgroup of $\mathbb{Q}^m$, from (4.23) we get $\sum_{j\in J} m_j g_j \in \mathbb{Q}^m_{\pi^q}$. Therefore,

$$b = -a + \sum_{j\in J} m_j g_j \in \mathbb{Q}^m_{\pi_0} + \mathbb{Q}^m_{\pi^q} \subseteq \mathbb{Q}^m.$$

Since $b \in H$, we get $b \in \mathbb{Q}^m \cap H$. Since the sum $\mathbb{Q}^m + H = \mathbb{Q}^m \oplus H$ is direct, we conclude that $b = 0$. Thus, $\sum_{j\in J} m_j g_j = a \in \mathbb{Q}^m_{\pi_0}$. Since $J \subseteq \{1, \dots, k\}$, we get

$$\sum_{j\in J} m_j g_j \in \langle\{g_j : j \in J\}\rangle \cap \mathbb{Q}^m_{\pi_0} \subseteq \langle\{g_1, \dots, g_k\}\rangle \cap \mathbb{Q}^m_{\pi_0} \subseteq \lceil s^p_{n^p}\rceil^m$$

by Lemma 4.3.4(A)(ii) applied with $s = s^p_{n^p}$. Combining this with (4.29), we get $x \in y + \lceil s^p_{n^p}\rceil^m + \lceil s^p_i\rceil^m$. Since $i \le n_p$, s^p_i divides $s^p_{n^p}$ by (8_p), so $\lceil s^p_{n^p}\rceil^m \subseteq \lceil s^p_i\rceil^m$. This gives $x \in y + \lceil s^p_i\rceil^m \subseteq U^p_i + \lceil s^p_i\rceil^m = U^p_i$ by (6_p). □

Claim 22. $q \le p$.

Proof. We need to check conditions (i^p_q)–(iv^p_q) from Definition 4.5.1(b). Condition (i^p_q) follows from (4.2). Since $n^q = n^p$ by our definition, (ii^p_q) holds. Condition (iii^p_q) is proved in Claim 21, and (iv^p_q) holds by the definition of s^q_i. □

The density of D_{g+h} in $(\mathbb{P}, \le)$ follows from Claims 18, 19 and 22. □

4.7 Main theorem

Recall the following concepts. Let X be a set and let $\mathcal{F}$ be a family of non-empty subsets of X, we say that $\mathbf{F}$ is a *filter-base* if for every pair $A, B \in \mathcal{F}$ there exists $C \in \mathcal{F}$ such that $C \subseteq A \cap B$. A filter-base $\mathcal{F}$ is said to be a *filter* if for every set F satisfying $F' \subseteq F$ for some $F \in \mathcal{F}$ the inclusion $F \in \mathcal{F}$ holds. Group topologies can be described in terms of filters and filter-bases as follows:

Theorem 4.7.1. *Let G be a group and $\mathcal{V}(e)$ be the filter of all neighbourhoods of e in some group topology τ for G. Then:*

(a) for every $U \in \mathcal{V}(e)$ there exists $V \in \mathcal{V}(e)$ such that $V \cdot V \subseteq U$;

(b) for every $U \in \mathcal{V}(e)$ there exists $V \in \mathcal{V}(e)$ such that $V^{-1} \subseteq U$;

(c) for every $U \in \mathcal{V}(e)$ and $a \in G$ there exists $V \in \mathcal{V}(e)$ such that $aVa^{-1} \subseteq U$.

If $\mathcal{V}$ is a filter(-base) on G satisfying (a),(b) and (c), then there exists a unique group topology τ on G such that $\mathcal{V}$ coincides with the filter of all τ-neighbourhoods of e_G in G.

The following is clear.

Remark 4.7.2. If G is an Abelian group then Theorem 4.7.1(a) implies item Theorem 4.7.1(c).

We shall also need the following folklore lemma.

Lemma 4.7.3. *If $\mathscr{D}$ is an at most countable family of dense subsets of a non-empty poset $(\mathbb{P}, \leq)$, then there exists an at most countable subset $\mathbb{F}$ of $\mathbb{P}$ such that $(\mathbb{F}, \leq)$ is a linearly ordered set and $\mathbb{F} \cap D \neq \emptyset$ for every $D \in \mathscr{D}$.*

Proof. Since the family $\mathscr{D}$ is at most countable, we can fix an enumeration $\mathscr{D} = \{D_n : n \in \mathbb{N}^+\}$ of elements of $\mathscr{D}$. Since $\mathbb{P} \neq \emptyset$, there exists $p_0 \in \mathbb{P}$. By induction on $n \in \mathbb{N}^+$, we can choose $p_n \in D_n$ such that $p_n \leq p_{n-1}$; this is possible because D_n is dense in $(\mathbb{P}, \leq)$. Now $\mathbb{F} = \{p_n : n \in \mathbb{N}^+\}$ is the desired subset of $\mathbb{P}$. □

The next theorem provides a positive answer to a more general version of [16, Question 13.1].

Theorem 4.7.4. *Suppose that $m \in \mathbb{N}^+$ and G is a wide subgroup of $\mathbb{Q}^m$. Then for each at most countable abelian group H, the direct sum $K = G \oplus H$ admits a metric* SSGP *topology.*

Proof. Fix $m \in \mathbb{N}^+$ and let G be a wide subgroup of $\mathbb{Q}^m$. Let H be an at most countable abelian group. We work in the direct sum $\mathbb{Q}^m \oplus H$ and consider its subgroup $K = G + H = G \oplus H$.

Let $(\mathbb{P}, \leq)$ be the poset from Definition 4.5.1 which uses our $m \in \mathbb{N}^+$, G and H as its parameters.

As a subgroup of $\mathbb{Q}^m$, G is at most countable. Since H is at most countable as well, so is the sum $K = G + H$. Therefore, the family

$$\mathscr{D} = \{C_x : x \in K \setminus \{0\}\} \cup \{A_n \cap D_x : n \in \mathbb{N}, x \in K\} \tag{4.32}$$

of subsets of $\mathbb{P}$ is at most countable.

Lst us check that all members of $\mathscr{D}$ are dense in $(\mathbb{P}, \leq)$. By Lemma 4.6.5, each C_x for $x \in K \setminus \{0\}$ is dense in $(\mathbb{P}, \leq)$. Let $n \in \mathbb{N}$ and $x \in K$. Since A_n is dense and downward-closed in $(\mathbb{P}, \leq)$ by

Lemma 4.6.3 and D_x is dense in $(\mathbb{P}, \leq)$ by Lemma 4.6.6, from Lemma 4.6.2 we conclude that $A_n \cap D_x$ is dense in $(\mathbb{P}, \leq)$.

Apply Lemma 4.7.3 to find an at most countable set $\mathbb{F} \subseteq \mathbb{P}$ such that $(\mathbb{F}, \leq)$ is a linearly ordered set and $\mathbb{F} \cap D \neq \emptyset$ for every $D \in \mathscr{D}$. For every $i \in \mathbb{N}$, define

$$U_i = \bigcup \{U_i^p : p \in \mathbb{F} \text{ and } i \leq n^p\}. \tag{4.33}$$

Our nearest goal is to show that the family

$$\mathcal{B} = \{U_i : i \in \mathbb{N}\} \tag{4.34}$$

is a neighbourhood base at 0 of a Hausdorff group topology $\mathscr{T}$ on K. The verification of this will be split into three claims.

Claim 23. $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. Since $A_n \cap D_0 \in \mathscr{D}$ by (4.32), there exists $p \in A_n \cap D_0 \cap \mathbb{F}$. Therefore, $n \leq n^p$ by the definition of A_n. Now $0 \in U_n^p$ by the condition (4_p) imposed on $\mathbb{P}$. Since $p \in \mathbb{F}$ and $n \leq n^p$, it follows from (4.33) that $U_n^p \subseteq U_n$. This shows that $0 \in U_n$. Since $n \in \mathbb{N}$ was chosen arbitrarily, we conclude that $0 \in \bigcap_{n \in \mathbb{N}} U_n$.

To prove the reverse inclusion $\bigcap_{n \in \mathbb{N}} U_n \subseteq \{0\}$, we choose $x \in K \setminus \{0\}$ arbitrarily and show that $x \notin \bigcap_{n \in \mathbb{N}} U_n$. Since $C_x \in \mathscr{D}$ by (4.32), our choice of $\mathbb{F}$ allows us to fix $p \in C_x \cap \mathbb{F}$. Then

$$x \in (\mathbb{Q}_{\pi^p}^m + H) \setminus U_{n^p}^p \tag{4.35}$$

by the definition of C_x.

Assume that $x \in U_{n^p}$. Thus, by (4.33), there exists $q \in \mathbb{F}$ such that $n^p \leq n^q$ and $x \in U_{n^p}^q$. Suppose that $p \leq q$. Then $U_{n^p}^p \cap (\mathbb{Q}_{\pi^q}^m + H) = U_{n^p}^q$ by (iii$_p^q$). Since $x \in U_{n^p}^q$, this implies $x \in U_{n^p}^p$, in contradiction with (4.35). Similarly, assume that $q \leq p$. Then $U_{n^p}^q \cap (\mathbb{Q}_{\pi^p}^m + H) = U_{n^p}^p$ by (iii$_q^p$), so $x \notin U_{n^p}^q$ by (4.35), in contradiction with $x \in U_{n^p}^q$. This contradiction shows that neither $p \leq q$ nor $q \leq p$ holds. Since $p, q \in \mathbb{F}$, this contradicts the fact that $\mathbb{F}$ is linearly ordered by $\leq$. This contradiction shows that our assumption that $x \in U_{n^p}$ is false, so $x \notin \bigcap_{n \in \mathbb{N}} U_n$. $\square$

Claim 24. $-U_i = U_i$ and $U_{i+1} + U_{i+1} \subseteq U_i$ for every $i \in \mathbb{N}$.

Proof. Let $x \in U_i$. By (4.33), this means that $x \in U_i^p$ for some $p \in \mathbb{F}$ satisfying $i \leq n^p$. By (5_p), we know that $-U_i^p = U_i^p$, so $-x \in U_i^p$. Thus, $-x \in U_i$ by (4.33). This proves that $-U_i \subseteq U_i$. The reverse inclusion is proved analogously.

Finally, consider $x, y \in U_{i+1}$. By (4.33), there exist $p, q \in \mathbb{F}$ such that $x \in U_{i+1}^p$, $y \in U_{i+1}^q$, $i+1 \leq n^p$ and $i+1 \leq n^q$. Since $\leq$ is a linear order on $\mathbb{F}$, we may assume, without loss of generality, that $q \leq p$. Since $i+1 \leq n^p$, (iii_q^p) implies $x \in U_{i+1}^p \subseteq U_{i+1}^q$. Therefore, $x, y \in U_{i+1}^q$. Since $i+1 \leq n^q$, we have $x + y \in U_{i+1}^q + U_{i+1}^q \subseteq U_i^q$ by (7_q). Then $x + y \in U_i$ by (4.33). This shows that $U_{i+1} + U_{i+1} \subseteq U_i$, as desired. □

Claim 25. The family $\mathcal{B}$ as in (4.34) is a neighbourhood base at 0 of some Hausdorff group topology $\mathscr{T}$ on K.

Proof. It easily follows from Claims 23 and 24 that $U_m \subseteq U_n$ whenever $n, m \in \mathbb{N}$ and $n \leq m$. Combined with (4.34), this implies that $\mathcal{B}$ is a filter base. By (4.34) and Claim 24, $\mathcal{B}$ has the following two properties.

- For every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $V + V \subseteq U$; and
- For every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $-V \subseteq U$.

By Theorem 4.7.1 and Remark 4.7.2

$$\mathscr{T} = \{O \subseteq K : \forall\ a \in O\ \exists\ U \in \mathcal{B}\ (a + U \subseteq O)\}$$

is a topology on K making it into a topological group such that the family $\mathcal{B}$ is a neighbourhood base at 0 comprised of $\mathscr{T}$-neighbourhoods of 0. It follows from Claim 23, Theorem 4.7.1 and Remark 4.7.2 that $\mathscr{T}$ is Hausdorff group topology for K. □

Claim 26. The topological group $(K, \mathscr{T})$ has the small subgroup generating property.

Proof. We are going to check that $(K, \mathscr{T})$ has the property (ii) from Proposition 4.2.3 (applied to $G = K$).

Let W be a neighbourhood of 0 in $(K, \mathscr{T})$. By Claim 25, there exists $i \in \mathbb{N}$ such that $U_i \subseteq W$.

Take an arbitrary $x \in K$. Since $A_i \cap D_x \in \mathscr{D}$ by (4.32), there exists $q \in \mathbb{F} \cap A_i \cap D_x$. Since $q \in D_x$, we can use (4.20) to find $k \in \mathbb{N}$ such that

$$x \in U^q_{n^q} + \langle \mathrm{Cyc}(U^q_{n^q}) \rangle_k. \tag{4.36}$$

Since $q \in A_i$, we have $i \leq n^q$, so $U^q_{n^q} \subseteq U^q_i$ by Remark 4.5.2 (i). Since $q \in \mathbb{F}$, from (4.33) we get $U^q_i \subseteq U_i$, so $U^q_{n^q} \subseteq U_i \subseteq W$, which implies that $\mathrm{Cyc}(U^q_{n^q}) \subseteq \mathrm{Cyc}(W)$. Therefore,

$$U^q_{n^q} + \langle \mathrm{Cyc}(U^q_{n^q}) \rangle_k \subseteq W + \langle \mathrm{Cyc}(W) \rangle_k. \tag{4.37}$$

Combining (4.36) and (4.37), we conclude that $x \in \bigcup_{k \in \mathbb{N}^+} W + \langle \mathrm{Cyc}(W) \rangle_k$. Since this holds for every $x \in K$, we get $K \subseteq \bigcup_{k \in \mathbb{N}^+} W + \langle \mathrm{Cyc}(W) \rangle_k$. The converse inclusion is clear. □

Since $(K, \mathscr{T})$ is Hausdorff and has a countable base at 0 by Claim 25, it is metrizable. □

Chapter 5

SSGP **topologies on free groups of infinite rank**

5.1 Introduction

The main goal of this chapter is to prove the following two theorems announced by Shakhmatov and the author in [40]:

Theorem 5.1.1. *The free group $F(X)$ over a countably infinite set X admits a metric* DW *group topology.*

Theorem 5.1.2. *Every free group with infinitely many generators admits an* DW *group topology.*

The proofs of these two theorems are postponed until Sections 5.8 and 5.9, respectively.

In view of the first implication in (1.6), Theorem 5.1.2 provides an answer (in a stronger form) to Question 1.5.10(a) for free groups of infinite rank.

Even a weaker version of Theorem 5.1.2 seems worthwhile stating explicitly.

Corollary 5.1.3. *Free group of infinite rank admit a minimally almost periodic group topology.*

Question 1.5.10(a) remains open for most other non-Abelian groups. To our knowledge, the only other result in the non-commutative case is the following. If a set X has at least two elements, then its symmetric group $S(X)$ does not admit an SSGP group topology [16, Example 5.4(c)].

Remark 5.1.4. Shortly before the arrival of galley proofs in the published version of these results, Dmitri Shakhmatov asked Vladimir Pestov about the novelty of Corollary 5.1.3. While this result does not seem to have appeared in published form, Vladimir Pestov informed Dmitri Shakhmatov that he can prove a stronger version of Corollary 5.1.3 in which minimally almost periodic is replaced with extremely amenable. Since extremely amenable groups need not have the SSGP property [1], the strengthening of Corollary 5.1.3 due to Vladimir Pestov implies neither Theorem 5.1.1 nor Theorem 5.1.2.

This chapter is organized as follows. Basic facts about free groups are recalled in Section 5.2. In Section 5.3 we introduce a notion of a finite neighbourhood system on a free group; this is basically a finite initial segment of a countable family of future neighbourhoods in some group topology on this group. In Section 5.4, a notion of an extension of a finite neighbourhood system is defined; this is a finite neighbourhood system on a bigger free group whose traces of new neighbourhoods to the smaller free group coincide with the original neighborhoods. In Section 5.5, we devise a technique for extending a finite neighbourhood system to a finite neighbourhood system on a bigger group.and a fixed set which can be viewed as a base for such an extension. Section 5.6 contains three auxiliary lemmas, the main of which is Lemma 5.6.3 responsible for the SSGP property of the topology under construction. In Section 5.7, we introduce a partially ordered set which is used in the proof of Theorem 5.1.1 (the countable case); the proof itself is carried out in Section 5.8. Theorem 5.1.2 (the general case) is proved in Section 5.9. Its proof simply provides a reduction of general case to the countable case. Finally, open questions are listed in Section 5.10.

In the proof of Theorem 5.1.1, we use a partially ordered set to produce a topology on the free group with a countably infinite set of generators. This technique was used by the Shakhmatov and the author recently in [39] and [42] (see Chapter 4 for the latter).

Theorems 5.1.1 and 5.1.2 were announced previously by Shakhmatov and the author in [40].

5.2 The free group $F(X)$ over a set X

Definition 5.2.1. Let X be a set.

[1]Indeed, it follows from [23, Theorem 1.1] that the group $\mathbb{Z}$ of integer numbers admits an extremely amenable group topology, yet it does not admit any SSGP group topology by [4, Corollary 3.14].

(i) Let $W_0 = \{\emptyset\}$. For $n \in \mathbb{N}^+$, let

$$W_n = \{x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} : x_i \in X, \varepsilon_i \in \{-1, 1\} \text{ for all } i = 1, \dots, n\}.$$

(ii) For $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \in W_n(X)$ and $v = y_1^{\delta_1} \dots y_m^{\delta_m} \in W_m(X)$, we let $w = v$ if and only if $n = m$ and $x_i = y_i$, $\varepsilon_i = \delta_i$ for all $i = 1, \dots, n$.

(iii) Elements of the set

$$W(X) = \bigcup_{n \in \mathbb{N}} W_n(X)$$

are called *words in alphabet* X. According to (ii), this union consists of pairwise disjoint sets, so for every word $w \in W(X)$, there exists a unique $n \in \mathbb{N}$ such that $w \in W_n(X)$; this n is called the *length* of w and denoted by $l(w)$.

(iv) Given a word $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \in W_n(X)$, a *sub-word of* w is a word $w' = x_k^{\varepsilon_k} \dots x_l^{\varepsilon_l}$ for some $k, l \in \mathbb{N}$ such that $1 \leq k \leq l \leq n$. The word w' is said to be an *initial sub-word* of w when $k = 1$ and a *final sub-word* of w when $l = n$.

The *empty word* $\emptyset$ will be denoted also by e. Clearly, we have that $l(e) = 0$.

Definition 5.2.2. Let X be a set.

(i) For $v = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \in W_n(X)$ and $w = y_1^{\delta_1} \dots y_m^{\delta_m} \in W_m(X)$, the word $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} y_1^{\delta_1} \dots y_m^{\delta_m} \in W_{n+m}(X)$ is called the (result of) *concatenation of* v *and* w; we denote this word by $v * w$. We also let $e * w = w * e = w$ for every word $w \in W(X)$.

(ii) For $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \in W_n(X)$, the word $w^{-1} = x_n^{-\varepsilon_n} \dots x_1^{-\varepsilon_1} \in W_n(X)$ is called the *inverse of* w. We also let $e^{-1} = e$.

The set $W(X)$ equipped with the binary operation $*$ is a semigroup with the identity e.

The proof of the following lemma is straightforward.

Lemma 5.2.3. *For every initial sub-word w' of a word $w \in W(X)$, there exists a unique final sub-word w'' of w such that $w = w' * w''$. Conversely, for every final sub-word w'' of w, there exists an initial sub-word w' of w such that $w = w' * w''$.*

Definition 5.2.4. Let X be a set. A word $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \in W(X)$ is *irreducible* provided that, for every $i = 1, \dots, n-1$, either $x_i \neq x_{i+1}$ or $\varepsilon_i = \varepsilon_{i+1}$. Observe that the empty word e is considered irreducible too. We shall denote by $F(X)$ the set of all irreducible words $w \in W(X)$.

Lemma 5.2.5. *For every pair of irreducible words $v, w \in F(X)$, there exist unique $v', v'', w', w'' \in W(X)$ such that $v = v' * v''$, $w = w' * w''$, v'' and w' are inverses of each other and $v' * w''$ is an irreducible word.*

Proof. Let v'' be the final sub-word of v of maximal length $l(v'')$ such that its inverse $w' = (v'')^{-1}$ is an initial sub-word of w. Use Lemma 5.2.3 to find a unique initial sub-word v' of v and a unique final sub-word w'' of w such that $v = v' * v''$ and $w = w' * w''$. Finally, note that the word $v' * w''$ is irreducible by the maximality of v'' and Definition 5.2.4. □

For a set X, we define a binary operation $\cdot$ on the set $F(X)$ as follows.

Definition 5.2.6. For a set X and $v, w \in F(X)$, we define $v \cdot w = v' * w''$, where $v', v'', w', w'' \in W(X)$ are the unique words as in the conclusion of Lemma 5.2.5.

The following fact is well-known.

Fact 5.2.7. The $\cdot$ operation on $F(X)$ is associative.

From this fact, Lemma 5.2.5 and observing that e behaves as an identity element, we obtain that $F(X)$ equipped with the operation $\cdot$ is a group:

Lemma 5.2.8. *For every set X, the set $F(X)$ equipped with the binary operation $\cdot$ is a group with the identity e. The inverse of an element $w \in F(X)$ in $F(X)$ coincides with the (irreducible) word w^{-1} defined in item (ii) of Definition 5.2.2.*

Definition 5.2.9. The group $F(X)$ from Lemma 5.2.8 is called the *free group over X*.

Let us observe the following fundamental property of the free group:

Lemma 5.2.10. *Let X be a set and G be any group. Every mapping $f : X \to G$ has an extension to a unique homomorphism $\hat{f} : F(X) \to G$ such that $\hat{f} \restriction_X = f$. (Here we identify each $x \in X$ with the word $x^1 \in F(X)$.)*

Proof. Let X, f and G be as in the hypotheses. Let $\varphi : W(X) \to G$ be the map defined by $\varphi(e) = e$ and $\varphi(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n}) = f(x_1)^{\varepsilon_1} \cdots f(x_n)^{\varepsilon_n}$ whenever $n \in \mathbb{N}^+$, $x_1, \dots, x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}$. Clearly, φ is a semigroup homomorphism from $(W(X), *)$ to G. We claim that $\hat{f} = \varphi \restriction_{F(X)}: (F(X), \cdot) \to G$ is the desired group homomorphism. (Recall that $F(X)$ is a subset of $W(X)$ by Definition 5.2.4.)

To show that $\hat{f}$ is a homomorphism, it suffices to fix $v, w \in F(X)$ and verify the equality $\hat{f}(v \cdot w) = \hat{f}(v)\hat{f}(w)$. We use notations from Definition 5.2.6. Since v'' and w' are inverses of each other by Lemma 5.2.5, from our definition of φ, Definition 5.2.2(ii) and the fact that G is a group, we obtain that $\varphi(v'' * w') = e$. Since $v, w \in F(X)$ and φ is a semigroup homomorphism, we have

$$\hat{f}(v)\hat{f}(w) = \varphi(v)\varphi(w) = \varphi(v*w) = \varphi(v'*v''*w'*w'') = \varphi(v')\varphi(v''*w')\varphi(w'') = \varphi(v')\varphi(w'') = \varphi(v'*w'').$$

Since $v' * w'' \in F(X)$ by Lemma 5.2.5, we get $\varphi(v' * w'') = \hat{f}(v' * w'')$. Finally, since $v \cdot w = v' * w''$ by Definition 5.2.6, the equality $\hat{f}(v' * w'') = \hat{f}(v \cdot w)$ trivially holds. We have proved that $\hat{f}(v)\hat{f}(w) = \hat{f}(v \cdot w)$.

Clearly, $\hat{f}(X) = \hat{f}(x^1) = \varphi(x^1) = f(x)^1 = f(x)$ for every $x \in X$, so $\hat{f} \restriction_X = f$. The uniqueness of $\hat{f}$ is straightforward. □

Lemma 5.2.10 shows that the group $(F(X), \cdot)$ coincides with the categorically defined $\mathscr{G}$-free group $F_{\mathscr{G}}(X)$ over X in the variety $\mathscr{G}$ of all groups.

Definition 5.2.11. Let X be a set. For a word $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \in W(X) \setminus \{e\}$,

$$\mathrm{lett}(w) = \{x_1, \dots, x_n\}$$

denotes the set of all letters x_i appearing in w. We also let $\mathrm{lett}(e) = \mathrm{lett}(\emptyset) = \emptyset$.

For an arbitrary variety $\mathscr{V}$ of groups, the notion of the support $\mathrm{supp}_{\mathscr{V}}(w)$ of an element w of a free group $F_{\mathscr{V}}(X)$ in the variety $\mathscr{V}$ was introduced in the text after [14, Lemma 7.1]. One can easily see that the set $\mathrm{lett}(w)$ from the above definition coincides with the support $\mathrm{supp}_{\mathscr{V}}(w)$ of the word $w \in F(X)$ in the variety $\mathscr{G}$ of all groups.

We finish this section with two lemmas which shall be needed in the future.

Lemma 5.2.12. *(i)* $\mathrm{lett}(v \cdot w) \subseteq \mathrm{lett}(v) \cup \mathrm{lett}(w)$ *for all* $v, w \in F(X)$.

(ii) $\mathrm{lett}(a_1 \cdot a_2 \cdot \dots \cdot a_m) \subseteq \bigcup_{l=1}^{m} \mathrm{lett}(a_l)$ *whenever* $m \in \mathbb{N}^+$ *and* $a_1, a_2, \dots, a_m \in F(X)$.

Proof. (i) Let $v', v'', w', w'' \in F(X)$ be as in Definition 5.2.6. Then $v \cdot w = v' * w''$, which implies $\mathrm{lett}(v \cdot w) = \mathrm{lett}(v' * w'') = \mathrm{lett}(v') \cup \mathrm{lett}(w'')$ by Definition 5.2.11. Since v' is a sub-word of v, we have $\mathrm{lett}(v') \subseteq \mathrm{lett}(v)$ by Definitions 5.2.1(iv) and 5.2.11. Similarly, since w'' is a sub-word of w, we have $\mathrm{lett}(w'') \subseteq \mathrm{lett}(w)$. This proves item (i).

Item (ii) is proved by induction making use of item (i). □

Lemma 5.2.13. *Suppose that* $x_1, x_2, \dots, x_s \in X$, $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s \in \{-1, 1\}$, $g_0 = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_s^{\varepsilon_s} \in F(X)$, $j = 1, \dots, s$ *and* $x_j \neq x_p$ *for* $p = 1, \dots, s$ *with* $j \neq p$. *Then* $x_j \in \mathrm{lett}(h)$ *for every* $h \in \langle g_0 \rangle \setminus \{e\}$.

Proof. Fix $h \in \langle g_0 \rangle$. Then $h = g_0^q$ for some integer q. Consider the map $f : X \to F(X)$ satisfying

$$f(x_j) = x_j \quad \text{and} \quad f(x) = e \text{ for } x \in X \setminus \{x_j\}. \tag{5.1}$$

Let $\hat{f} : F(X) \to F(X)$ be the homomorphism such that $\hat{f} \restriction_X = f$. Since $\hat{f}$ is a homomorphism extending f and (5.1) holds, we have $\hat{f}(h) = \hat{f}(g_0^q) = x_j^q$.

Suppose that $x_j \notin \mathrm{lett}(h)$. Then $\mathrm{lett}(h) \subseteq X \setminus \{x_j\}$, and so $\hat{f}(h) = e$ by (5.1). This shows that $x_j^q = e$, which implies $q = 0$. Thus, $h = g_0^q = e$. □

5.3 Finite neighbourhood systems

Definition 5.3.1. Let X be a set. Given any set $A \subseteq F(X)$, we define $A^{-1} = \{a^{-1} : a \in A\}$, and $\bar{A} = A \cup A^{-1} \cup \{e\}$.

Definition 5.3.2. Let X be a set. A *finite neighbourhood system of* $F(X)$ is a finite sequence $\mathscr{U} = \{U_i : i \leq n\}$ (where $n \in \mathbb{N}^+$) satisfying the following conditions:

($1_{\mathscr{U}}$) $U_i \subseteq F(X)$ for every $i \leq n$,

($2_{\mathscr{U}}$) $U_i^{-1} = U_i$ for every $i \leq n$,

($3_{\mathscr{U}}$) $\bigcup_{x \in \bar{X}} x \cdot U_{i+1} \cdot U_{i+1} \cdot x^{-1} \subseteq U_i$ for every $i < n$,

($4_{\mathscr{U}}$) $e \in U_n$.

Remark 5.3.3. *If X is a set and $\mathscr{U} = \{U_i : i \le n\}$ is a finite neighbourhood system for $F(X)$, then $e \in U_i$ for every $i \le n$.* This statement is proved by finite reverse induction on $i = n, n-1, \dots, 0$. For $i = n$, the statement holds by ($4_{\mathscr{U}}$). Suppose now that $i < n$ and we have already proved that $e \in U_{i+1}$. Since $e \in \bar{X}$ by Definition 5.3.1, $e = e \cdot e \cdot e \cdot e \in \bigcup_{x \in \bar{X}} x \cdot U_{i+1} \cdot U_{i+1} \cdot x^{-1} \subseteq U_i$ by ($3_{\mathscr{U}}$).

Definition 5.3.4. Let $\mathscr{U} = \{U_i : i \le n\}$ be a finite sequence of subsets of $F(X)$ for some set X.

(i) Let $B \subseteq F(X)$. Define

$$V_n = U_n \cup B. \tag{5.2}$$

By finite reverse induction on $i = n-1, n-2, \dots, 0$, define

$$V_i = U_i \cup \bigcup_{x \in \bar{X}} x \cdot V_{i+1} \cdot V_{i+1} \cdot x^{-1}. \tag{5.3}$$

We shall call the sequence $\mathscr{V} = \{V_i : i \le n\}$ the *B-enrichment of the sequence $\mathscr{U}$ in $F(X)$*.

(ii) For a set $C \subseteq F(X)$, we shall call the $\left(\bigcup_{c \in C} \langle c \rangle\right)$-enrichment of $\mathscr{U}$ in $F(X)$ the *cyclic C-enrichment of $\mathscr{U}$ in $F(X)$*.

Lemma 5.3.5. *Let X be a set and $\mathscr{U} = \{U_i : i \le n\}$ be a finite sequence such that:*

(a) $U_i \subseteq F(X)$ for all $i \le n$,

(b) $U_i^{-1} = U_i$ for every $i \le n$,

(c) $e \in U_n$.

Furthermore, let $B \subseteq F(X)$ be a set satisfying

(d) $B^{-1} = B$.

Then the B-enrichment of $\mathscr{U}$ in $F(X)$ is a finite neighbourhood system for $F(X)$.

Proof. Let $\mathscr{V} = \{V_i : i \le n\}$ be the B-enrichment of the sequence $\mathscr{U}$ in $F(X)$. It suffices to check conditions ($1_{\mathscr{V}}$)–($4_{\mathscr{V}}$) of Definition 5.3.2.

($1_{\mathscr{V}}$) Since U_n and B are subsets of $F(X)$ by our assumption, $V_n \subseteq F(X)$ by (5.2). Note that $\bar{X} \subseteq F(X)$. Applying finite reverse induction on $i = n-1, n-2, \ldots, 0$, one concludes from this, (5.3) and $\bigcup_{i \leq n} U_i \subseteq F(X)$ that $V_i \subseteq F(X)$ for all $i \leq n$.

($2_{\mathscr{V}}$) We shall prove by finite reverse induction on $i = n, n-1, \ldots, 0$ that $V_i^{-1} = V_i$. First, note that $V_n^{-1} = U_n^{-1} \cup B^{-1} = U_n \cup B = V_n$ by (5.2), (b) and (d). Assume now that $i < n$ and the equation $V_{i+1}^{-1} = V_{i+1}$ has already been proved. It easily follows from this inductive assumption that

$$\left(\bigcup_{x \in \bar{X}} x \cdot V_{i+1} \cdot V_{i+1} \cdot x^{-1}\right)^{-1} = \bigcup_{x \in \bar{X}} x \cdot V_{i+1}^{-1} \cdot V_{i+1}^{-1} \cdot x^{-1} = \bigcup_{x \in \bar{X}} x \cdot V_{i+1} \cdot V_{i+1} \cdot x^{-1}. \tag{5.4}$$

Since $U_i^{-1} = U_i$ by (b), from (5.3) and (5.4) we conclude that $V_i^{-1} = V_i$.

($3_{\mathscr{V}}$) is straightforward from (5.3).

($4_{\mathscr{V}}$) is straightforward from (c) and (5.2). □

Remark 5.3.6. *Every finite neighbourhood system $\mathscr{U} = \{U_i : i \leq n\}$ for $F(X)$ satisfies the assumptions of Lemma 5.3.5.* Indeed, item (a) follows from ($1_{\mathscr{U}}$), item (b) follows from ($2_{\mathscr{U}}$), and item (c) follows from ($4_{\mathscr{U}}$).

Corollary 5.3.7. *For every symmetric subset B of $F(X)$ and each finite neighbourhood system $\mathscr{U} = \{U_i : i \leq n\}$ for $F(X)$, the B-enrichment of $\mathscr{U}$ in $F(X)$ is a finite neighbourhood system for $F(X)$.*

Proof. By Remark 5.3.6, $\mathscr{U}$ satisfies items (a)–(c) of Lemma 5.3.5. Item (d) of this lemma holds because B is symmetric by our assumption. Now the conclusion of our corollary follows from the conclusion of Lemma 5.3.5. □

Definition 5.3.8. Let X and Y be sets such that $X \subseteq Y$. For a finite neighbourhood system $\mathscr{U}$ for $F(X)$, we shall denote by $\mathscr{U}_Y$ the cyclic $(Y \setminus X)$-enrichment of $\mathscr{U}$ in $F(Y)$.

Corollary 5.3.9. *Let X and Y be sets such that $X \subseteq Y$. For each finite neighbourhood system $\mathscr{U}$ for $F(X)$, its cyclic $(Y \setminus X)$-enrichment $\mathscr{U}_Y$ is a finite neighbourhood system for $F(Y)$.*

Proof. Since $F(X) \subseteq F(Y)$, it follows from Remark 5.3.6 that $\mathscr{U}$ satisfies items (a)–(c) of Lemma 5.3.5 (with X replaced by Y). Since $B = \bigcup_{y \in Y \setminus X} \langle y \rangle$ is a symmetric subset of $F(Y)$, item (d) of

Lemma 5.3.5 is satisfied as well. Applying this lemma, we conclude that the B-enrichment of $\mathscr{U}$ in $F(Y)$ is a finite neighbourhood system for $F(Y)$. It remains only to note that this B-enrichment coincides with $\mathscr{U}_Y$ by Definitions 5.3.4(ii) and 5.3.8. □

5.4 Extension of finite neighbourhood systems

Definition 5.4.1. Given two sets X and Y, we shall say that a finite neighbourhood system $\mathscr{V} = \{V_i : i \leq m\}$ for $F(Y)$ is an *extension* of a finite neighbourhood system $\mathscr{U} = \{U_i : i \leq n\}$ for $F(X)$ if and only if the following conditions are satisfied:

(i$^{\mathscr{U}}_{\mathscr{V}}$) $X \subseteq Y$, so $F(X) \subseteq F(Y)$,

(ii$^{\mathscr{U}}_{\mathscr{V}}$) $n \leq m$,

(iii$^{\mathscr{U}}_{\mathscr{V}}$) $V_i \cap F(X) = U_i$ for every $i \leq n$.

A straightforward proof of the next lemma is left to the reader.

Lemma 5.4.2. *Let X be a set, and let $\mathscr{U} = \{U_i : i \leq n\}$ be a finite neighbourhood system for $F(X)$. Let $m \in \mathbb{N}$ and $m > n$. Define $V_i = U_i$ for $i \leq n$ and $V_i = \{e\}$ for $n < i \leq m$. Then $\mathscr{V} = \{V_i : i \leq m\}$ is a finite neighbourhood system for $F(X)$ extending $\mathscr{U}$.*

Lemma 5.4.3. *Let X be a set and $\mathscr{U}$ be a finite neighbourhood system for $F(X)$. Then for every set Y containing X and each set $C \subseteq \langle Y \setminus X\rangle$, the cyclic C-enrichment $\mathscr{V}$ of $\mathscr{U}$ extends it.*

Proof. Let $\mathscr{U} = \{U_i : i \leq n\}$ be a finite neighbourhood system for $F(X)$ and let $\mathscr{V} = \{V_i : i \leq n\}$ be its cyclic C-enrichment in $F(X)$. By Definition 5.3.4, we have

$$V_n = U_n \cup \bigcup_{c \in C} \langle c\rangle \tag{5.5}$$

and

$$V_i = U_i \cup \bigcup_{y \in \bar{Y}} y \cdot V_{i+1} \cdot V_{i+1} \cdot y^{-1} \tag{5.6}$$

for $i = 0, \dots, n-1$.

Let $p_X : Y \to F(X)$ be the map which sends each $y \in Y \setminus X$ to e and coincides with the identity map on X, and let $\pi_X : F(Y) \to F(X)$ be the homomorphism which extends p_X. Then

$$\pi_X \restriction_{F(X)} = \mathrm{id}_{F(X)} \quad \text{and} \quad \pi_X[\langle Y \setminus X\rangle] = \{e\}, \tag{5.7}$$

where $\mathrm{id}_{F(X)}$ denotes the identity map of $F(X)$. Since $U_i \subseteq F(X)$ by $(1_{\mathscr{U}})$, the first equation in (5.7) implies that

$$\pi_X(U_i) = U_i \text{ for every } i = 0, 1, \dots, n. \tag{5.8}$$

Claim 27. $\pi_X[V_i] \subseteq U_i$ for every $i \le n$.

Proof. We shall prove our claim by reverse induction on $i = n, n-1, \dots, 0$.

Since π_X is a homomorphism and $C \subseteq \langle Y \setminus X\rangle$, from (5.5), (5.7), (5.8) and $(4_{\mathscr{U}})$, we get

$$\pi_X[V_n] = \pi_X\left[U_n \cup \bigcup_{c\in C} \langle c\rangle\right] = \pi_X[U_n] \cup \{e\} = U_n \cup \{e\} = U_n.$$

Suppose that $i < n$ and the inclusion $\pi_X[V_{i+1}] \subseteq U_{i+1}$ has already been proved. We shall show that $\pi_X[V_i] \subseteq U_i$.

Let $y \in \bar{Y}$ and $v_1, v_2 \in V_{i+1}$ be arbitrary. By inductive hypothesis we have that $\pi_X(v_k) \in \pi_X[V_{i+1}] \subseteq U_{i+1}$ for $k = 1, 2$. Note that $\pi_X(y) \in \bar{X}$ by the definition of the homomorphism π_X and Definition 5.3.1. Therefore,

$$\pi_X(y \cdot v_1 \cdot v_2 \cdot y^{-1}) = \pi_X(y) \cdot \pi_X(v_1) \cdot \pi_X(v_2) \cdot \pi_X(y)^{-1} \in \bigcup_{x\in\bar{X}} x \cdot U_{i+1} \cdot U_{i+1} \cdot x^{-1} \subseteq U_i$$

by $(3_{\mathscr{U}})$. This shows that

$$\pi_X\left[\bigcup_{y\in\bar{Y}} y \cdot V_{i+1} \cdot V_{i+1} \cdot y^{-1}\right] \subseteq U_i. \tag{5.9}$$

Since π_X is a homomorphism, combining (5.6), (5.8), (5.9), we obtain

$$\pi_X[V_i] = \pi_X\left[U_i \cup \bigcup_{y\in\bar{Y}} y \cdot V_{i+1} \cdot V_{i+1} \cdot y^{-1}\right] = \pi_X[U_i] \cup \pi_X\left[\bigcup_{y\in\bar{Y}} y \cdot V_{i+1} \cdot V_{i+1} \cdot y^{-1}\right] \subseteq U_i \cup U_i = U_i.$$

This finishes the inductive step. □

Claim 28. $V_i \cap F(X) = U_i$ for every $i \leq n$.

Proof. The inclusion $U_i \subseteq V_i \cap F(X)$ follows from (5.5), (5.6) and ($1_{\mathscr{U}}$). To show the inverse inclusion $V_i \cap F(X) \subseteq U_i$, let $h \in V_i \cap F(X)$ be arbitrary. Then $h = \pi_X(h) \in \pi_X[V_i] \subseteq U_i$ by the first equation in (5.7) and Claim 27. □

Since $n \leq n$ and $X \subseteq Y$, conditions ($\mathrm{i}^{\mathscr{U}}_{\mathscr{V}}$) and ($\mathrm{ii}^{\mathscr{U}}_{\mathscr{V}}$) of Definition 5.4.1 are satisfied. Condition ($\mathrm{iii}^{\mathscr{U}}_{\mathscr{V}}$) holds by the previous claim. Since all conditions from Definition 5.4.1 are met, $\mathscr{V}$ extends $\mathscr{U}$. □

Corollary 5.4.4. *If X and Y are sets such that $X \subseteq Y$, then for every finite neighbourhood system $\mathscr{U}$ for $F(X)$, its cyclic $(Y \setminus X)$-enrichment $\mathscr{U}_Y$ extends $\mathscr{U}$.*

5.5 Canonical representations for elements of neighbourhoods of B-enrichments

Definition 5.5.1. Assume that $\mathscr{U} = \{U_i : i \leq n\}$ is a finite sequence of subsets of $F(X)$, $B \subseteq F(X)$ and $\mathscr{V} = \{V_i : i \leq n\}$ is the B-enrichment of $\mathscr{U}$ in $F(X)$. By finite reverse induction on $i = n, n-1, \dots, 0$, we shall define a (not necessarily unique) *canonical representation*

$$h = a_1 \cdot a_2 \cdot \dots \cdot a_m \quad \text{(for a suitable } m \in \mathbb{N}^+) \tag{5.10}$$

of every element $h \in V_i$ as follows.

Basis of induction. For $h \in V_n$, we let $h = a_1$ be a canonical representation of h.

Inductive step. Suppose that i is an integer satisfying $0 \leq i < n$ and we have already defined a canonical representation of every element $h \in V_{i+1}$. We fix $h \in V_i$ and define its canonical representation as in (5.10) according to the rules outlined below. By (5.3), at least one (perhaps both) of the following cases hold.

Case 1. $h \in U_i$. In this case, we let $h = a_1$ be the canonical representation of h.

Case 2. $h = x \cdot u \cdot v \cdot x^{-1}$ for suitable $x \in \bar{X}$ and $u, v \in V_{i+1}$. Suppose also that

$$u = b_1 \cdot b_2 \cdot \dots \cdot b_{m_1} \quad \text{and} \quad v = c_1 \cdot c_2 \cdot \dots \cdot c_{m_2}, \tag{5.11}$$

are some canonical representations of u and v, respectively. (These canonical representations were already defined, as $u, v \in V_{i+1}$.) Then we call

$$h = x \cdot b_1 \cdot b_2 \cdot \dots \cdot b_{m_1} \cdot c_1 \cdot c_2 \cdot \dots \cdot c_{m_2} \cdot x^{-1} \tag{5.12}$$

a canonical representation of h.

Lemma 5.5.2. *Let X be a set and $\mathscr{U} = \{U_i : i \le n\}$ be a finite sequence satisfying items (a), (b) and (c) of Lemma 5.3.5. Assume that $B' \subseteq B \subseteq F(X)$ and B' is symmetric. Let $\mathscr{V}' = \{V_i' : i \le n\}$ and $\mathscr{V} = \{V_i : i \le n\}$ be the B'-enrichment and the B-enrichment of $\mathscr{U}$ in $F(X)$, respectively. Suppose that*

$$(B \setminus B') \cap \left(\bar{X} \cup \bigcup_{i=1}^{n} V_i' \right) \subseteq \{e\}. \tag{5.13}$$

Let $\eta : F(X) \to F(X)$ be the map defined by

$$\eta(g) = \begin{cases} e & \text{if } g \in B \setminus B' \\ g & \text{otherwise.} \end{cases} \tag{5.14}$$

If $i \le n$ and $h = a_1 \cdot a_2 \cdot \dots \cdot a_m$ is a canonical representation of some element $h \in V_i$, then $h' = \eta(a_1) \cdot \eta(a_2) \cdot \dots \cdot \eta(a_m)$ is a canonical representation of $h' \in V_i'$.

Proof. We shall prove this lemma by finite reverse induction on $i = n, n-1, \dots, 0$.

First, we shall prove the statement of our lemma for $i = n$. By Definition 5.3.4(i), we have

$$V_n' = U_n \cup B' \text{ and } V_n = U_n \cup B. \tag{5.15}$$

Fix $h \in V_n$. By Definition 5.5.1, $h = a_1$ is a canonical representation of h.

If $a_1 \in B \setminus B'$, then $\eta(a_1) = e \in U_n \subseteq V_n'$ by (5.14), (c) and (5.15). Thus, $h' = \eta(a_1)$ is a canonical representation of the element $h' = e \in V_n'$ by Definition 5.5.1.

Suppose now that $a_1 \notin B \setminus B'$. Then $\eta(a_1) = a_1$ by (5.14), so $h = a_1 = \eta(a_1) = h'$. Since $h \in V_n$, (5.15) implies that either $h \in U_n$ or $h \in B$. In the former case, $h' = h \in U_n \subseteq V_n'$ by (5.15). In the latter case, from $a_1 = h \in B$ and $a_1 \notin B \setminus B'$, we conclude that $a_1 \in B'$. Since $B' \subseteq V_n'$ by (5.15), it follows that $h' = a_1 \in V_n'$. Thus, $h' = \eta(a_1)$ is a canonical representation of the element

$h' \in V'_n$ by Definition 5.5.1.

Suppose that $i < n$ and the statement of our lemma has already been proved for $i+1$. Fix an arbitrary $h \in V_i$. We consider two cases as in the inductive step of Definition 5.5.1.

Case 1. $h \in U_i$. In this case $h = a_1$ is a canonical representation of h by Case 1 of Definition 5.5.1. Since $U_i \subseteq V'_i$, we have $h = a_1 \in V'_i$.

If $a_1 \in B \setminus B'$, then $a_1 = e$ by (5.13) and $\eta(a_1) = e$ by (5.14). In particular, $h' = \eta(a_1) = e = a_1 = h$. Since $h \in U_i$, we conclude that $h' = \eta(a_1)$ is a canonical representation of $h' \in V'_i$ by Case 1 of the inductive step of Definition 5.5.1.

Suppose now that $a_1 \notin B \setminus B'$. Then $\eta(a_1) = a_1$ by (5.14), so $h' = \eta(a_1) = a_1 = h \in U_i \subseteq V'_i$. Therefore, $h' = \eta(a_1)$ is a canonical representation of $h' \in V'_i$ by Case 1 of the inductive step of Definition 5.5.1.

Case 2. $h = x \cdot u \cdot v \cdot x^{-1}$ for some $x \in \bar{X}$ and $u, v \in V_{i+1}$. Consider arbitrary canonical representations of u and v as in (5.11), so that (5.12) becomes a canonical representation of h.

The following claim holds by our inductive assumption.

Claim 29. $u' = \eta(b_1) \cdot \eta(b_2) \cdot \dots \cdot \eta(b_{m_1})$ and $v' = \eta(c_1) \cdot \eta(c_2) \cdot \dots \cdot \eta(c_{m_2})$ are canonical representations of elements $u' \in V'_{i+1}$ and $v' \in V'_{i+1}$, respectively.

Claim 30. $h^* = x \cdot u' \cdot v' \cdot x^{-1} \in V'_i$.

Proof. Note that $\mathscr{V}'$ is a finite neighbourhood system for $F(X)$ by the assumptions of our lemma and Lemma 5.3.5. In particular, condition $(3_{\mathscr{V}'})$ of Definition 5.3.2 holds, so $x \cdot V'_{i+1} \cdot V'_{i+1} \cdot x^{-1} \subseteq V'_i$, as $x \in \bar{X}$. Since $u', v' \in V'_{i+1}$ by Claim 29, we have $h^* = x \cdot u' \cdot v' \cdot x^{-1} \in x \cdot V'_{i+1} \cdot V'_{i+1} \cdot x^{-1}$, which implies that $h^* \in V'_i$. □

Claim 31. $h^* = x \cdot \eta(b_1) \cdot \eta(b_2) \cdot \dots \cdot \eta(b_{m_1}) \cdot \eta(c_1) \cdot \eta(c_2) \cdot \dots \cdot \eta(c_{m_2}) \cdot x^{-1}$ is a canonical representation of $h^* \in V'_i$.

Proof. This follows from Claims 29, 30 and Case 2 of the inductive step of Definition 5.5.1. □

Claim 32. $\eta(x) = x$ and $\eta(x^{-1}) = x^{-1}$.

Proof. Indeed, if $x \notin B \setminus B'$, then $\eta(x) = x$ by (5.14). If $x \in B \setminus B'$, then $x = e$ by (5.13), so $\eta(x) = \eta(e) = e = x$ by (5.14). Similarly, if $x^{-1} \notin B \setminus B'$, then $\eta(x^{-1}) = x^{-1}$ by (5.14). If $x^{-1} \in B \setminus B'$, then $x^{-1} = e$ by (5.13), so $\eta(x^{-1}) = \eta(e) = e = x^{-1}$ by (5.14). □

Since (5.12) is a canonical representation of h that is now being considered, in order to finish the inductive step, it suffices to show that

$$h' = \eta(x) \cdot \eta(b_1) \cdot \eta(b_2) \cdot \dots \cdot \eta(b_{m_1}) \cdot \eta(c_1) \cdot \eta(c_2) \cdot \dots \cdot \eta(c_{m_2}) \cdot \eta(x^{-1}) \tag{5.16}$$

is a canonical representation of $h' \in V_i'$. This follows from Claims 31, 32 and (5.16), as $h' = h^* \in V_i'$. The inductive step is now complete. □

Lemma 5.5.3. *Let X and Y be sets such that $X \neq \emptyset$ and $X \subseteq Y$. Let $\mathscr{U} = \{U_i : i \leq n\}$ be a finite neighbourhood system for $F(X)$ and let $\mathscr{V} = \{V_i : i \leq n\}$ be its cyclic $(Y \setminus X)$-enrichment in $F(Y)$. Then*

$$\sum_{l=1}^{m} |\mathrm{lett}(a_l)| \leq |X| \cdot 4^{n-i} \tag{5.17}$$

whenever $i \leq n$ and $h = a_1 \cdot a_2 \cdot \dots \cdot a_m$ is a canonical representation of $h \in V_i$ as in Definition 5.5.1.

Proof. We shall prove the statement of our lemma by finite reverse induction on $i = n, n-1, \dots, 0$.

First, we shall prove the statement of our lemma for $i = n$. By Definition 5.3.4, we have

$$V_n = U_n \cup \bigcup_{y \in Y \setminus X} \langle y \rangle. \tag{5.18}$$

Fix $h \in V_n$. By Definition 5.5.1, $h = a_1$ is the unique canonical representation of h, so in order to check (5.17), it suffices to show that $|\mathrm{lett}(h)| \leq |X|$. By (5.18), we need to consider two cases. If $h \in U_n$, then $\mathrm{lett}(h) \subseteq X$, as $U_n \subseteq F(X)$ by $(1_{\mathscr{U}})$, so $|\mathrm{lett}(h)| \leq |X|$. If $h \in \langle y \rangle$ for some $y \in Y$, then $|\mathrm{lett}(h)| \leq 1 \leq |X|$ holds, as X is non-empty.

Suppose that $i < n$ and the statement of our lemma has already been proved for $i + 1$. We shall prove the statement of our lemma for i. Fix $h \in V_i$ and consider an arbitrary canonical representation $h = a_1 \cdot a_1 \cdot \dots \cdot a_m$ of h. By Definition 5.5.1, we need to consider two cases.

If $h \in U_i$, then $m = 1$ by Case 1 of the inductive step of Definition 5.5.1. Since $U_i \subseteq F(X)$ by $(1_{\mathscr{U}})$, we have $\text{lett}(a_1) = \text{lett}(h) \subseteq X$, which implies $|\text{lett}(a_1)| \leq |X| \leq |X| \cdot 4^{n-i}$ because $i < n$.

Suppose now that there exist $x \in \bar{X}$, $u, v \in V_{i+1}$ and their canonical representations as in (5.11) such that $m = m_1 + m_2 + 2$, $a_1 = x$, $a_m = x^{-1}$, $a_{1+s} = b_s$ for $1 \leq s \leq m_1$, and $a_{m_1+1+t} = c_t$ for $1 \leq t \leq m_2$. Therefore,

$$\sum_{l=1}^{m} |\text{lett}(a_l)| = |\text{lett}(x)| + \sum_{l=1}^{m_1} |\text{lett}(b_l)| + \sum_{l=1}^{m_2} |\text{lett}(c_l)| + |\text{lett}(x^{-1})|. \tag{5.19}$$

By inductive hypothesis,

$$\sum_{l=1}^{m_1} |\text{lett}(b_l)| \leq |X| \cdot 4^{n-(i+1)} \text{ and } \sum_{l=1}^{m_2} |\text{lett}(c_l)| \leq |X| \cdot 4^{n-(i+1)}. \tag{5.20}$$

Since X is non-empty and $i + 1 \leq n$, we have

$$|\text{lett}(x)| + |\text{lett}(x^{-1})| = 2 \leq 2|X| \cdot 4^{n-(i+1)}. \tag{5.21}$$

Combining (5.19), (5.20) and (5.21), we conclude that

$$\sum_{l=1}^{m} |\text{lett}(a_l)| \leq 4|X| \cdot 4^{n-(i+1)} = |X| \cdot 4^{n-i}.$$

This finishes the inductive step. □

5.6 Three auxiliary lemmas

Lemma 5.6.1. *Let X be a set. Suppose that $n, s \in \mathbb{N}^+$, $x_1, x_2, \ldots, x_s \in X$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s \in \{-1, 1\}$,*

$$g_0 = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_s^{\varepsilon_s} \in F(X), \tag{5.22}$$

$j \in \{1, \ldots, s\}, v_1, \ldots, v_n, w_1, \ldots, w_{n+1} \in F(X)$ and

$$h^* = w_1 \cdot v_1 \cdot w_2 \cdot v_2 \cdot \cdots \cdot w_n \cdot v_n \cdot w_{n+1} \tag{5.23}$$

satisfy the following conditions:

(i) $x_j \neq x_p$ *for all* $p = 1, \ldots, s$ *with* $j \neq p$*;*

(ii) $x_j \notin \mathrm{lett}(w_i)$ *for all* $i = 1, \ldots, n+1$*;*

(iii) $x_j \notin \mathrm{lett}(h^*)$*;*

(iv) for each $i = 1, \ldots, n$*, either* $v_i = g_0$ *or* $v_i = g_0^{-1}$*.*

Then $h^* = w^*$*, where*

$$w^* = w_1 \cdot e \cdot w_2 \cdot \cdots \cdot w_n \cdot e \cdot w_{n+1} \tag{5.24}$$

is the word obtained from h^* *by replacing all* v_i *in it with* e*.*

Proof. We use the principle of minimal counter-example. Suppose that the conclusion of our lemma fails. Among all counter-examples to our lemma, we choose a counter-example h^* for which the number n is the smallest. The goal is to derive a contradiction from this assumption.

It follows from (5.22), (i) and (iv) that the variable x_j appears exactly once in each v_i for $i = 1, \ldots, n$. It follows from (ii) that the variable x_j does not appear in any of $w_1, \ldots, w_{n+1}$. Since $x_j \notin \mathrm{lett}(h^*)$ by (iii), all appearances of the terms x_j and x_j^{-1} in (5.23) cancel out as a result of computation in $F(X)$ on the right-hand side of (5.23). Therefore, there exist $i = 1, \ldots, n-1$ and $\delta \in \{-1, 1\}$ such that the unique term x_j^{δ} in v_i cancels in (5.23) with the unique term $x_j^{-\delta}$ in v_{i+1}. From (5.22), (i) and (iv) one easily concludes that the words v_i and v_{i+1} are inverses of each other and $v_i \cdot w_{i+1} \cdot v_{i+1} = e$. Therefore, $w_{i+1} = v_i^{-1} \cdot v_{i+1}^{-1} = (v_{i+1} \cdot v_i)^{-1} = e^{-1} = e$, and so

$$w = w_i \cdot w_{i+2} = w_i \cdot e \cdot w_{i+2} = w_i \cdot v_i \cdot w_{i+1} \cdot v_{i+1} \cdot w_{i+2}. \tag{5.25}$$

Claim 33. The product

$$h' = w_1 \cdot v_1 \cdot \cdots \cdot w_{i-1} \cdot v_{i-1} \cdot w \cdot v_{i+2} \cdot w_{i+3} \cdot \cdots \cdot w_n \cdot v_n \cdot w_{n+1} \tag{5.26}$$

satisfies conditions (i)–(iv) of our lemma, after an obvious re-labelling of its elements.

Proof. Indeed, conditions (i) and (iv) are not affected by the change from h^* to h'.

Since $\mathrm{lett}(w) \subseteq \mathrm{lett}(w_i) \cup \mathrm{lett}(w_{i+2})$ by the first equation in (5.25) and Lemma 5.2.12(i), from item (ii) we conclude that $x_j \notin \mathrm{lett}(w)$. This shows that the representation of h' in (5.26) satisfies item (ii) of our lemma.

It remains only to check condition (iii). From (5.23), (5.25) and (5.26), we obtain that

$$h^* = h'. \tag{5.27}$$

Since $x_j \notin \mathrm{lett}(h^*)$ by (iii), from (5.27) we conclude that $x_j \notin \mathrm{lett}(h')$. □

Since h' is "shorter" than the minimal counter-example h^* to Lemma 5.6.1, from Claim 33 we conclude that

$$h' = w', \tag{5.28}$$

where

$$w' = w_1 \cdot e \cdot \dots \cdot w_{i-1} \cdot e \cdot w \cdot e \cdot w_{i+3} \cdot \dots \cdot w_n \cdot e \cdot w_{n+1} \tag{5.29}$$

is the word obtained from h' by replacing all v_i in it with e.

From (5.24), (5.25) and (5.29), we get $w^* = w'$. Combining this with (5.27) and (5.28), we deduce that $h^* = w^*$. However, this contradicts the assumption that h^* is a counter-example to our lemma. □

Lemma 5.6.2. *Let X be a set. Suppose that $m, s \in \mathbb{N}^+$, $x_1, x_2, \dots, x_s \in X$, $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s \in \{-1, 1\}$,*

$$g_0 = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_s^{\varepsilon_s} \in F(X), \tag{5.30}$$

$j \in \{1, \dots, s\}, a_1, a_2, \dots, a_m \in F(X)$ and

$$L = \{l = 1, \dots, m : x_j \in \mathrm{lett}(a_l)\} \tag{5.31}$$

satisfy the following conditions:

(a) $x_j \neq x_p$ for all $p = 1, \dots, s$ with $j \neq p$;

(b) $x_j \notin \mathrm{lett}(a_1 \cdot a_2 \cdot \dots \cdot a_m)$;

(c) $a_l \in \langle g_0 \rangle \setminus \{e\}$ *for each* $l \in L$.

Let $\eta : F(X) \to F(X)$ *be the map which sends each element of the cyclic group* $\langle g_0 \rangle$ *to* e *and does not move elements of its complement* $F(X) \setminus \langle g_0 \rangle$. *Then*

$$a_1 \cdot a_2 \cdot \dots \cdot a_m = \eta(a_1) \cdot \eta(a_2) \cdot \dots \cdot \eta(a_m). \tag{5.32}$$

Proof. Suppose first that $L = \emptyset$. Then $x_j \notin \mathrm{lett}(a_1) \cup \mathrm{lett}(a_2) \cup \dots \cup \mathrm{lett}(a_m)$ by (5.31). On the other hand, $x_j \in \mathrm{lett}(h)$ for every $h \in \langle g_0 \rangle \setminus \{e\}$ by Lemma 5.2.13. Therefore, $a_l \notin \langle g_0 \rangle \setminus \{e\}$ for every $l = 1, \dots, m$. By our definition of η, this means that $\eta(a_l) = a_l$ for all $l = 1, \dots, m$. Thus, (5.32) holds in this case. In the rest of the proof we assume that $L \neq \emptyset$.

Claim 34. $L = \{l = 1, \dots, m : a_l \in \langle g_0 \rangle \setminus \{e\}\}$.

Proof. The inclusion $L \subseteq \{l = 1, \dots, m : a_l \in \langle g_0 \rangle \setminus \{e\}\}$ follows from (c). To prove the inverse inclusion, suppose that $l \in \{1, \dots, m\}$ and $a_l \in \langle g_0 \rangle \setminus \{e\}$. Then $x_j \in \mathrm{lett}(a_l)$ by Lemma 5.2.13. Therefore $l \in L$ by (5.31). □

For every $l \in L$, use item (c) to fix a non-zero integer q_l such that $a_l = g_0^{q_l}$ and define $\delta_l = q_l/|q_l| \in \{-1, 1\}$. It is clear that

$$a_l = g_0^{\delta_l} \cdot e \cdot g_0^{\delta_l} \cdot e \cdot \dots \cdot e \cdot g_0^{\delta_l}, \tag{5.33}$$

where the term $g_0^{\delta_l}$ appears $|q_l|$-many times in (5.33). Note that $\eta(e) = e$. Since $g_0^{q_l}, g_0^{\delta_l} \in \langle g_0 \rangle$, we have $\eta(g_0^{q_l}) = \eta(g_0^{\delta_l}) = e$, and so

$$\eta(a_l) = \eta(g_0^{q_l}) = e = \eta(g_0^{\delta_l}) \cdot \eta(e) \cdot \eta(g_0^{\delta_l}) \cdot \eta(e) \cdot \dots \cdot \eta(e) \cdot \eta(g_0^{\delta_l}). \tag{5.34}$$

Let $b_1 \cdot b_2 \cdot \dots \cdot b_t$ be the formal expression obtained from the product $a_1 \cdot a_2 \cdots a_m$ by replacing in it every element a_l for $l \in L$ with the formal expression on the right-hand side of (5.33). Clearly,

$$a_1 \cdot a_2 \cdot \dots \cdot a_m = b_1 \cdot b_2 \cdot \dots \cdot b_t. \tag{5.35}$$

Since (5.34) holds for every $l \in L$, we obtain

$$\eta(a_1) \cdot \eta(a_2) \cdot \dots \cdot \eta(a_m) = \eta(b_1) \cdot \eta(b_2) \cdot \dots \cdot \eta(b_t). \tag{5.36}$$

It follows from our definition of b_r's that

$$b_r \in \{e, g_0, g_0^{-1}\} \cup \{a_l : l \in \{1, \dots, m\} \setminus L\} \text{ for every } r = 1, \dots, t. \tag{5.37}$$

Since $L \neq \emptyset$, we can select $l \in L$. Then a_l has the form as in (5.33), and so $g_0^{\delta_l}$ appears as one of b_r's. This shows that the set

$$M = \{r = 1, \dots, t : \text{either } b_r = g_0 \text{ or } b_r = g_0^{-1}\} \tag{5.38}$$

is non-empty, so we can fix an enumeration $M = \{r_i : i \leq n\}$ of M such that $r_1 < r_2 < \dots < r_n$. For $i = 1, \dots, n$, define $v_i = b_{r_i}$.

If $r_1 = 1$, then we let $w_1 = e$; otherwise, we let $w_1 = b_1 \cdot \dots \cdot b_{r_1 - 1}$. Similarly, if $r_n = t$, then we let $w_{n+1} = e$; otherwise, we let $w_{n+1} = b_{r_n+1} \cdot \dots \cdot b_t$. For $i = 2, \dots, n$, we let $w_i = b_{r_{i-1}+1} \cdot \dots \cdot b_{r_i - 1}$.

Claim 35. $x_j \notin \text{lett}(w_i)$ for all $i = 1, \dots, n+1$.

Proof. Let $i = 1, \dots, n+1$ be arbitrary. Combining our definition of w_i with (5.37) and (5.38), we conclude that w_i is a finite product of elements of the set $\{e\} \cup \{a_l : l \in \{1, \dots, m\} \setminus L\}$. Since $\text{lett}(e) = \emptyset$ and $x_j \notin \text{lett}(a_l)$ for $l \in \{1, \dots, m\} \setminus L$ by (5.31), applying Lemma 5.2.12(ii) we obtain that $x_j \notin \text{lett}(w_i)$. □

It follows from our definition of v_i and w_i that

$$b_1 \cdot b_2 \cdot \dots \cdot b_t = w_1 \cdot v_1 \cdot w_2 \cdot v_2 \cdot \dots \cdot w_n \cdot v_n \cdot w_{n+1}. \tag{5.39}$$

Let h^* be the element defined in (5.23). Note that

$$a_1 \cdot a_2 \cdot \dots \cdot a_m = h^* \tag{5.40}$$

by (5.23), (5.35) and (5.39).

Claim 36. h^* satisfies conditions (i)–(iv) of Lemma 5.6.1.

Proof. Condition (i) of Lemma 5.6.1 coincides with condition (a) of our lemma.

Item (ii) holds by Claim 35.

(iii) Note that $x_j \notin \mathrm{lett}(h^*)$ by (5.40) and (b).

(iv) If $i = 1, \ldots, n$, then $r_i \in M$ and $v_i = b_{r_i}$ by our construction, so (5.38) implies that either $v_i = g_0$ or $v_i = g_0^{-1}$. □

By the previous claim, we can apply Lemma 5.6.1 to conclude that

$$h^* = w_1 \cdot e \cdot w_2 \cdot \cdots \cdot w_n \cdot e \cdot w_{n+1}. \tag{5.41}$$

Claim 37. $\eta(b_r) = e$ for $r \in M$ and $\eta(b_r) = b_r$ for $r \in \{1, \ldots, t\} \setminus M$.

Proof. If $r \in M$ then $b_r \in \{g_0, g_0^{-1}\} \subseteq \langle g_0 \rangle$ by (5.38), so $\eta(b_r) = e$ by our definition of η.

Suppose now that $r \in \{1, \ldots, t\} \setminus M$. Then $b_r \notin \{g_0, g_0^{-1}\}$ by (5.38). From this and (5.37), we get $b_r \in \{e\} \cup \{a_l : l \in \{1, \ldots, m\} \setminus L\}$. If $b_r = e$, then $\eta(b_r) = \eta(e) = e = b_r$. Assume now that $b_r = a_l$ for some $l \in \{1, \ldots, m\} \setminus L$. Then $b_r = a_l \notin \langle g_0 \rangle \setminus \{e\}$ by Claim 34, which yields $\eta(b_r) = b_r$ by our definition of η. □

Claim 37 and our definition of w_r's implies that

$$w_1 \cdot e \cdot w_2 \cdot \ldots \cdot w_n \cdot e \cdot w_{n+1} = \eta(b_1) \cdot \eta(b_2) \cdot \ldots \cdot \eta(b_t). \tag{5.42}$$

Now (5.32) follows from (5.40), (5.41), (5.42) and (5.36) (in this order). □

Lemma 5.6.3. *Assume that X is a non-empty finite set, $g \in F(X)$, $\mathscr{U} = \{U_i : i \leq n\}$ is a finite neighbourhood system for $F(X)$ and Y is a finite set containing X satisfying the inequality $|Y \setminus X| > |X| \cdot 4^n$. Then there exists a finite neighbourhood system $\mathscr{V} = \{V_i : i \leq n\}$ for $F(Y)$ extending $\mathscr{U}$ such that $g \in \langle \mathrm{Cyc}(V_n) \rangle$.*

Proof. Let $|Y \setminus X| = k$. By the assumption of our lemma, we have

$$k > |X| \cdot 4^n. \tag{5.43}$$

Fix faithful enumeration $Y \setminus X = \{y_1, \ldots, y_k\}$ of the set $Y \setminus X$ and define

$$g_0 = (y_1 y_2 \cdots y_k) \cdot g. \tag{5.44}$$

Note that the product in (5.44) does not undergo any cancellations, as $\mathrm{lett}(g) \subseteq X$ and the set $\{y_1, \ldots, y_k\} = Y \setminus X$ is disjoint from X. For the same reason, for every non-zero integer p, the power g_0^p of g_0 does not undergo any cancellations as well. Therefore,

$$\{y_1, y_2, \ldots, y_k\} \subseteq \mathrm{lett}(g_0^p) \text{ for every non-zero integer } p. \tag{5.45}$$

Since $X \subseteq Y$, we have $F(X) \subseteq F(Y)$. Therefore, the finite sequence $\mathscr{U}$ satisfies conditions (a)–(c) of Lemma 5.3.5, where in condition (a) one has to replace X by Y. Define

$$B' = \bigcup_{y \in Y \setminus X} \langle y \rangle \text{ and } B = B' \cup \langle g_0 \rangle. \tag{5.46}$$

Clearly, $B' \subseteq B \subseteq F(Y)$ and B' is symmetric.

Claim 38. The B'-enrichment $\mathscr{V}' = \{V_i' : i \leq n\}$ of $\mathscr{U}$ in $F(Y)$ is a finite neighbourhood system for $F(Y)$ extending $\mathscr{U}$.

Proof. From the first equation in (5.46), Definition 5.3.4(ii) and definition of $\mathscr{V}'$, it follows that $\mathscr{V}'$ coincides with the cyclic $(Y \setminus X)$-enrichment $\mathscr{U}_Y$ of $\mathscr{U}$. Now the conclusion of our claim follows from Corollaries 5.3.9 and 5.4.4. □

Claim 39. The B-enrichment $\mathscr{V} = \{V_i : i \leq n\}$ of $\mathscr{U}$ in $F(Y)$ is a finite neighbourhood system for $F(Y)$.

Proof. It follows from (5.46) that $B^{-1} = B$. Now the conclusion of our claim follows from Lemma 5.3.5 (in which one has to replace X by Y). □

Claim 40. $g \in \langle \mathrm{Cyc}(V_n) \rangle$.

Proof. Since $Y \setminus X = \{y_1, \dots, y_k\}$, from (5.46) we conclude that $\langle y_i^{-1} \rangle = \langle y_i \rangle \subseteq B$ for all $i = 1, \dots, k$. Similarly, $\langle g_0 \rangle \subseteq B$ by (5.46). Since $\mathscr{V}$ is the B-enrichment of $\mathscr{U}$ in $F(Y)$, we conclude from equation (5.2) of Definition 5.3.4 that $B \subseteq V_n$. This implies that $y_1^{-1}, \dots, y_k^{-1}, g_0 \in \mathrm{Cyc}(V_n)$. From this and (5.44), we get $g = y_k^{-1} \cdot y_{k-1}^{-1} \cdot \dots \cdot y_1^{-1} \cdot g_0 \in \langle \mathrm{Cyc}(V_n) \rangle$. □

Claim 41. The map $\eta : F(X) \to F(X)$ defined in Lemma 5.6.2 coincides with the map η defined in Lemma 5.5.2.

Proof. From (5.46) we get $B \setminus B' = \langle g_0 \rangle \setminus \{e\}$. The conclusion of our claim follows from this observation and our definitions of both maps. □

In the rest of the proof we shall denote both of the maps from the above claim by η.

Claim 42. $(B \setminus B') \cap \left(\bar{Y} \cup \bigcup_{i=1}^n V_i'\right) = \emptyset$.

Proof. Suppose that $h \in (B \setminus B') \cap \left(\bar{Y} \cup \bigcup_{i=1}^n V_i'\right)$. From $h \in B \setminus B'$ and (5.46), we conclude that $h \in \langle g_0 \rangle \setminus \{e\}$, so $h = g_0^p$ for some non-zero integer p. Then $|\mathrm{lett}(h)| \geq |Y \setminus X| = k$ by (5.45). Note that $h \notin \bar{Y}$, as $|\mathrm{lett}(y)| \leq 1 < k$ for every $y \in \bar{Y}$. Since $h \in \bar{Y} \cup \bigcup_{i=1}^n V_i'$, it follows that $h \in V_i'$ for some $i \leq n$. Since $\mathscr{V}'$ is the B'-enrichment of $\mathscr{U}$ in $F(Y)$, the element $h \in V_i'$ has a canonical representation

$$h = a_1 \cdot a_2 \cdot \dots \cdot a_m. \tag{5.47}$$

Then $\mathrm{lett}(h) \subseteq \bigcup_{l=1}^m \mathrm{lett}(a_l)$ by (5.47) and Lemma 5.2.12(ii). Furthermore,

$$|\mathrm{lett}(h)| \leq \left| \bigcup_{l=1}^m \mathrm{lett}(a_l) \right| \leq \sum_{l=1}^m |\mathrm{lett}(a_l)| \leq |X| \cdot 4^{n-i} \leq |X| \cdot 4^n < k$$

by Lemma 5.5.3 and (5.43), in contradiction with $|\mathrm{lett}(h)| \geq k$. □

Claim 43. Suppose $h \in V_i \cap F(X)$ has a canonical representation (5.47). Then

$$h' = \eta(a_1) \cdot \eta(a_2) \cdot \dots \cdot \eta(a_m) \tag{5.48}$$

is a canonical representation of $h' \in V_i'$ satisfying the inequality $|\bigcup_{l=1}^m \mathrm{lett}(\eta(a_l))| < k$.

Proof. Let $h \in V_i \cap F(X)$ be arbitrary. From Claim 42, we conclude that (5.13) holds (with X replaced by Y). Therefore, all the assumptions of Lemma 5.5.2 are satisfied (with X replaced by Y in this lemma). This implies that $h' = \eta(a_1) \cdot \eta(a_2) \cdot \cdots \cdot \eta(a_m)$ is a canonical representation of $h' \in V_i'$. Finally,

$$\left| \bigcup_{l=1}^{m} \operatorname{lett}(\eta(a_l)) \right| \leq |X| \cdot 4^{n-i} \leq |X| \cdot 4^n < k$$

by Lemma 5.5.3 and (5.43). □

Claim 44. If $h \in V_i \cap F(X)$ has a canonical representation (5.47) and $a_l \neq \eta(a_l)$ for some $l = 1, \ldots, m$, then $a_l \in \langle g_0 \rangle \setminus \{e\}$.

Proof. Suppose that $a_l \neq \eta(a_l)$ for some $l = 1, \ldots, m$. By (5.14), this implies that $a_l \neq e$ and furthermore $a_l \in B \setminus B'$. Finally, $B \setminus B' \subseteq \langle g_0 \rangle$ by (5.46). This establishes that $a_l \in \langle g_0 \rangle \setminus \{e\}$. □

Claim 45. If $h \in V_i \cap F(X)$ has a canonical representation (5.47), then elements $g_0, a_1, \ldots, a_m \in F(Y)$ satisfy all assumptions of Lemma 5.6.2, with Y substituted for X in its statement.

Proof. Since $g_0 \in F(Y)$, there exist $s \in \mathbb{N}^+, x_1, \ldots, x_s \in Y$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ such that (5.30) holds with Y substituted for X in it; in particular, the word in (5.30) is irreducible. Since $|\bigcup_{l=1}^{m} \operatorname{lett}(\eta(a_l))| < k$ by Claim 43, we can use (5.44) to choose some variable $y_j \in \{y_1, \ldots, y_k\}$ such that $y_j \notin \operatorname{lett}(\eta(a_l))$ for all $l = 1, \ldots m$. Let

$$L = \{l = 1, \ldots, m : y_j \in \operatorname{lett}(a_l)\} \tag{5.49}$$

and

$$M = \{l = 1, \ldots, m : a_l \neq \eta(a_l)\}. \tag{5.50}$$

Let $l \in M$. Then $a_l \neq \eta(a_l)$, so $a_l \in \langle g_0 \rangle \setminus \{e\}$ by Claim 44. Therefore, $y_j \in \operatorname{lett}(a_l)$ by (5.45), which implies $l \in L$ by (5.49). This proves that $M \subseteq L$. To show the reverse inclusion, let us suppose that $l \in L$. Then $y_j \in \operatorname{lett}(a_l)$ by (5.49). Since $y_j \notin \operatorname{lett}(\eta(a_l))$ by our choice of a_l, this implies that $a_l \neq \eta(a_l)$; that is, $l \in M$. This shows that $L \subseteq M$. From these two inclusions we get $L = M$.

Recall that $g \in F(X)$ and $Y \setminus X = \{y_1, \dots, y_k\}$ is a faithful enumeration of $Y \setminus X$. Combining this with (5.30) and (5.44), we conclude that $k \leq s$ and

$$x_p = y_p, \varepsilon_p = 1 \text{ for all } p = 1, \dots, k. \tag{5.51}$$

In particular, $x_j = y_j$.

We are now ready to check the assumptions (a)–(c) of Lemma 5.6.2 with X replaced by Y in it.

(a) Let $p \in \{1, \dots, s\}$ and $j \neq p$. If $p \leq k$, then $x_p = y_p \neq y_j = x_j$ by (5.51) and the faithfulness of the enumeration $Y \setminus X = \{y_1, \dots, y_k\}$. Suppose now that $p > k$. Then $x_p \in \text{lett}(g) \subseteq X$, while $x_j = y_j \in Y \setminus X$, which yields $x_j \neq x_p$.

(b) Recall that $h \in V_i \cap F(X)$ by hypothesis, so $\text{lett}(h) \subseteq X$. Since $y_j \in Y \setminus X$, we have $y_j \notin \text{lett}(h)$. It remains only to note that $h = a_1 \cdot a_2 \cdot \dots \cdot a_m$.

(c) Given a_l with $l \in L$, we have $a_l \neq \eta(a_l)$, as $L = M$. Now $a_l \in \langle g_0 \rangle \setminus \{e\}$ by Claim 44. □

Claim 46. $V_i \cap F(X) \subseteq U_i$ for every $i \leq n$.

Proof. Let $h \in V_i \cap F(X)$ be an arbitrary element, and let (5.47) be one of its canonical representations. Claim 45 allows us to make use of Lemma 5.6.2 to show that $h = \eta(a_1) \cdot \eta(a_2) \cdot \dots \cdot \eta(a_m)$. From Claim 43 and (5.48), we can conclude that $h = h'$, where $h' = \eta(a_1) \cdot \eta(a_2) \cdot \dots \cdot \eta(a_m)$ is the canonical representation of $h' \in V_i'$. In particular, the equality $h = h'$ shows that $h = h' \in V_i' \cap F(X)$. Since $\mathscr{V}'$ extends $\mathscr{U}$ by Claim 38, $V_i' \cap F(X) = U_i$ by $(\text{iii}^{\mathscr{U}}_{\mathscr{V}'})$. This shows that $h \in U_i$. □

Claim 47. The finite neighbourhood system $\mathscr{V}$ for $F(Y)$ extends $\mathscr{U}$.

Proof. Since $n \leq n$ and $X \subseteq Y$, conditions$(\text{i}^{\mathscr{U}}_{\mathscr{V}})$ and $(\text{ii}^{\mathscr{U}}_{\mathscr{V}})$ of Definition 5.4.1 are both satisfied.

Let $i \leq n$ be arbitrary. Since $\mathscr{V}$ is the B-enrichment of $\mathscr{U}$ in $F(Y)$, we have $U_i \subseteq V_i$ by equation (5.3) of Definition 5.3.4. Since $\mathscr{U}$ is a finite neighbourhood system for $F(X)$, we have $U_i \subseteq F(X)$ by item $(1_{\mathscr{U}})$ of Definition 5.3.2. This shows that $U_i \subseteq V_i \cap F(X)$. The inverse inclusion holds by Claim 46. Thus, $V_i \cap F(X) = U_i$. Since this equation holds for an arbitrary $i \leq n$, condition $(\text{iii}^{\mathscr{U}}_{\mathscr{V}})$ also holds. □

The conclusion of our lemma follows from Claims 39, 40 and 47. □

5.7 The partially ordered set and density lemmas

As usual, the symbol $[X]^{<\omega}$ denotes the set of all finite subsets of a set X.

Definition 5.7.1. Let X be an infinite set.

(a) We denote by $\mathbb{P}$ the set of all triples $p = \langle\langle X^p, n^p, \mathscr{U}^p \rangle\rangle$ satisfying the following conditions:

(1_p) $X^p \in [X]^{<\omega}$,

(2_p) $n^p \in \mathbb{N}$,

(3_p) $\mathscr{U}^p = \{U_i^p : i \leq n^p\}$ is a finite neighbourhood system for $F(X^p)$.

(b) Given triples $p = \langle\langle X^p, n^p, \mathscr{U}^p \rangle\rangle \in \mathbb{P}$ and $q = \langle\langle X^q, n^q, \mathscr{U}^q, \rangle\rangle \in \mathbb{P}$, we define $q \leq p$ if and only if $\mathscr{U}^q$ is an extension of $\mathscr{U}^p$ in the sense of Definition 5.4.1.

The following lemma easily follows from item (b) of Definition 5.7.1.

Lemma 5.7.2. *The pair* $(\mathbb{P}, \leq)$ *is a partially ordered set.*

Lemma 5.7.3. *There exists* $p \in \mathbb{P}$ *such that* $X^p \neq \emptyset$*. In particular,* $\mathbb{P} \neq \emptyset$*.*

Proof. Fix $x \in X$ and let $X^p = \{x\}$. Furthermore, let $n^p = 0$, $U_0^p = \{e\}$ and $\mathscr{U}^p = \{U_i^p : i \leq 0\}$. Then $p = \langle\langle X^p, n^p, \mathscr{U}^p \rangle\rangle$ clearly satisfies conditions (1_p)–(3_p), so $p \in \mathbb{P}$ by Definition 5.7.1(a). □

In what follows, we shall need the concept of partial order *density and downward-closedness* (see Definition 4.6.1 and Lemma 4.6.2)

Lemma 5.7.4. *(i) For every* $n \in \mathbb{N}$*, the set* $A_n = \{q \in \mathbb{P} : n \leq n^q\}$ *is dense and downward-closed in* $(\mathbb{P}, \leq)$*.*

(ii) For every $S \in [X]^{<\omega}$*, the set* $B_S = \{q \in \mathbb{P} : S \subseteq X^q\}$ *is dense in* $(\mathbb{P}, \leq)$*.*

(iii) For every $g \in F(X) \setminus \{e\}$*, the set* $C_g = \{q \in \mathbb{P} : g \in F(X^q) \setminus U_{n^q}^q\}$ *is dense in* $(\mathbb{P}, \leq)$*.*

(iv) For every $g \in F(X)$*, the set* $D_g = \{q \in \mathbb{P} : g \in \langle \mathrm{Cyc}(U_{n^q}^q) \rangle\}$ *is dense in* $(\mathbb{P}, \leq)$*.*

Proof. (i) Fix $n \in \mathbb{N}$. To prove that A_n is dense in $(\mathbb{P}, \leq)$, we consider an arbitrary $p \in \mathbb{P}$. If $n \leq n^p$, the inclusion $p \in A_n$ holds trivially. For this reason, we can assume that $n^p < n$. By

Lemma 5.4.2, there exists a finite neighbourhood system $\mathscr{V} = \{V_i : i \leq n\}$ for $F(X^p)$ extending $\mathscr{U}^p$. If we define $X^q = X^p$, $n^q = n$ and $\mathscr{U}^q = \mathscr{V}$, then $q = \langle\langle X^q, n^q, \mathscr{U}^q \rangle\rangle \in \mathbb{P}$. Since $\mathscr{U}^q$ extends $\mathscr{U}^p$, we have $q \leq p$ by Definition 5.7.1(b). Finally, $n = n^q$ implies $q \in A_n$ by the definition of A_n, thereby showing that A_n is dense in $(\mathbb{P}, \leq)$.

To check that A_n is downward-closed, consider $p \in A_n$ and $q \in \mathbb{P}$ which satisfy $q \leq p$. Since $\mathscr{U}^q$ extends $\mathscr{U}^p$ by Definition 5.7.1(b), the inequality $n^p \leq n^q$ holds by item $(\mathrm{ii}^{\mathscr{U}^p}_{\mathscr{U}^q})$ of Definition 5.4.1. Since $p \in A_n$, we also have $n \leq n^p$. By transitivity, this implies that $n \leq n^q$, and therefore $q \in A_n$ by definition of A_n. This shows that A_n is downward-closed.

(ii) Let $S \in [X]^{<\omega}$ and $p \in \mathbb{P}$ be arbitrary. Then $X^p \in [X]^{<\omega}$ by (1_p), so $X^q = X^p \cup S \in [X]^{<\omega}$ holds as well. Let $n^q = n^p$. Since $\mathscr{U}^p$ is a finite neighbourhood system for $F(X^p)$ by (3_p), the cyclic $(S \setminus X^p)$-enrichment $\mathscr{U}^q$ of $\mathscr{U}^p$ is a finite neighbourhood system for $F(X^q)$ by Corollary 5.3.9. We have shown that $q = \langle\langle X^q, n^q, \mathscr{U}^q \rangle\rangle \in \mathbb{P}$. Since $\mathscr{U}^q$ is an extension for $\mathscr{U}^p$ by Corollary 5.4.4, we have $q \leq p$ by Definition 5.7.1(b). Finally $S \subseteq X^q$ by definition of X^q, so $q \in B_S$ by definition of B_S. This shows that B_S is dense in $(\mathbb{P}, \leq)$.

(iii) Let $g \in F(X) \setminus \{e\}$ and $p \in \mathbb{P}$ be arbitrary. Since g is generated by the symbols in X, there exists some finite set $X_g \subseteq X$ such that $g \in F(X_g)$. Since the set B_{X_g} is dense in $\mathbb{P}$ by (ii), we may assume without loss of generality that $p \in B_{X_g}$. Since $p \in B_{X_g}$, this implies that $X_g \subseteq X^p$, so $g \in F(X^p)$, as $F(X_g) \subseteq F(X^p)$. By by Lemma 5.4.2, there exists a finite neighbourhood system $\mathscr{U}^q = \{U^q_i : i \leq n^p + 1\}$ on X^p which extends $\mathscr{U}^p$ such that $g \notin U^q_{n^p+1} = \{e\}$. If we define $n^q = n^p + 1$ and $X^q = X^p$, then $q = \langle\langle X^q, n^q, \mathscr{U}^q \rangle\rangle \in \mathbb{P}$. Since $\mathscr{U}^q$ extends $\mathscr{U}^p$, we have $q \leq p$ by Definition 5.7.1(b).

(iv) Let $g \in F(X)$ and $p \in \mathbb{P}$. Arguing as in the proof (iii), we may find a finite subset X_g of X such that $g \in F(X_g)$. By (ii), without loss of generality, we may assume that $p \in B_{X_g}$. Since the set $X^p \subseteq X$ is finite, we can fix a set $X^q \subseteq X$ containing X^p such that $|X^q \setminus X^p| = k > |X^p| \cdot 4^{n^p}$. By Lemma 5.6.3, there exists a finite neighbourhood system $\mathscr{U}^q = \{U^q_i : i \leq n^p\}$ on X^q extending $\mathscr{U}^p$ such that $g \in \langle \mathrm{Cyc}(U^q_n) \rangle$. If we define $n^q = n^p$, then we have that $q = \langle\langle X^q, n^q, \mathscr{U}^q \rangle\rangle \in \mathbb{P}$. Since $\mathscr{U}^q$ extends $\mathscr{U}^p$, we have $q \leq p$ by Definition 5.7.1(b). Since the condition $g \in \langle \mathrm{Cyc}(U^q_n) \rangle$ is satisfied, we have $q \in D_g$. □

5.8 Proof of Theorem 5.1.1

For this section we shall make use of Lemma 4.7.3.

Let X be a countably infinite set. Since X algebraically generates the free group $F(X)$, the latter is at most countable.

Let $(\mathbb{P}, \leq)$ be the poset from Definition 5.7.1 which uses the set X as its parameter. Clearly, the family

$$\mathscr{D} = \{C_g : g \in F(X) \setminus \{e\}\} \cup \{A_n \cap D_g : n \in \mathbb{N}, g \in F(X)\} \cup \{B_S : S \in [X]^{<\omega}\} \tag{5.52}$$

of subsets of $\mathbb{P}$ is at most countable.

Let us check that all members of $\mathscr{D}$ are dense in $(\mathbb{P}, \leq)$. By Lemma 5.7.4(iii), each C_g for $g \in F(X) \setminus \{e\}$ is dense in $(\mathbb{P}, \leq)$. Let $n \in \mathbb{N}$ and $g \in F(X)$ be arbitrary. Since A_n is dense and downward-closed in $(\mathbb{P}, \leq)$ by Lemma 5.7.4(i), and D_g is dense in $(\mathbb{P}, \leq)$ by Lemma 5.7.4(iv), by Lemma 4.6.2 we can conclude that $A_n \cap D_g$ is dense in $(\mathbb{P}, \leq)$. Finally, the density in $(\mathbb{P}, \leq)$ of each B_S for $S \in [X]^{<\omega}$ follows from Lemma 5.7.4(ii).

Since we have shown that all members of $\mathscr{D}$ are dense in $\mathbb{P}$, we can apply Lemma 4.7.3 to find a countable set $\mathbb{F} \subseteq \mathbb{P}$ such that $(\mathbb{F}, \leq)$ is a linearly ordered set and $\mathbb{F} \cap D \neq \emptyset$ for every $D \in \mathscr{D}$. For every $n \in \mathbb{N}$, define

$$U_n = \bigcup \{U_n^p : p \in \mathbb{F} \text{ and } n \leq n^p\}. \tag{5.53}$$

Our nearest goal is to show that the family

$$\mathcal{B} = \{U_n : n \in \mathbb{N}\} \tag{5.54}$$

is a neighbourhood base at e of a Hausdorff group topology $\mathscr{T}$ on the free group $F(X)$. The verification of this will be split into a sequence of claims.

Claim 48. The equality $\bigcap_{n \in \mathbb{N}} U_n = \{e\}$ holds.

Proof. Let us show that $e \in \bigcap_{n \in \mathbb{N}} U_n$. Take an arbitrary $n \in \mathbb{N}$. Since $A_n \in \mathscr{D}$ by (5.52), the choice of $\mathbb{F} \subseteq \mathbb{P}$ allows us to find some $p \in A_n \cap \mathbb{F}$. Then $n \leq n^p$ by the definition of A_n. Since $p \in \mathbb{P}$, the family $\mathscr{U}^p = \{U_i^p : i \leq n^p\}$ is a finite neighbourhood system for $F(X^p)$ by condition (3_p) of

Definition 5.7.1(a). Therefore, we can apply Remark 5.3.3 to conclude that $e \in U_n^p$. Since $p \in \mathbb{F}$ and $n \leq n^p$, we have $U_n^p \subseteq U_n$ by (5.53). Thus, $e \in U_n$. Since $n \in \mathbb{N}$ was arbitrary, we conclude that $e \in \bigcap_{n\in\mathbb{N}} U_n$.

Suppose that there exists some $g \in \bigcap_{n\in\mathbb{N}} U_n$ with $g \neq e$. Then $C_g \in \mathscr{D}$ by (5.52). The choice of $\mathbb{F}$ allows us to find some $p \in C_g \cap \mathbb{F}$. By the definition of C_g, this automatically implies that

$$g \in F(X^p) \setminus U^p_{n^p}. \tag{5.55}$$

Since $g \in \bigcap_{n\in\mathbb{N}} U_n$, the inclusion $g \in U_{n^p}$ holds. By (5.53), this implies the existence of $q \in \mathbb{F}$ such that $n \leq n^q$ and $g \in U^q_{n^p}$. Since $\mathbb{F}$ is linearly ordered, either $q \leq p$ or $p \leq q$. We shall show that both of these two conditions lead to a contradiction.

If $q \leq p$ holds, then $\mathscr{U}^q$ is an extension of $\mathscr{U}^p$ by Definition 5.7.1(b), so conditions ($\mathrm{i}^{\mathscr{U}^p}_{\mathscr{U}^q}$) and ($\mathrm{iii}^{\mathscr{U}^p}_{\mathscr{U}^q}$) of Definition 5.4.1 imply that $F(X^p) \subseteq F(X^q)$ and $U^q_{n^p} \cap F(X^p) = U^p_{n^p}$. Since g belongs to the set on the left-hand side of the last equation, we get $g \in U^p_{n^p}$. This is a direct contradiction to (5.55), so this case cannot hold.

Suppose that $p \leq q$ holds. Then this time $\mathscr{U}^p$ is an extension of $\mathscr{U}^q$ by Definition 5.7.1(b), so $n^q \leq n^p$ by condition ($\mathrm{ii}^{\mathscr{U}^q}_{\mathscr{U}^p}$) of Definition 5.4.1. Since $n^p \leq n^q$ also holds, we have $n^p = n^q$. Conditions ($\mathrm{i}^{\mathscr{U}^q}_{\mathscr{U}^p}$) and ($\mathrm{iii}^{\mathscr{U}^q}_{\mathscr{U}^p}$) of Definition 5.4.1 imply that $F(X^q) \subseteq F(X^p)$ and $U^p_{n^p} \cap F(X^q) = U^q_{n^p}$. Since $g \in U^q_{n^p}$, the last equation implies that $g \in U^p_{n^p}$. Once more, this contradicts (5.55), so this case cannot hold either.

The obtained contradiction finishes the proof of our claim. □

Claim 49. $U_n^{-1} = U_n$ and $U_{n+1} \cdot U_{n+1} \subseteq U_n$ for every $n \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$.

Consider an arbitrary $p \in \mathbb{F}$ satisfying $n \leq n^p$. Then $U_n^p = (U_n^p)^{-1}$ by condition ($2_{\mathscr{U}}$) of Definition 5.3.2. If we apply this to (5.53), we obtain that $U_n = U_n^{-1}$.

For the second inclusion, consider some arbitrary $g_1, g_2 \in U_{n+1}$. By (5.53), there exist $p_1, p_2 \in \mathbb{F}$ such that $n + 1 \leq n^{p_j}$ and $g_j \in U^{p_j}_{n+1}$ for $j = 1, 2$. Since $\mathbb{F}$ is linearly ordered, without loss of generality, we may assume that $p_1 \leq p_2$. Then $\mathscr{U}^{p_1}$ extends $\mathscr{U}^{p_2}$ by Definition 5.7.1(b). Now conditions ($\mathrm{i}^{\mathscr{U}^{p_2}}_{\mathscr{U}^{p_1}}$) and ($\mathrm{iii}^{\mathscr{U}^{p_2}}_{\mathscr{U}^{p_1}}$) of Definition 5.4.1 imply $F(X^{p_2}) \subseteq F(X^{p_1})$ and $U^{p_1}_{n+1} \cap F(X^{p_2}) =$

$U_{n+1}^{p_2}$. From the last equation, we obtain $g_2 \in U_{n+1}^{p_1}$. The inclusion $g_1 \in U_{n+1}^{p_1}$ holds by the choice of p_1.

Note that $\mathscr{U}^{p_1}$ is a finite neighbourhood system for $F(X^{p_1})$ by condition (3_{p_1}) of Definition 5.7.1(a). Since $e \in \overline{X^{p_1}}$, from condition $(3_{\mathscr{U}^{p_1}})$ of Definition 5.3.2 we obtain that

$$g_1 \cdot g_2 \in U_{n+1}^{p_1} \cdot U_{n+1}^{p_1} = e \cdot U_{n+1}^{p_1} \cdot U_{n+1}^{p_1} \cdot e^{-1} \subseteq \bigcup_{x \in \overline{X^{p_1}}} x \cdot U_{n+1}^{p_1} \cdot U_{n+1}^{p_1} \cdot x^{-1} \subseteq U_n^{p_1}. \tag{5.56}$$

Since $p_1 \in \mathbb{F}$ and $n < n+1 \leq n^{p_1}$, we have $U_n^{p_1} \subseteq U_n$ by (5.53). From this inclusion and (5.56), we obtain $g_1 \cdot g_2 \in U_n$. Since $g_1, g_2 \in U_{n+1}$ were arbitrary, this establishes the inclusion $U_{n+1} \cdot U_{n+1} \subseteq U_n$. □

Claim 50. For every $n \in \mathbb{N}$ and each $y \in \bar{X}$, the inclusion $y \cdot U_{n+1} \cdot y^{-1} \subseteq U_n$ holds.

Proof. Let $n \in \mathbb{N}$ and $y \in \bar{X}$ be arbitrary. It follows from Definition 5.3.1 that $y = x^{\varepsilon}$ for some $x \in X$ and $\varepsilon \in \{-1, 1\}$. Since $B_{\{x\}} \in \mathscr{D}$, there exists $p \in B_{\{x\}} \cap \mathbb{F}$ by the choice of $\mathbb{F}$. Let $h \in U_{n+1}$. By (5.53), there exists some $q \in \mathbb{F}$ such that $n+1 \leq n^q$ and $h \in U_{n+1}^q$. Since $\mathbb{F}$ is linearly ordered, there exists some $r \in \mathbb{F}$ such that $r \leq p$ and $r \leq q$. Then $\mathscr{U}^r$ extends both $\mathscr{U}^p$ and $\mathscr{U}^q$ by Definition 5.7.1(b). Since $\mathscr{U}^r$ extends $\mathscr{U}^q$, by condition $(\mathrm{iii}_{\mathscr{U}^r}^{\mathscr{U}^q})$ of Definition 5.4.1, the equality $U_{n+1}^r \cap F(X^q) = U_{n+1}^q$ holds, implying that $h \in U_{n+1}^r$. Similarly, by condition $(\mathrm{ii}_{\mathscr{U}^r}^{\mathscr{U}^q})$ of Definition 5.4.1, we have $n+1 \leq n^q \leq n^r$. Since $\mathscr{U}^r$ is a finite neighbourhood system for $F(X^r)$ by condition (3_r) of Definition 5.7.1(a), we have $e \in U_{n+1}^r$ by Remark 5.3.3. Since $p \in B_{\{x\}}$, we have $x \in X^p$ by definition of $B_{\{x\}}$. Since $\mathscr{U}^r$ extends $\mathscr{U}^p$, we have $X^p \subseteq X^r$ by condition $(\mathrm{i}_{\mathscr{U}^r}^{\mathscr{U}^p})$ of Definition 5.4.1. Therefore, $x \in X^r$, and so $y = x^{\varepsilon} \in \overline{X^r}$ by Definition 5.3.1. Since $\mathscr{U}^r$ is a finite neighbourhood system for $F(X^r)$, by applying condition $(3_{\mathscr{U}^r})$ of Definition 5.3.2, we get that

$$y \cdot h \cdot y^{-1} = y \cdot h \cdot e \cdot y^{-1} \subseteq y \cdot U_{n+1}^r \cdot U_{n+1}^r \cdot y^{-1} \subseteq \bigcup_{z \in \overline{X^{p_1}}} z \cdot U_{n+1}^r \cdot U_{n+1}^r \cdot z^{-1} \subseteq U_n^r. \tag{5.57}$$

Since $r \in \mathbb{F}$ and $n \leq n^r$, we have $U_n^r \subseteq U_n$ by (5.53). Combining this with (5.57), we conclude that $y \cdot h \cdot y^{-1} \in U_n$. Since $h \in U_{n+1}$ was chosen arbitrarily, this proves the inclusion $y \cdot U_{n+1} \cdot y^{-1} \subseteq U_n$. □

Claim 51. For every $n \in \mathbb{N}$ and each $g \in F(X)$, there exists $k \in \mathbb{N}$ such that $g \cdot U_k \cdot g^{-1} \subseteq U_n$.

Proof. Let $n \in \mathbb{N}$ and $g \in F(X)$ be arbitrary. Then $g = y_1 \cdot y_2 \cdot \cdots \cdot y_m$ for some $y_1, y_2, \ldots, y_m \in \bar{X}$. Note that

$$y_m \cdot U_{n+m} \cdot y_m^{-1} \subseteq U_{n+m-1}$$

by Claim 50. Applying this claim once again, we get

$$y_{m-1} \cdot (y_m \cdot U_{n+m} \cdot y_m^{-1}) \cdot y_{m-1}^{-1} \subseteq y_{m-1} \cdot U_{n+m-1} \cdot y_{m-1}^{-1} \subseteq U_{n+m-2}.$$

By inductively applying Claim 50 finitely many times, we obtain the inclusion

$$g \cdot U_{n+m} \cdot g^{-1} = y_1 \cdot \cdots \cdot y_m \cdot U_{n+m} \cdot y_m^{-1} \cdot \cdots \cdot y_1^{-1} \subseteq U_n.$$

Therefore, it suffices to let $k = n + m$. □

Claim 52. The family $\mathcal{B}$ as in (5.54) is a neighbourhood base at e of some Hausdorff group topology $\mathscr{T}$ on $F(X)$.

Proof. It easily follows from Claims 48 and 49 that

$$U_m \subseteq U_n \text{ whenever } n, m \in \mathbb{N} \text{ and } n \leq m. \tag{5.58}$$

Combined with (5.54), this implies that $\mathcal{B}$ is a filter base. By (5.54), Claim 49 and Claim 51, $\mathcal{B}$ has the following three properties.

- For every $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V \cdot V \subseteq U$;
- For every $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V^{-1} \subseteq U$;
- For every $U \in \mathcal{B}$ and each $g \in F(X)$, there exists $V \in \mathcal{B}$ such that $g \cdot V \cdot g^{-1} \subseteq U$.

By [8, Theorem 3.1.5], the family

$$\mathscr{T} = \{O \subseteq F(X) : \forall\, g \in O\ \exists\, U \in \mathcal{B}\ (gU \subseteq O)\}$$

is a topology on the free group $F(X)$ making it into a topological group such that the family $\mathcal{B}$ is a neighbourhood base at e comprised of $\mathscr{T}$-neighborhoods of e. It follows from Claim 48, Theorem 4.7.1 and Remark 4.7.2 that $\mathscr{T}$ is a Hausdorff group topology for $F(X)$. $\square$

Claim 53. The topological group $(F(X), \mathscr{T})$ has the DW property.

Proof. We are going to check that the topological group $(F(X), \mathscr{T})$ satisfies Definition 1.5.2.

Let W be a neighbourhood of e in $(F(X), \mathscr{T})$. By Claim 52, there exists $n \in \mathbb{N}$ such that $U_n \subseteq W$.

Fix $g \in F(X) \setminus \{e\}$ and recall that $A_n \cap D_g \in \mathscr{D}$. By the choice of $\mathbb{F}$, there exists some $q \in A_n \cap D_g \cap \mathbb{F}$. Since $q \in A_n$, the inequality $n \leq n^q$ holds by definition of A_n, and therefore

$$U^q_{n^q} \subseteq U^q_n \subseteq U_n \subseteq W \tag{5.59}$$

by (5.53) and (5.58). From (5.59) we obtain that $\langle \mathrm{Cyc}(U^q_{n^q})\rangle \subseteq \langle \mathrm{Cyc}(W)\rangle$. Finally, from $q \in D_g$ and our definition of D_g, we get $g \in \langle \mathrm{Cyc}(U^q_{n^q})\rangle$. Therefore, $g \in \langle \mathrm{Cyc}(W)\rangle$. Since this holds for every $g \in F(X)$, we have proved that $F(X) \subseteq \langle \mathrm{Cyc}(W)\rangle$. Since the converse inclusion clearly holds, we get $F(X) = \langle \mathrm{Cyc}(W)\rangle$.

Since the last equality holds for every neighbourhood W of e in $(F(X), \mathscr{T})$, we conclude that $(F(X), \mathscr{T})$, has the DW property. $\square$

Since $(F(X), \mathscr{T})$ is Hausdorff and has a countable base at e by Claim 52, it is metrizable. This concludes the proof of Theorem 5.1.1.

5.9 Proof of Theorem 5.1.2

Fix an infinite cardinal κ. We are going to show that the free group with κ many generators admits an DW group topology.

Let H be the free group with a countably infinite set of generators equipped with the metric DW group topology constructed in Theorem 5.1.1. Let $\mathcal{B}$ be a countable local base of H at its identity e_H consisting of open neighbourhoods of e_H.

Define $\mathbb{E} = [\kappa \times \mathbb{N}]^{<\omega} \setminus \{\emptyset\}$. Since κ is infinite, $|\mathbb{E}| = \kappa$ and $|[\mathbb{E}]^{<\omega}| = \kappa$. Since H and $\mathcal{B}$ are countable, we can fix a listing $\{(\mathscr{E}_\alpha, h_\alpha, B_\alpha) : \alpha < \kappa\}$ of triples $(\mathscr{E}_\alpha, h_\alpha, B_\alpha)$, where $\mathscr{E}_\alpha \in [\mathbb{E}]^{<\omega} \setminus \{\emptyset\}$, $h_\alpha \in H^{\mathscr{E}_\alpha}$ and $B_\alpha \in \mathcal{B}$ for every $\alpha < \kappa$, having the following property:

($\diamond$) If $\mathscr{E} \in [\mathbb{E}]^{<\omega} \setminus \{\emptyset\}$, $h \in H^{\mathscr{E}}$ and $B \in \mathcal{B}$, then $(\mathscr{E}_\gamma, h_\gamma, B_\gamma) = (\mathscr{E}, h, B)$ for some $\gamma < \kappa$.

For every $E \in \mathbb{E}$ fix an injection $\theta_E : E \to H$ such that $\theta_E(E)$ is an independent subset of H. (It suffices to send E injectively into a subset of the infinite set of generators of H.)

By transfinite induction on $\alpha < \kappa$, we shall define $x_{\alpha,n} \in H^{\mathbb{E}}$ for every $n \in \mathbb{N}$ satisfying two properties:

(i_α) If $E \in \mathbb{E}$ and $(\alpha, n) \in E$ for some $n \in \mathbb{N}$, then $x_{\alpha,n}(E) = \theta_E(\alpha, n)$

(ii_α) There exist $n_0, k \in \mathbb{N}$ such that $x_{\alpha,n_0+i} \in \mathrm{Cyc}(W_\alpha)$ for $i = 0, \dots, k$ and $h_\alpha(E) = \prod_{i=0}^{k} x_{\alpha,n_0+i}(E)$ for every $E \in \mathscr{E}_\alpha$, where $W_\alpha = B_\alpha^{\mathscr{E}_\alpha} \times H^{\mathbb{E}\setminus\mathscr{E}_\alpha}$.

Suppose that $\alpha < \kappa$ is an ordinal and $x_{\beta,n} \in H^{\mathbb{E}}$ were already defined for all $\beta < \alpha$ and $n \in \mathbb{N}$ in such a way that properties (i_β) and (ii_β) hold. We shall define $x_{\alpha,n} \in H^{\mathbb{E}}$ for every $n \in \mathbb{N}$ satisfying (i_α) and (ii_α).

Recall that $\mathscr{E}_\alpha \in [\mathbb{E}]^{<\omega} \setminus \{\emptyset\}$, so $\mathscr{E}_\alpha$ is a (non-empty) finite subset of $\mathbb{E} = [\kappa \times \mathbb{N}]^{<\omega} \setminus \{\emptyset\}$. Therefore, $\bigcup \mathscr{E}_\alpha$ is a finite subset of $\kappa \times \mathbb{N}$, so we can fix $n_0 \in \mathbb{N}$ such that

$$(\alpha, n) \notin \bigcup \mathscr{E}_\alpha \text{ whenever } n \in \mathbb{N} \text{ and } n \geq n_0. \tag{5.60}$$

Let $\mathscr{E}_\alpha = \{E_1, \dots, E_l\}$ be a faithfully indexed enumeration, where $E_j \in \mathbb{E}$ for every $j = 1, \dots, l$.

Let $j = 1, \dots, l$ be arbitrary. Since $h_\alpha \in H^{\mathscr{E}_\alpha}$, we have $h_\alpha(E_j) \in H$. Since $B_\alpha \in \mathcal{B}$, it is an open neighbourhood of e_H in H. Since H is DW, we have $H = \langle \mathrm{Cyc}(B_\alpha) \rangle$ by Definition 1.5.2. Since $h_\alpha(E_j) \in H$, this allows us to choose $k_j \in \mathbb{N}^+$ and $g_{j,0}, \dots, g_{j,k_j} \in \mathrm{Cyc}(B_\alpha)$ such that $h_\alpha(E_j) = \prod_{i=0}^{k_j} g_{j,i}$.

Define $k = \max\{k_j : j = 1, \dots, l\}$.

Let $j = 1, \dots, l$ be arbitrary. For every integer i satisfying $k_j < i \leq k$, define $g_{j,i} = e_H$. Then

$$g_{j,0}, \dots, g_{j,k} \in \mathrm{Cyc}(B_\alpha) \quad \text{and} \quad h_\alpha(E_j) = \prod_{i=0}^{k} g_{j,i}. \tag{5.61}$$

For $n \in \mathbb{N}$ satisfying $n_0 \leq n \leq n_0 + k$, define $x_{\alpha,n} \in H^{\mathbb{E}}$ by

$$x_{\alpha,n}(E) = \begin{cases} g_{j,n-n_0} & \text{if } E = E_j \text{ for some } j = 1, \dots, l \\ \theta_E(\alpha, n) & \text{if } (\alpha, n) \in E \\ e & \text{otherwise} \end{cases} \qquad \text{for } E \in \mathbb{E}. \tag{5.62}$$

It should be noted that the first and the second line of (5.62) do not contradict each other. Indeed, if $E = E_j$ for some $j = 1, \dots, l$, then $E \in \mathscr{E}_\alpha$. Since $n \geq n_0$, we have $(\alpha, n) \notin E$ by (5.60).

For $n \in \mathbb{N}$ satisfying either $n < n_0$ or $n > n_0 + k$, define $x_{\alpha,n} \in H^{\mathbb{E}}$ by

$$x_{\alpha,n}(E) = \begin{cases} \theta_E(\alpha, n) & \text{if } (\alpha, n) \in E \\ e & \text{otherwise} \end{cases} \qquad \text{for } E \in \mathbb{E}. \tag{5.63}$$

Condition (i_α) is satisfied by (5.62) and (5.63).

Let us check the condition (ii_α). Let $E \in \mathscr{E}_\alpha$ be arbitrary. Then $E = E_j$ for a unique $j = 1, \dots, l$. It follows from (5.62) that $x_{\alpha,n_0+i}(E) = g_{j,i}$ for all $i = 0, \dots, k$, so from (5.61) we get

$$h_\alpha(E) = h_\alpha(E_j) = \prod_{i=0}^{k} g_{j,i} = \prod_{i=1}^{k} x_{\alpha,n_0+i}.$$

and

$$x_{\alpha,n_0+i}(E) \in \mathrm{Cyc}(B_\alpha) \quad \text{for} \quad i = 0, \dots, k \tag{5.64}$$

Since $W_\alpha = B_\alpha^{\mathscr{E}_\alpha} \times H^{\mathbb{E} \setminus \mathscr{E}_\alpha}$ and (5.64) holds for every $E \in \mathscr{E}_\alpha$, we conclude that $x_{\alpha,n_0+i} \in \mathrm{Cyc}(W_\alpha)$ for every $i = 0, \dots, k$. This finishes the check of condition (ii_α).

The inductive construction is complete.

Claim 54. $X = \{x_{\alpha,n} : \alpha < \kappa, n \in \mathbb{N}\}$ is an independent subset of $H^{\mathbb{E}}$.

Proof. By [10, Lemma 2.3], it suffices to show that every non-empty finite subset Y of X is independent in $H^{\mathbb{E}}$. Fix a non-empty finite set $E \subseteq \kappa \times \mathbb{N}$ such that $Y = \{x_{\alpha,n} : (\alpha, n) \in E\}$. Note that $E \in \mathbb{E}$ by our definition of $\mathbb{E}$. Let $\pi_E : H^{\mathbb{E}} \to H$ be the projection on E'th coordinate.

Let $\eta : E \to Y$ be a surjective map defined by $\eta(\alpha, n) = x_{\alpha,n}$ for $(\alpha, n) \in E$. If $(\alpha, n) \in E$, then $\pi_E(\eta(\alpha, n)) = \pi_E(x_{\alpha,n}) = x_{\alpha,n}(E) = \theta_E(\alpha, n)$ by (i_α). This shows that $\pi_E \restriction_Y \circ \eta = \theta_E$. Since θ_E

is an injection, so is η. Since η is also surjective, η is a bijection between E and Y, so it has the inverse map $\eta^{-1} : Y \to E$. Now $\pi_E \restriction_Y \circ \eta = \theta_E$ implies $\pi_E \restriction_Y = \theta_E \circ \eta^{-1}$. Since both η^{-1} and θ_E are injections, so is $\pi_E \restriction_Y$. Finally, it follows from our choice of θ_E that $\pi_E \restriction_Y (Y) = \theta_E \circ \eta^{-1}(Y) = \theta_E(\eta^{-1}(Y)) = \theta_E(E)$ is an independent subset of H. It follows from [10, Lemma 2.4] that Y is an independent subset of $H^{\mathbb{E}}$. □

Our goal is to show that the subgroup $G = \langle X \rangle$ of $H^{\mathbb{E}}$ generated by X is DW; see Claim 57 below. To achieve this, we fix arbitrarily an open neighbourhood U of the identity e of $H^{\mathbb{E}}$ and an element $g \in G$. Since G is a subgroup of $H^{\mathbb{E}}$, by Definition 1.5.2, it suffices to check that $g \in \langle \mathrm{Cyc}(U \cap G) \rangle$. Since $g \in G = \langle X \rangle$, we can find $l \in \mathbb{N}$ such that

$$g = \prod_{i=1}^{l} x_{\alpha_i, n_i}^{\varepsilon_i} \tag{5.65}$$

where $\alpha_i \in \kappa, n_i \in \mathbb{N}$ and $\varepsilon_i \in \{-1, 1\}$ for every $i = 1, \ldots, l$. Let

Let

$$\mathscr{E}_1 = \{(\alpha_i, n_i) : i = 1, \ldots, l\}.$$

Since U is open in $H^{\mathbb{E}}$, there exists a non-empty finite subset $\mathscr{E}_2$ of $\mathbb{E}$ and $B \in \mathcal{B}$ such that $B^{\mathscr{E}_2} \times H^{\mathbb{E} \setminus \mathscr{E}_2} \subseteq U$. Let $\mathscr{E} = \mathscr{E}_1 \cup \mathscr{E}_2$. Clearly $\mathscr{E} \in [\mathbb{E}]^{<\omega} \setminus \{\emptyset\}$ and $h = g \restriction_{\mathscr{E}} \in H^{\mathscr{E}}$. By ($\diamond$), we can find an ordinal $\alpha < \kappa$ such that $(\mathscr{E}, h, B) = (\mathscr{E}_\alpha, h_\alpha, B_\alpha)$. Let $n_0, k \in \mathbb{N}, x_{\alpha, n_0}, x_{\alpha, n_0+1}, \ldots, x_{\alpha, n_0+k}$ and W_α be as in condition (ii$_\alpha$).

Claim 55. $g_1 = \prod_{i=0}^{k} x_{\alpha, n_0+i} \in \langle \mathrm{Cyc}(U \cap G) \rangle$ and $g_1 \restriction_{\mathscr{E}} = g \restriction_{\mathscr{E}}$.

Proof. Since $\mathscr{E}_2 \subseteq \mathscr{E} = \mathscr{E}_\alpha$ and $B_\alpha = B$, we have

$$W_\alpha = B_\alpha^{\mathscr{E}_\alpha} \times H^{\mathbb{E} \setminus \mathscr{E}_\alpha} = B^{\mathscr{E}} \times H^{\mathbb{E} \setminus \mathscr{E}} \subseteq B^{\mathscr{E}_2} \times H^{\mathbb{E} \setminus \mathscr{E}_2} \subseteq U; \tag{5.66}$$

in particular, $\mathrm{Cyc}(W_\alpha) \subseteq \mathrm{Cyc}(U)$ by (1.2). Combining this with (ii$_\alpha$), we conclude that $x_{\alpha, n_0+i} \in \mathrm{Cyc}(U)$ for $i = 0, \ldots, k$. Since $x_{\alpha, n_0+i} \in X \subseteq G$, we have $x_{\alpha, n_0+i} \in \mathrm{Cyc}(U \cap G)$ for $i = 0, \ldots, k$. Therefore, $g_1 \in \langle \mathrm{Cyc}(U \cap G) \rangle$ by our definition of g_1.

Let $E \in \mathscr{E}$ be arbitrary. It follows from $\mathscr{E} \in \mathbb{E}, g \restriction_{\mathscr{E}} = h = h_\alpha$ and (ii$_\alpha$) that

$$g(E) = g\restriction_{\mathscr{E}}(E) = h(E) = h_\alpha(E) = \prod_{i=0}^{k} x_{\alpha,n_0+i}(E) = g_1(E).$$

This implies the equality $g_1 \restriction_{\mathscr{E}} = g \restriction_{\mathscr{E}}$. □

Claim 56. $g_2 = g \cdot g_1^{-1} \in \mathrm{Cyc}(U \cap G)$.

Proof. If $E \in \mathscr{E}$, then $g_2(E) = g(E) \cdot (g_1(E))^{-1} = g(E) \cdot g(E)^{-1} = e_H$ by definition of g_2 and the second statement of Claim 55. Thus, $g_2 \in \{e_H\}^{\mathscr{E}} \times H^{\mathbb{E}\setminus\mathscr{E}}$. Note that $\{e_H\}^{\mathscr{E}} \times H^{\mathbb{E}\setminus\mathscr{E}} \subseteq B^{\mathscr{E}} \times H^{\mathbb{E}\setminus\mathscr{E}} \subseteq U$ by (5.66). Since $\{e_H\}^{\mathscr{E}} \times H^{\mathbb{E}\setminus\mathscr{E}}$ is a subgroup of $H^{\mathscr{E}}$, the stricter inclusion $\{e_H\}^{\mathscr{E}} \times H^{\mathbb{E}\setminus\mathscr{E}} \subseteq \mathrm{Cyc}(U)$ holds as well. This argument shows that $g_2 \in \mathrm{Cyc}(U)$.

Note that $g_1 \in \langle \mathrm{Cyc}(U \cap G)\rangle \subseteq G$ by Claim 55 and the fact that G is a group. Since $g \in G$, we obtain $g_2 = g \cdot g_1^{-1} \in G$. Combining this with $g_2 \in \mathrm{Cyc}(U)$, we get $g_2 \in \mathrm{Cyc}(U \cap G)$. □

It follows from Claims 55 and 56 that $g = g_2 \cdot g_1 \in \mathrm{Cyc}(U \cap G) \cdot \langle \mathrm{Cyc}(U \cap G)\rangle \subseteq \langle \mathrm{Cyc}(U \cap G)\rangle$. Since this inclusion holds for an arbitrary $g \in G$, we conclude that $G \subseteq \langle \mathrm{Cyc}(U \cap G)\rangle$. The converse inclusion $\langle \mathrm{Cyc}(U \cap G)\rangle \subseteq G$ trivially holds. We have proved that $G = \langle \mathrm{Cyc}(U \cap G)\rangle$. Since this equation holds for an arbitrary open neighbourhood U of e in $H^{\mathbb{E}}$ and G is a subgroup of $H^{\mathbb{E}}$, from Definition 1.5.2 we obtain the following.

Claim 57. The subgroup $G = \langle X \rangle$ of $H^{\mathbb{E}}$ generated by X is DW.

It follows from Claim 54 that G is a free group with generating set X. Since κ is an infinite cardinal, $|X| = |\kappa \times \mathbb{N}| = \kappa$. By Claim 57, the subspace topology G inherits from $H^{\mathbb{E}}$ is DW.

5.10 Open questions

The free group with one generator is isomorphic to the group $\mathbb{Z}$ of integer numbers, so it does not admit an SSGP group topology by [4, Corollary 3.14], and therefore it also cannot have an DW group topology either. In view of this remark, Theorem 5.1.2 motivates the following question.

Question 5.10.1. Let $n \in \mathbb{N}$ with $n \geq 2$. Can the free group with n generators admit either an DW or an SSGP group topology?

Comparison of Theorems 5.1.1 and 5.1.2 suggests the following question:

Question 5.10.2. Can the DW group topology in Theorem 5.1.2 be chosen to be metric?

In fact, a more general questions seems quite intriguing.

Question 5.10.3. If a group G admits an SSGP group topology, must G also admit a metric SSGP group topology? What if G is abelian?

The DW version of this question also makes sense.

Question 5.10.4. If a group G admits an DW group topology, must G also admit a metric DW group topology? What if G is abelian?

The following problem may be considered as a “heir” of Question 1.5.10(a):

Problem 5.10.5. Describe the algebraic structure of (abelian) groups which admit an DW group topology.

Shakhmatov and the author made some progress on this problem in [45].

Chapter 6

On algebraic structure of groups with a property of Dierolf and Warken

6.1 Introduction

In this chapter, we investigate the class of groups which carry property DW (please see Section 1.5.2 for the historical overview of this property). During our investigation, we also obtain some information about their algebraic structure. To the best of the knowledge of Shakhmatov and the author, the results presented here mark a humble first attempt at a close investigation of Dierolf and Warken's original unmodified property DW. As we shall see in Section 6.3, the class of topological groups defined by this property does not coincide with the class of SSGP groups, so it becomes of importance to understand just how different these two classes may be.

This chapter is organized as follows: in Section 6.2 we cover algebraic preliminaries and definitions which will be needed throughout this chapter, in particular, we also cover some folklore facts for minimally almost periodic groups. Each higlighted concept also comes with a literature recommendation for the interested reader. In Section 6.3 we give a necessary condition for torsion Abelian groups to admit an DW group topology, in particular, we are able to show some elementary examples to distinguish this class from the SSGP class. In the intermediate Section 6.4 we show that the so called *maximal* 0*-rank* subgroups of an Abelian group inherit a *precompact* group topology if they happen to be finitely generated; in particular, this allows us to deduce that a topological group quotient by one of such subgroups cannot be trivial. In Section 6.5 we utilize the main result of Section 6.4 to obtain a necessary condition on groups of finite free rank to have an DW group topology; as a consequence, we are able to show that $\mathbb{Q}^n$ (where n is a positive integer) contains no non-trivial DW subgroups. In the latter Section 6.6 we dedicate some time to show some basic topological properties of the DW class; in particular, we cover some loose ends in the topic of subgroup inheritance. In the self-contained Appendices A.1 and A.2 we cover some purely algebraic results that are utilized in Section 6.5; to be precise, we utilize the notion of finitely generated (or Prüfer) rank to obtain an upper bound for the p-rank in group quotients where ALL p-ranks are finite.

6.2 Preliminary lemmas

Definition 6.2.1. Let G be a group. We say that a subgroup H of G is *essential* in G if and only if for every $g \in G \setminus \{e\}$ we have that $\langle g \rangle \cap H \neq \{e\}$ holds.

For convenience, we shall be utilizing the following terminology as seen in [16][Section 3]:

Definition 6.2.2. Let G be an Abelian group. A subgroup H of G is called a *maximal* 0*-rank* subgroup of G provided that the algebraic quotient G/H is a torsion group.

Proposition 6.2.3. *Let G be a non-torsion Abelian group. The following are equivalent for a subgroup H of G:*

(i) H is a maximal 0*-rank subgroup of G,*

(ii) H contains a maximal independent subset of G,

(iii) For every non-torsion $g \in G$ the inequality $\langle g \rangle \cap H \neq \{e\}$ holds.

We invite the reader to compare item (iii) of Proposition 6.2.3 with Definition 6.2.1. Indeed, one can easily see that a maximal 0-rank subgroup of a group G is essential if and only if the group G is torsion-free. In essence, one can think of maximal 0-rank subgroups as subgroups which are *almost* essential.

The following lemma is well known. We include its easy proof only for the convenience of the reader.

Lemma 6.2.4. *A precompact minimally almost periodic group is trivial.*

Proof. Let G be a precompact minimally almost periodic group. Since G is precompact, its completion K is a compact group. Let $\iota : G \to K$ be the natural inclusion monomorphism. Since ι is continuous and G is minimally almost periodic, the image $\iota(G)$ must be trivial. Since ι is a monomorphism, G itself must be trivial as well. □

6.3 The DW property in torsion groups

In this section we shall be focusing on the behavior of the DW property in torsion groups. The next results shows that the implications in (1.6) all become reversible in the realm of bounded groups. We begin as follows:

Proposition 6.3.1. *Every* SSGP *topological group of bounded order satisfies the* DW *property.*

Proof. Suppose $n \in \mathbb{N}$ is the exponent of G and let $U \subseteq G$ be any open neighbourhood of the identity of G. Choose another open neighbourhood V of e such that $V^n \subseteq U$ and let $x \in V$ be arbitrary. Clearly the inclusion

$$\langle x \rangle = \{x^m : m = 1, \ldots, n\} \subseteq V^n \subseteq U$$

holds. Since the previous inclusion implies that $x \in \mathrm{Cyc}(U)$, we have shown that $V \subseteq \mathrm{Cyc}(U)$ holds. Naturally, the inclusion $\mathrm{Cyc}(U) \subseteq \langle \mathrm{Cyc}(U) \rangle$ also holds. Since the subgroup $\langle \mathrm{Cyc}(U) \rangle$ contains the non-empty open set V, it is a clopen subset of G. Since it is also dense in G by the SSGP property in Definition 1.5.5, $\langle \mathrm{Cyc}(U) \rangle$ must coincide with G. □

By combining Proposition 6.3.1 with Theorem 1.5.7 we obtain the following:

Corollary 6.3.2. *Every Abelian minimally almost periodic topological group of bounded order satisfies the* DW *property.*

As a consequence, the implications shown in (1.6) become reversible in the realm of Abelian groups of bounded order. As we had highlighted before in Remark 1.2.6, an Abelian group has null divisible rank if and only if it is a bounded group. We can then modify Theorem 1.4.11 to re-state it in the following form:

Theorem 6.3.3. *A non-trivial Abelian group G of bounded order admits an* DW *group topology if and only if all leading Ulm-Kaplanski invariants of G are infinite.*

With the case of bounded groups out of the way, we can now begin to study the general case of torsion groups. The following lemma begins to show the algebraic structure of torsion groups that can admit an DW group topology.

Lemma 6.3.4. *If an Abelian torsion group admits an* DW *group topology, then all of its non-zero p-ranks are infinite.*

Proof. For all prime numbers $p \in P$ let us denote by G_p the p-component of G. In order to proceed by contradiction, let us suppose that G admits an DW group topology and that the non-trivial component G_p (for some fixed $p \in \mathbf{P}$) has finite rank. Since G_p has finite rank, we can fix $E \subseteq G_p$ to be its finite essential subgroup of rank $r_p(G)$. For convenience, let us denote $G = G_p \oplus H$ where

$$H = \bigoplus_{q \in \mathbf{P} \setminus \{p\}} G_q.$$

Define $E^* = E \setminus \{e\}$ and consider V any open neighbourhood of e in G. Since E is finite, the set $U = V \setminus E^*$ is also an open neighbourhood of e in G.

Claim 58. The inclusion $\langle Cyc(U) \rangle \subseteq H$ holds.

Proof. Let $g + h \in Cyc(U)$ be arbitrary and suppose that $g \neq e$. By the definition of G and its p-components, we may find integers $n, m \in \mathbb{N}$ be natural numbers such that n is a power of p, p does not divide m, and the equalities $ng = e$ and $mh = e$ are satisfied. By the choice of m we have that $mh = e$ holds, implying that

$$mg = mg + mh = m(g + h) \in Cyc(U), \tag{6.1}$$

as $g + h \in \mathrm{Cyc}(U)$. Now, since n and m are relatively prime and $g \neq e$, we have that $mg \neq e$. By hypothesis, E is an essential subgroup of G, and thus we have that $\langle mg \rangle \cap E \neq \{e\}$; this implies that $U \cap E \neq \{e\}$ which contradicts the definition of U. We then have that $g = e$ and so $\mathrm{Cyc}(U) \subseteq H$. Since H is a group we have that $\langle Cyc(U) \rangle \subseteq H$, and the assertion is proved. □

By the previous claim we have that $\langle Cyc(U)\rangle \subseteq H \subsetneq G_p \oplus H = G$, and so G does not have the DW property. □

Theorem 6.3.5. *If an Abelian torsion group G admits an* DW *group topology, then every non-trivial p-component G_p of G is either bounded with all its leading Ulm-Kaplansky invariants infinite, or it is of infinite divisible rank.*

Proof. We begin with the following claim:

Claim 59. For every prime $p \in \mathbf{P}$ the cardinal $r(nG_p)$ is either infinite or null for all $n \in \mathbb{N}$.

Proof. First, let us observe that Lemma 6.3.4 implies that for every torsion Abelian group H and every prime $p \in \mathbf{P}$, if H admits an DW group topology then the rank of its p-components, $r(H_p) = r_p(H)$, are either infinite or null. Since G admits an DW group topology, Proposition 6.6.2(i) implies that for every $n \in \mathbb{N}$ the group nG also admits an DW group topology. By combining these two results we obtain that for every $n \in \mathbb{N}$ the cardinal $r(nG_p)$ is either infinite or null, as desired. □

Recall that for any group H the inequality $r(H) < |H|$ holds, so we can use Claim 59 to deduce that for every prime $p \in \mathbf{P}$ and $n \in \mathbb{N}$, either $|nG_p| \geq \omega$ or $nG_p = \{0\}$ holds. Let p be a prime $p \in \mathbf{P}$, if the p-component G_p is bounded, then the previous observation and [11][Proposition 2.5] imply that all leading Ulm-Kaplansky invariants of G_p are infinite.

On the other hand, if the p-component G_p is unbounded, then $r(nG_p)$ is non-zero for every positive integer $n \in \mathbb{N}$. By using Claim 59 we obtain that $r(nG_p)$ is infinite for all positive integers $n \in \mathbb{N}$. This shows that G_p has infinite divisible rank by definition. □

Example 6.3.6. Let $P \subseteq \mathbf{P}$ be an infinite set of primes. The following groups admit an SSGP group topology but do not admit an DW group topology:

(i) $G_1 = \bigoplus_{p\in P} \mathbb{Z}(p)$ a direct sum of $Z(p)$ groups.

(ii) $G_2 = \bigoplus_{p\in P} \mathbb{Z}(p^\infty)$ a direct sum of p-Prüfer groups.

Since the set P is countably infinite, the inequalities $r_d(G_1) \geq \omega$ and $r_d(G_2) \geq \omega$ can easily be verified from Definition 1.2.4 for both groups. By Theorem 1.5.12, both G_1 and G_2 admit an

SSGP group topology. If we fix some prime $q \in P$ it suffices to observe that neither of the groups $\mathbb{Z}(p)$ and $\mathbb{Z}(p^\infty)$ satisfy the necessary conditions from Theorem 6.3.3. This shows that neither G_1 nor G_2 can admit an DW group topology.

These two examples show that DW groups form a proper subclass of SSGP groups, and therefore the first implication in (1.6) is not reversible. As previously discussed in Section 1.5.5, the notion of divisible rank played a key role in the characterization of Abelian SSGP groups. In the case of groups with the DW property however, it is quite likely that a different approach will be needed.

6.4 Precompactness in maximal 0-rank subgroups of DW groups

In this section we shall be focusing on maximal 0-rank subgroups and the properties they inherit. As we shall see, if a maximal 0-rank subgroup *also happens* to be *finitely generated*, then this subgroup will inherit a precompact subgroup topology. As a result, this shall give us some more insight when we study Abelian groups with finite free rank. Before we begin, we shall be needing the following lemma for certain vector spaces.

Lemma 6.4.1. *Let $n \in \mathbb{N}$ and consider $V = \mathbb{Q}^n$ as a vector space over $\mathbb{Q}$. If $\{x_1, \dots, x_n\} \subseteq \mathbb{Z}^n$ is a linearly independent subset of V, there exists a finite subset $F \subseteq \mathbb{Z}^n$ such that*

$$\bigoplus_{j=1}^{n} \langle x_i \rangle + F = \mathbb{Z}^n, \tag{6.2}$$

where $\langle x_i \rangle$ is the smallest subgroup generated by x_i with integer coefficients for all $i = 1, \dots, n$.

Proof. Define

$$B = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \in (-1,1) \cap \mathbb{Q} \right\}. \tag{6.3}$$

Claim 60. $\mathbb{Q}^n = \bigoplus_{j=1}^{n} \langle x_i \rangle + B$.

Proof. Let $v \in \mathbb{Q}^n$ be arbitrary. Since $\{x_1, \dots, x_n\}$ is a basis for V, for every $i = 1, \dots, n$ there exists $\beta_i \in \mathbb{Q}$ such that $v = \sum_{j=1}^{n} \beta_i x_i$. Finally, for every $i = 1, \dots, n$ there exist $z_i \in \mathbb{Z}$ and $r_i \in (-1, 1)$ such that $\beta_i = z_i + r_i$. This implies that

$$v = \sum_{j=1}^{n} z_i x_i + \sum_{j=1}^{n} r_i x_i \in \bigoplus_{j=1}^{n} \langle x_i \rangle + B. \tag{6.4}$$

This proves that $\mathbb{Q}^n \subseteq \bigoplus_{j=1}^n \langle x_i \rangle + B$. The converse inclusion clearly holds, showing the equality. $\square$

It is clear from (6.3) that B is a bounded set in $\mathbb{Q}^n$, and therefore the set $F = B \cap \mathbb{Z}^n$ is finite.

Let $v \in \mathbb{Z}^n$ be arbitrary. By the previous claim, there exists $v_1 \in \bigoplus_{j=1}^n \langle x_i \rangle$ and $v_2 \in B$ such that $v = v_1 + v_2$. Since $\mathbb{Z}^n$ is a subgroup of $\mathbb{Q}^n$, we have that $v_2 = v - v_1 \in \mathbb{Z}^n$. This implies that $v_2 \in B \cap \mathbb{Z}^n = F$, proving that $\mathbb{Z}^n \subseteq \bigoplus_{j=1}^n \langle x_i \rangle + F$. The converse inclusion clearly holds, showing the equality in (6.2). $\square$

Lemma 6.4.2. *Let G be an abelian* DW *group and suppose that $E \subseteq G$ is a maximal 0-rank subgroup of G. For every neighbourhood U of 0 in G and each finite sequence $g_1, \ldots, g_k \in G$, there exists $m \in \mathbb{N}^+$ such that $mg_j \in \langle \mathrm{Cyc}(U \cap E) \rangle$ for all $j = 1, \ldots, k$.*

Proof. We check the following elemental claim.

Claim 61. If $g_1, \ldots, g_k \in G$ is a sequence such that $m_j g_j \in \mathrm{Cyc}(U \cap E)$ for some $m_1, \ldots m_j \in \mathbb{Z}$ then there exists $m \in \mathbb{N}^+$ such that $mg_j \in \mathrm{Cyc}(U \cap E)$ for all $j = 1, \ldots, k$.

Proof. Let $g_1, \ldots, g_k \in G$ be a sequence as above. By hypothesis, we can consider a finite sequence $m_1, \ldots, m_k \in \mathbb{N}^+$ such that for every $i = 1, \ldots, k$ we have that the inclusion

$$m_j g_j \in \mathrm{Cyc}(U \cap E)$$

holds. Let us then take

$$m = m_1 \cdot \cdots \cdot m_k \in \mathbb{N}^+.$$

Let $j = 1, \ldots, k$ be arbitrary, observe that since $m_j g_j \in E$ and E is a subgroup of G, the inclusion

$$mg_j = (m/m_j) m_j g_j \in E$$

holds. Finally, since $g_j \in \mathrm{Cyc}(U)$ holds, this implies that $mg_j \in \mathrm{Cyc}(U)$ by (1.2). With this we have proved that the inclusion

$$mg_j \in Cyc(U \cap E)$$

holds for every $j = 1, \ldots, k$. $\square$

It now suffices to verify the hypotheses of our previous claim. Let us fix a sequence $g_1, \dots, g_k \in G$ and take U to be some neighbourhood of 0. Since G is DW, for every $i = 1, \dots, k$ we can get a sequence $h_{i,1}, \dots, h_{i,n_i} \in \mathrm{Cyc}(U)$ such that

$$g_i = \sum_{j=1}^{n_i} h_{i,j}. \tag{6.5}$$

We now utilize our hypothesis over E with Proposition 6.2.3(iii). Observe that if some $h_{i,j}$ happens to be a torsion element of G, we can find some $n_{i,j} \in \mathbb{N}^+$ such that $n_{i,j}h_{i,j} = 0 \in \mathrm{Cyc}(U) \cap E$. In the case where $h_{i,j}$ is a torsion free element, we simply take $n_{i,j}$ such that $0 \neq n_{i,j}h_{i,j} \in \mathrm{Cyc}(U) \cap E$. In either case, we can always take some $n_{i,j} \in \mathbb{N}^+$ such that

$$n_{i,j}h_{i,j} \in \mathrm{Cyc}(U) \cap E = \mathrm{Cyc}(U \cap E)$$

holds for every $j = 1, \dots, n_i$. From this, the sequences $\{h_{i,j} : i = 1, \dots, k \text{ and } j = 1, \dots, n_i\} \subseteq G$ and $\{n_{i,j} : i = 1, \dots, k \text{ and } j = 1, \dots, n_i\} \subseteq \mathbb{N}^+$ satisfy the hypotheses of our claim. This implies the existence of some $m \in \mathbb{N}^+$ such that

$$mh_{i,j} \in \mathrm{Cyc}(U \cap E) \tag{6.6}$$

for all $i = 1, \dots, k$ and $j = 1, \dots, n_i$. If we combine this with Equation 6.5, we get that

$$mg_i = \sum_{j=1}^{n_i} mh_{i,j} \in \langle \mathrm{Cyc}(U \cap E) \rangle.$$

From this, our proof has been completed. □

Theorem 6.4.3. *Suppose that an Abelian* DW *group G contains a free maximal* 0*-rank subgroup E of finite rank. The following hold:*

(i) E is precompact in the subgroup topology inheried from G;

(ii) E is not dense in G.

Since E is free and of finite rank, it is isomorphic to a power of the group of integers, $\mathbb{Z}^n$, where n is the rank of E. For this reason (and for convenience), let us assume that $\mathbb{Z}^n$ is precisely this subgroup E.

Proof. We begin with the following claim.

Claim 62. For every neighbourhood U of 0, the set $\mathrm{Cyc}(U \cap \mathbb{Z}^n)$ generates $\mathbb{Q}^n$ as a vector space over $\mathbb{Q}$.

Proof. Let U be any neighbourhood of 0.

For every $i = 1, \ldots, n$ we denote by $e_i \in \mathbb{Z}^n$ the basic canonical vector of $\mathbb{Q}^n$ of the i-th coordinate. By our hypothesis over $\mathbb{Z}^n$, Lemma 6.4.2 implies that we can find $m \in \mathbb{N}^+$ such that $me_i \in \langle \mathrm{Cyc}(U \cap \mathbb{Z}^n)\rangle$ for all $i = 1, \ldots, n$.

Since $m \neq 0$, the system $\{me_1, \ldots, me_n\}$ is a linearly independent set of size n in $\mathbb{Q}^n$, and therefore a basis for $\mathbb{Q}^n$ as a vector space over $\mathbb{Q}$. Since $\{me_1, \ldots, me_n\} \subseteq \langle \mathrm{Cyc}(U \cap \mathbb{Z}^n)\rangle$ and the latter is a subset of the vector subspace of $\mathbb{Q}^n$ spanned by $\mathrm{Cyc}(U \cap \mathbb{Z}^n)$, this shows that $\mathrm{Cyc}(U \cap \mathbb{Z}^n)$ is a generating set for $\mathbb{Q}^n$ as a vector space over $\mathbb{Q}$. □

(i) First, we shall prove that $\mathbb{Z}^n$ inherits a precompact topology as a subgroup of G. For this, it suffices to check that for every neighbourhood V of 0, there exists a finite set $F \subseteq \mathbb{Z}^n$ such that $\mathbb{Z}^n = V + F$. Let U be a neighbourhood of 0 in G such that $U^n = \{\sum_{j=1}^n x_j : x_j \in U\} \subseteq V$ holds. By our previous claim, $\mathrm{Cyc}(U \cap \mathbb{Z}^n)$ is a generating set for $\mathbb{Q}^n$. Therefore, we can find a subset $\{x_1, \ldots, x_n\} \subseteq \mathrm{Cyc}(U \cap \mathbb{Z}^n)$ which is a basis for $\mathbb{Q}^n$ as a vector space over $\mathbb{Q}$. Since the set $\{x_1, \ldots, x_n\}$ satisfies the hypotheses of Lemma 6.4.1, there exists a finite set $F \subseteq \mathbb{Z}^n$ such that

$$\mathbb{Z}^n = \bigoplus_{j=1}^{n} \langle x_j \rangle + F \subseteq U^n + F \subseteq V + F. \tag{6.7}$$

This proves that $\mathbb{Z}^n$ inherits a precompact group topology from G.

(ii) Suppose $\mathbb{Z}^n$ happens to be dense in G. Observe that $\mathbb{Z}^n$ is precompact, and therefore G as well, as they both have the same (compact) completion. Since G is DW, we have that G must be the trivial group by Lemma 6.2.4. This gives a contradiction, as G contains the non-trivial subgroup $\mathbb{Z}^n$.

□

Corollary 6.4.4. *Suppose that an Abelian* DW *group* G *contains a finitely generated subgroup* E *which is of maximal* 0*-rank . The following hold:*

(i) E *is precompact in the subgroup topology inherited from* G*; and*

(ii) E *is dense in* G *if and only if* G *is the trivial group.*

Proof. Since E is finitely generated, we can apply the fundamental representation theorem of finitely generated groups. From this, we obtain that the isomorphism

$$E \simeq \mathbb{Z}^n \oplus \bigoplus_{j=i}^{m} \mathbb{Z}(p_j^{k_j}) \tag{6.8}$$

holds for some primes $p_1, \ldots p_m \in \mathbf{P}$ and non-negative integers $n, m, k_1, \ldots, k_m \in \mathbb{N}$. From this, we can assume without loss of generality that $\mathbb{Z}^n$ is a subgroup of E.

Claim 63. $\mathbb{Z}^n$ is a maximal 0-rank subgroup of G.

Proof. To prove this, it is enough to show that Proposition 6.2.3(iii) holds. Indeed, if we take a non-torsion $g \in G$, we know there exists some $n \in \mathbb{N}^+$ such that $ng \in E$ since E is of maximal 0-rank . By Equation 6.8 we have that $ng = z + h$ where $z \in \mathbb{Z}^n$ and h is a torsion element of E. Since h is a torsion element of E, there exists some $m \in \mathbb{N}^+$ such that $mh = 0$. We then have that

$$0 \neq (mn)g = mz + mh = mz \in \mathbb{Z}^n.$$

This shows that there exists $l \in \mathbb{N}^+$ such that $0 \neq lg \in \mathbb{Z}^n \cap \langle g \rangle$ holds. Since we began with an arbitrary non-torsion $g \in G$, we can conclude the proof of this claim. □

(i) Since $\mathbb{Z}^n$ is of maximal 0-rank then we arrive to the conditions of Theorem 6.4.3. From this, we obtain that $\mathbb{Z}^n$ is precompact. Finally, the torsion part $t(E)$ is a finite group (as evidenced by Equation 6.8), therefore the direct sum $\mathbb{Z}^n \oplus t(E) = E$ is also a precompact group.

(ii) Suppose that E happens to be dense in G. Since E is precompact by (i), G itself must be a precompact group by virtue of having the same completion as E. By Lemma 6.2.4, we have that G must be the trivial group.

Conversely, if G is the trivial group, then every subgroup of G coincides with G itself and is therefore dense. This concludes our proof. □

We close this section with the following question:

Question 6.4.5. Suppose that an abelian SSGP group G contains a maximal 0-rank subgroup E. Must E be precompact in the inherited subspace topology? What if E happens to be essential in G?

6.5 A necessary condition of the existence of an DW group topology on groups of finite free rank

Theorem 6.5.1. *Let G be an Abelian group of finite free rank. If G admits an* DW *group topology, then either one of the following holds:*

(i) If $r_0(G) = 0$ then every non-trivial p-component of G admits an DW *group topology.*

(ii) If $0 < r_0(G)$ then there exists $p \in \mathbf{P}$ such that the p-component of G has infinite divisible rank.

Proof. The first case coincides with G being a torsion group, which has been covered in Theorem 6.3.4. We proceed to solve the second case by contradiction. For this reason, let us consider G to have finite p-ranks and a non-zero free rank n for some $n \in \mathbb{N}^+$. Since G has free rank n, we can fix some maximal independent subset $I \subseteq G$ of size n and comprised of torsion free elements of G. Clearly, the subgroup $H = \langle I \rangle$ generated by I is a maximal 0-rank subgroup of G. Since H is finitely generated, it satisfies the hypotheses of Corollary 6.4.4. Therefore, by item (ii) of Corollary 6.4.4 we have that H cannot be dense in G. Since H is not dense in G, the topological quotient $K = G/cl_G(H)$ is a non-trivial group, we shall proceed as follows.

Claim 64. The quotient K is a torsion group.

Proof. Indeed, since H contains a maximal independent subset of torsion-free elements of G, and $H \subseteq cl_G(H)$, then Proposition 6.2.3 (ii) implies that $cl_G(H)$ is also a maximal 0-rank subgroup of G. By Definition 6.2.2 the quotient group K is torsion. □

Claim 65. For every $p \in \mathbf{P}$ the p-rank of K is finite.

Proof. Let $p \in \mathbf{P}$ be an arbitrary prime, since K is a quotient group of G, the inequality

$$r_p(K) \leq r_0(G) + r_p(G)$$

holds by Corollary A.2.2. Since the ranks $r_0(G)$ and $r_p(G)$ are finite by hypothesis, the previous inequality shows that $r_p(K)$ is finite. □

Claim 66. *K* does not admit an DW group topology.

Proof. Since K is a non-trivial torsion group, there exists some $p_0 \in \mathbf{P}$ such that its p_0-rank $r_{p_0}(K)$ is non-zero. If K admits an DW group topology, then by case 1 of this theorem the cardinal $r_{p_0}(K)$ must be infinite in contradiction to our previous claim. We have then shown that K does not admit an DW group topology. □

Finally, since G was equipped with an DW group topology, its topological quotient K would inherit the DW property, which contradicts the above claim. We can conclude that G itself cannot admit an DW group topology. □

Corollary 6.5.2. *If an unbounded Abelian topological group G admits an* DW *group topology, then G is of infinite divisible rank.*

Proof. Observe that if G is unbounded then for every $n \in \mathbb{N}$ the cardinal $r(nG)$ is non-zero. It then remains to prove that $r(nG)$ is infinite for all $n \in \mathbb{N}$, so let us assume the contrary. Since the group nG is a continuous homomorphic image of G, nG also admits an DW group topology by our hypothesis and Proposition 6.6.2(i). By our assumption, $r(nG)$ is finite, and therefore, nG is a group of finite rank. This implies that

$$r(nG) = r_0(nG) + \sum_{p \in \mathbf{P}} r_p(nG) < \omega$$

holds. Clearly, this implies that nG has finite free rank and finite p-ranks, and therefore nG does not admit an DW group topology by Theorem 6.5.1. □

As particular cases, we have the following:

Corollary 6.5.3. *The following holds:*

(i) Every divisible group with finite free rank and finite p-ranks does not admit an DW *group topology.*

(ii) For every $n \in \mathbb{N}$ the group $\mathbb{Q}^n$ does not admit an DW *group topology.*

While (ii) is a particular case from (i), we highlight it for the following reason: previously, Comfort and Gould [4],[25] had shown that $\mathbb{Q}$ itself (and its finite powers) would admit an SSGP topology. Finally, let us recall that *wide subgroups* of $\mathbb{Q}^n$ played a big role in the characterization of Abelian SSGP groups (see Definition 4.4.1 and Chapter 4)

In particular, we can state the following version of Theorems 4.7.4 and 4.1.1 as follows:

Theorem 6.5.4. *Let $n \in \mathbb{N}^+$. A subgroup $G \subseteq \mathbb{Q}^n$ admits an* SSGP *group topology if and only if G is a wide subgroup of $\mathbb{Q}^n$.*

Since all wide subgroups of $\mathbb{Q}^n$ satisfy the hypotheses of Theorem 6.5.1, we can combine these results to obtain the following:

Corollary 6.5.5. *For every $n \in \mathbb{N}^+$ the following hold:*

(i) The only subgroup of $\mathbb{Q}^n$ which admits an DW *group topology is the trivial group.*

(ii) Every wide subgroup of $\mathbb{Q}^n$ admits an SSGP *group topology but not an* DW *one.*

6.6 Topological operations of DW groups

In this final section we shall be covering some basic topological properties of the DW class. We highlight that Theorem 6.6.1 and Example 6.6.6 are more involved than usual standard proofs. In the former case, this example highlights that it is not enough to contain a dense DW subgroup to belong to the SSGP class.

Theorem 6.6.1. *Let $\{G_i : i \in I\}$ be a family of topological groups all of which have the* DW *property. Consider $G = \prod_{i \in I} G_i$ and let $H \subseteq G$ be a subgroup of G such that the direct sum $G^+ = \bigoplus_{i \in I} G_i$ satisfies $G^+ \subseteq H$. Then H has the* DW *property.*

Proof. It suffices to prove that for every open neighbourhood U of the identity e_G the equality $H = \langle \mathrm{Cyc}(U \cap H) \rangle$ holds. Let U be a open neighbourhood of the identity e_G and let $h \in H$ be arbitrary. Since U is open, we can find a finite set $J \subseteq I$ and open sets $U_j \subseteq G_j$ for $j \in J$ such that

$$V = \prod_{j \in J} U_j \times \prod_{i \in I \setminus J} G_i \subseteq U \tag{6.9}$$

For every $j \in J$ let $\pi_j : G \to G_j$ denote the projection from G to G_j. Let $j \in J$ be arbitrary and observe that $\pi_j(h) \in G_j$. Since $\pi_j[U] = U_j \subseteq G_j$ is an open neighbourhood of e_{G_j} and G_j is DW, there exists $n_j \in \mathbb{N}$ and $x^j_1, \ldots, x^j_{n_j} \in \mathrm{Cyc}(U_j)$ such that

$$\pi_j(h) = \prod_{i=1}^{n_j} x^j_i. \tag{6.10}$$

For every $j \in J$ and $i = 1, \ldots, n_j$ let $\hat{x}^j_i \in G$ be the mapping such that:

$$\pi_k(\hat{x}^j_i) = \begin{cases} x^j_i \text{ if } k = j; and \\ e_{G_k} \text{ otherwise.} \end{cases} \tag{6.11}$$

By our definition of $\hat{x}^j_i$ and Equation 6.9 we have that

$$\hat{x}^j_i \in \mathrm{Cyc}(V \cap G^+) \subseteq \mathrm{Cyc}(U) \tag{6.12}$$

for every $j \in J$ and $i = 1, \ldots, n_j$. Using that $G^+ \subseteq H$ and H is a subgroup of G, define $h_r \in H$ as

$$h_r = h \cdot (\prod_{j \in J} \prod_{i=1}^{n^j} \hat{x}^j_i)^{-1} \tag{6.13}$$

By Equations 6.10 and 6.11 for every $j \in J$ we have that $\pi_j(h_r) = e_{G_j}$ and so $h_r \in \mathrm{Cyc}(V \cap H)$ by Equation 6.9. It is clear then that

$$h = h_r \cdot \prod_{j \in J} \prod_{i=1}^{n^j} \hat{x}^j_i \tag{6.14}$$

by construction. From this, we have proven that $h \in \langle \mathrm{Cyc}(V \cap H) \rangle \subseteq \langle \mathrm{Cyc}(U \cap H) \rangle$ and so $H = \langle \mathrm{Cyc}(U \cap H) \rangle$, showing that H has the DW property. □

The next proposition is an analogue of the correspondent results for the class of SSGP groups due to Gould [25, Theorem 3.2.1 and 3.2.2]

Proposition 6.6.2. *The following statements hold in the class of* DW *groups.*

(i) The DW *class is closed under continuous surjective homomorphisms,*

(ii) The DW *class is closed under topological products,*

(iii) The DW *class is closed under direct sums; and*

(iv) The DW *class is closed under topological quotients.*

Proof. (i) Let G be an DW group. Suppose H is a topological group and $f : G \to H$ is a continuous surjective homomorphism. We shall prove that H has the DW property. Let $U \subseteq H$ be an open neighbourhood of e_H in H, by continuity of f the inverse image $V = f^{-1}[U] \subseteq G$ is an open neighbourhood of e_G in G.

Let $h \in H$ be arbitrary, since f is surjective let $g \in G$ such that $f(g) = h$. Since G is DW and V is an open neighbourhood of e_G, there exist $x_1 \ldots x_n \in \mathrm{Cyc}(V)$ such that $g = \prod_{j=1}^{n} x_j$. Since f is an homomorphism, we have that

$$f(g) = f(\prod_{j=1}^{n} x_j) = \prod_{j=1}^{n} f(x_j). \tag{6.15}$$

Finally, since f also maps subgroups of G to subgroups of H then for every $j = 1, \ldots, n$ we have that $f(x_j) \in \mathrm{Cyc}(f[V]) = \mathrm{Cyc}(U)$. Therefore $f(g) = h \in \langle Cyc(U) \rangle$, proving that the inclusion $H \subseteq \langle \mathrm{Cyc}(U) \rangle$ holds. Since the converse inclusion clearly holds, we conclude then that H has the DW property by definition.

(ii) If $\{G_i : i \in I\}$ is a family of indexed DW groups for some index set I then by Theorem 6.6.1 the product $G = \prod_{i \in I} G_i$ has the DW property.

(iii) Similar to (ii) by Theorem 6.6.1 the sum $G^+ = \bigoplus_{i \in I} G_i$ has the DW property.

(iv) Follows directly from (i). □

For the topic of inheritance, it is important to remember that minimally almost periodic groups cannot contain proper open subgroups [25, Observation 2.2.3]. From the get-go, we can see that our choice of additional properties for subgroups is a little restricted. As we shall see in this section, the topic of inheritance is rather bleak for DW groups, as even closed or dense subgroups fail to inehrit the property. To show this, we shall make use of the Hartman-Mycielski construction.

We pay attention to the following result by Dikranjan and the second listed author [16]:

Lemma 6.6.3 ([16] Lemma 8.2)**.** *Let $G = \bigoplus_{n\in\mathbb{N}} C_n$ where each C_n is a cyclic subgroup of order $a_n \in N^+ \cup \infty$. If infinitely many $a_n = \infty$ or $lim_n a_n = \infty$, then there exists a monomorphism $\varphi : G \to \mathrm{HM}(\mathbb{T})$ such that the subgroup $\varphi(G)$ of $\mathrm{HM}(\mathbb{T})$ is dense in $\mathrm{HM}(\mathbb{T})$ and has the* SSGP *property.*

Finally, we remind the reader of the following:

Fact 6.6.4. Suppose $G = \bigoplus_{p\in\mathbf{P}} G_p$ is a torsion group decomposed into its p-components. If $H \leq G$ is any subgroup of G and $H = \bigoplus_{p\in\mathbf{P}} H_p$ is its decomposition into p-components, then for every $p \in \mathbf{P}$ we have that $H_p \leq G_p$. In particular, any subgroup of G is a direct sum of subgroups of G.

With these tools in mind, we can show that the DW property may not be inherited to subgroups in general:

Example 6.6.5. Let $G = \bigoplus_{p\in\mathbf{P}} \mathbb{Z}(p)$ as in Example 6.3.6.

(i) G is naturally embedded as a closed subgroup of $\mathrm{HM}(G)$. By Fact 1.5.1 we have that $\mathrm{HM}(G)$ is DW. However, G itself does not admit an DW group topology, so it is not always inherited to closed subgroups.

(ii) Since G satisfies the hypotheses of Lemma 6.6.3, there exists an embedding $\varphi : G \to \mathrm{HM}(\mathbb{T})$ such that $\varphi(G)$ is dense in $\mathrm{HM}(\mathbb{T})$ and it has the SSGP property. By Fact 1.5.1 we have that $\mathrm{HM}(\mathbb{T})$ has the DW property. However, since G does not admit an DW topology then the DW property is not always inherited to dense subgroups.

(iii) Let $H \subseteq G$ be any non-trivial subgroup. By Fact 6.6.4, for every $p \in \mathbf{P}$ there exists a subgroup $H_p \leq \mathbb{Z}(p)$ such that $H = \bigoplus_{p\in\mathbf{P}} H_p$. If H is finite, then it is compact group and therefore cannot have an DW group topology. If H is infinite then by Theorem 6.3.4 we have that H does not admit an DW topology. With this we can conclude that G contains no non-trivial DW subgroups.

Finally there is one more possibility to explore. It is a well-known result that if a topological group has a dense SSGP subgroup, then it itself has the SSGP property. This, sadly, is not the case for DW groups as we shall see in the following example.

First, recall that a function called an *n-step function* if there exist a collection of exactly n disjoint intervals where the function attains a constant value. Finally, if we are given some function in $\mathrm{HM}(\mathbb{T})$, then by its *support* we mean the subset of I where it attains a non-zero value.

Example 6.6.6. There exists a non-DW Abelian torsion topological group with the SSGP property which contains dense DW subgroup.

Proof. Consider $\mathrm{HM}(\mathbb{T})$ and let X be the set of all 2-step functions on their support with rational end-points, but which also take values in $\bigcup_{p\in\mathbb{P}\setminus\{2\}} \mathbb{T}[p]$; where $\mathbb{T}[p] = \{t \in \mathbb{T} : pt = 0\}$. Consider the subgroup $H = \langle X\rangle$ of $\mathrm{HM}(\mathbb{T})$ generated by X and let f be the constant function on $[0, 1]$ which takes the unique non-zero value in $\mathbb{T}[2]$. By construction, H can be seen to be a dense subgroup of $\mathrm{HM}(\mathbb{T})$, and that $\langle f\rangle \cap H = \langle e\rangle$ holds.

We shall make use of the following claim in what follows.

Claim 67. Let $\epsilon > 0$ and U be an arbitrary neighbourhood of identity of G. Furthermore, suppose $J = [a, b)$ is some interval with rational end-points such that $b - a < \epsilon$. If we define

$$B(J) = \{f \in \mathrm{HM}(\mathbb{T}) : \text{f is constant on J and } f(t) = 0 \text{ for } t \in [0, 1) \setminus J\}$$

then $B(J)$ is a subgroup of X and $B(J) \subseteq O(U, \epsilon)$.

Proof. $B(J)$ can easily be seen to a subgroup of $HM(G)$ as we are utilizing the coordinate-wise operation. Additionally, $B(J)$ is a subset of X by virtue of every constant function on J also being a 2-step function on J. Let us then verify that the inclusion $B(J) \subseteq O(U, \epsilon)$ holds. Given $f \in B(J)$ we have that for all $t \in [0, 1) \setminus J$ the equality $f(t) = e \in U$ holds. This implies that

$$\mu(\{t \in I : f(t) \notin U\}) \leq b - a < \epsilon.$$

This shows that $f \in O(U, \epsilon)$ by Equation 1.5. Since $f \in B(J)$ was arbitrary, we have shown that $B(J) \subseteq O(U, \epsilon)$ holds, as desired. ☐

Claim 68. *H has the DW property.*

Proof. Observe that it suffices to show that $X \subseteq \langle \mathrm{Cyc}(O(U,\epsilon) \cap X) \rangle$ holds for every neighbourhood of identity U of G and every $\epsilon > 0$. Indeed, the previous would imply that

$$H = \langle X \rangle \subseteq \langle \mathrm{Cyc}(O(U,\epsilon) \cap X) \rangle \subseteq \langle \mathrm{Cyc}(O(U,\epsilon) \cap H) \rangle,$$

which implies that H inherits an DW group topology from $\mathrm{HM}(\mathbb{T})$. Let us then fix some function $f \in X$, a neighbourhood of identity U of G, and some $\epsilon > 0$. Denote by J the support of the function f. Since f is a two-step function in its support, there exist two disjoint subintervals J_1 and J_2 of J such that the restriction of f to J_i is constant for $i = 1, 2$ and $J = J_1 \cup J_2$.

For $i = 1, 2$ let us define the function $f_i \in G^I$ which satisfies that $f_i \restriction_{J_i} = f \restriction_{J_i}$ and $f_i(t) = e$ for every $t \in [0, 1) \setminus J_i$. By definition we have that $f_j \in B(J_i) \subseteq B(J)$ for $j = 1, 2$. Since $B(J) \subseteq O(U,\epsilon)$ and it is a subgroup of $\mathrm{HM}(\mathbb{T})$ by the previous claim, we have that $f_j \in \mathrm{Cyc}(O(U,\epsilon)$ for $j = 1, 2$. Since f_j is constant in its support, we also have that $f_j \in X$ for $j = 1, 2$. This implies that $f_j \in \mathrm{Cyc}(O(U,\epsilon) \cap X)$ for $j = 1, 2$. Finally, it is easily seen that $f = f_1 \cdot f_2$ by our construction, implying that $f \in \langle \mathrm{Cyc}(O(U,\epsilon) \cap X) \rangle$. Since $f \in X$ was arbitrary, we conclude that $X \subseteq \langle \mathrm{Cyc}(O(U,\epsilon) \cap X) \rangle$. This shows that H has an DW group topology. □

Consider the subgroup $G = \langle \{f\} \cup H \rangle = \langle f \rangle \oplus H$ of $\mathrm{HM}(\mathbb{T})$. Observe that H is also dense in G since it was already dense in $\mathrm{HM}(\mathbb{T})$, so G is a group that has a dense DW subgroup. Finally, observe that G is a torsion group with finite 2-rank equal to one, which is witnessed by its direct summand $\langle f \rangle$. By Theorem 6.3.4, the group G cannot admit an DW group topology. □

Appendix A

Reference of results on algebraic ranks

A.1 Special rank, p-groups, and a bound for the rank of their quotients

The goal of this section is to prove that the p-rank of a quotient group of an Abelian p-group of finite p-rank does not exceed the p-rank of the original group; see Lemma A.1.5. Even though this fact seems to be a part of folklore in group theory, we were unable to find any reference for it. The following notion will play a key role in the proof of Lemma A.1.5.

Definition A.1.1. For an Abelian group G, we use $r_{\text{fg}}(G)$ to denote the smallest natural number r such that every finitely generated subgroup H of G can be generated by at most r many of its elements, if such r exists; otherwise, we let $r_{\text{fg}}(G) = \omega$. We shall call $r_{\text{fg}}(G)$ the *finitely generated rank* of G.

In the literature, the cardinal $r_{\text{fg}}(G)$ is called also the *Prüfer rank* or *special rank* of G.

The following lemma relates ranks from Definitions 1.2.1(iv) and A.1.1.

Lemma A.1.2. *$r_{\text{fg}}(G) \leq r(G)$ for every Abelian group G.*

Proof. The inequality obviously holds if $r(G)$ is infinite, as $r_{\text{fg}}(G) \leq \omega$ by definition.

Suppose now that $r(G)$ is finite and let $r = r(G)$.

Take H to be an arbitrary finitely generated subgroup of G. Since G is Abelian, so is H. By the Frobenius-Stickelberger representation theorem, the isomorphism

$$H \simeq \mathbb{Z}^n \oplus \bigoplus_{j=i}^{m} \mathbb{Z}(p_j^{k_j}) \tag{A.1}$$

holds for some primes $p_1, \dots p_m \in \mathbf{P}$ and positive integers $n, k_1, \dots, k_m \in \mathbb{N}^+$. It easily follows from (A.1) that $r(H) = n + m$ and H is generated by $(n+m)$-many elements of H. Since $n + m = r(H) \leq r(G) = r$ by Fact 1.2.2, it follows that H can be generated by at most r-many of its elements. Since this holds for an arbitrary finitely generated subgroup H of G, this means that $r_{\mathrm{fg}}(G) \leq r$ by Definition A.1.1. □

The equality in Lemma A.1.2 need not hold even for torsion Abelian groups. Indeed, for different primes $p_1, \dots, p_k$, the torsion group $G = \mathbb{Z}(p_1 \cdots p_k)$ satisfies $r_{\mathrm{fg}}(G) = 1$ and $r(G) = k$.

The next result shows that the inequality in Lemma A.1.2 does become the equality for Abelian p-groups whose finitely generated rank is finite.

Lemma A.1.3. *Let p be a prime number and G be an Abelian p-group such that $r_{\mathrm{fg}}(G)$ is finite. Then $r_{\mathrm{fg}}(G) = r(G) = r_p(G)$.*

Proof. Since G is a p-group, $r_q(G) = 0$ for all $q \in (\mathbf{P} \cup \{0\}) \setminus \{p\}$, which implies $r(G) = r_p(G)$ by Definition 1.2.1(iv). By Lemma A.1.2, it remains only to establish the inequality $r_p(G) \leq r_{\mathrm{fg}}(G)$.

Let $S = \{g \in G : pg = 0\}$ be the socle of G. Clearly, $r_p(S) = r_p(G)$. By our assumption, $r_{\mathrm{fg}}(G) = r \in \mathbb{N}$. Suppose that $r_p(S) > r$. It follows from Definition 1.2.1(ii) that we can choose a p-independent subset X of S such that $|X| = r + 1$. Now $\langle X \rangle$ is a finitely generated subgroup of G which cannot be generated by at most r-many of its elements. [1] This means that $r_{\mathrm{fg}}(G) \geq r + 1$ holds, in contradiction with $r = r_{\mathrm{fg}}(G)$. This contradiction shows that $r_p(G) = r_p(S) \leq r = r_{\mathrm{fg}}(G)$. □

It is well-known that the rank of an Abelian group can increase when passing to its quotient group. Indeed, $r(\mathbb{Q}) = 1$, yet $r(\mathbb{Q}/\mathbb{Z}) = \omega$. This problem does not happen for finitely generated rank.

[1] Indeed, every element of S has order p. Since X is an independent subset of S, this implies that $|\langle X \rangle| = p^{|X|} = p^{r+1}$. On the other hand, if Y is a subset of $\langle X \rangle \subseteq S$ such that $|Y| \leq r$, then $|\langle Y \rangle| \leq p^r < p^{r+1} = |\langle X \rangle|$, so $\langle Y \rangle$ must be a proper subgroup of $\langle X \rangle$, which implies that Y does not generate $\langle X \rangle$.

Lemma A.1.4. *If H is a quotient group of an Abelian group G, then $r_{\mathrm{fg}}(H) \leq r_{\mathrm{fg}}(G)$.*

Proof. If $r_{\mathrm{fg}}(G) = \omega$, then $r_{\mathrm{fg}}(H) \leq \omega = r_{\mathrm{fg}}(G)$ by Definition A.1.1. From now on we can assume that $r_{\mathrm{fg}}(G) = n \in \mathbb{N}$.

Let $f : G \to H$ be a quotient homomorphism. and A be a finitely generated subgroup of H. Finally, fix Y to be a finite set of generators of A. Since f is a surjection, we can find a finite set X such that $f(X) = Y$. Since the subgroup $\langle X \rangle$ of G is obviously finitely generated, there exists a set $Z \subseteq G$ such that $|Z| \leq r_{\mathrm{fg}}(G) = n$ and $\langle Z \rangle = \langle X \rangle$. Since f is a homomorphism, we have that

$$f(Z) \subseteq \langle f(Z) \rangle = f(\langle Z \rangle) = f(\langle X \rangle) = \langle f(X) \rangle = \langle Y \rangle = A.$$

Since $|f(Z)| \leq |Z| \leq n$, we conclude that A can be generated by at most n-many elements of A. Since this holds for an arbitrary finitely generated subgroup A of H, we have $r_{\mathrm{fg}}(H) \leq n = r_{\mathrm{fg}}(G)$ by Definition A.1.1. □

If we observe the behavior of p-ranks, then the following problem persists in arbitrary groups: Let p be a fixed a prime number. Clearly $r_p(\mathbb{Q}) = 0$, yet $r_p(\mathbb{Q}/Z) = 1$ occurs. Our next result shows that this phenomenon does not happen in the class of Abelian p-groups.

Lemma A.1.5. *Let G be an Abelian p-group of finite p-rank. Then $r_p(H) \leq r_p(G)$ for every quotient group H of G.*

Proof. Suppose that $r_p(G) = n \in \mathbb{N}$. Then $r_{\mathrm{fg}}(G) \leq n$ by Lemma A.1.2.

Let H be a quotient group of G. Then $r_{\mathrm{fg}}(H) \leq r_{\mathrm{fg}}(G) \leq n$ by Lemma A.1.4. Being the quotient group of an Abelian p-group G, H itself is an Abelian p-group. Now $r_p(H) = r_{\mathrm{fg}}(H)$ by Lemma A.1.3. Thus, $r_p(H) \leq n = r_p(G)$. □

Corollary A.1.6. *The inequality $r(H) \leq r(G)$ holds for every quotient group H of an Abelian p-group G.*

A.2 An upper bound for the p-ranks of quotients of groups with finite ranks

Lemma A.2.1. *Let G be a divisible group with finite free rank and finite p-ranks. If H is a quotient subgroup of G then for every $p \in \mathbf{P}$ the inequality $r_p(H) \leq r_0(G) + r_p(G)$ holds.*

Proof. Since G is divisible, the isomorphism

$$G \simeq \mathbb{Q}^{r_0(G)} \oplus \bigoplus_{p \in \mathbf{P}} \mathbb{Z}(p^\infty)^{r_p(G)} \tag{A.2}$$

holds where $r_0(G)$ and $r_p(G)$ are used to denote the free rank of G and the p-rank of G for a prime $p \in \mathbf{P}$ respectively. Suppose that H is a quotient group of G, and that $\phi : G \to H$ is its natural quotient mapping.

Since H is a quotient of a divisible group, it must also be a divisible group. By utilizing Equation A.2 for H, we can see that H_p, the p-component of H, is a direct summand of H. This implies that the projection mapping $\pi_p : H \to H_p$ is a surjective homomorphism. This implies that the composition $\pi_p \circ \phi : G \to H_p$ is a surjective homomorphism.

Claim 69. Let $q \in \mathbf{P}$ be a prime distinct from p. For every element $g_q \in G_q$ of the q-component of G we have that $\pi_p \circ \phi(g_q) = 0$.

Proof. Since $g_q \in G_q$, there exists some $n \in \mathbb{N}^+$ such that q^n is the order of g_q, or g_q is the trivial element. Since the latter implies our desired conclusion, let us suppose that the former holds. We would then have that

$$e = \pi_p \circ \phi((g_q)^{q^n}) = [\pi_p \circ \phi(g_q)]^{q^n}.$$

Since $\pi_p \circ \phi(g_q)$ is an element of a p-group, its order is a power of the prime p or it is trivial. If the former holds, then by combining this with the previous equation, we would have that p must divide q^n. Since p and q are relatively prime, then this case is not possible. We can then conclude that $\pi_p \circ \phi(g_q) = 0$ as desired. □

Claim 70. The equality

$$\pi_p \circ \phi[\mathbb{Q}^{r_0(G)} \oplus Z(p^\infty)^{r_p(G)}] = H_p$$

holds.

Proof. Take some $h \in H_p$, since the mapping $\pi_p \circ \phi : G \to H_p$ is surjective there exists some $g \in G$ such that $\pi_p \circ \phi(g) = h$. By Equation A.2, we can find a finite subset of primes $P \subseteq \mathbf{P}$ such that $q \in \mathbf{P}$ and

$$g = g_0 + \sum_{q \in P} g_q$$

where g_0 is torsion-free (or neutral) and $g_q \in G_q$ is an element of the q-component of G. By our previous claim the equalities

$$h = \pi_p \circ \phi(g) = \pi_p \circ \phi(g_0) + \sum_{q \in P} \pi_p \circ f(g_q) = \pi_p \circ \phi(g_0) + \pi_p \circ \phi(g_q)$$

hold. This implies that $h = \pi_p \circ \phi(g_o + g_q)$ and therefore $h \in \pi_p \circ \phi[\mathbb{Q}^{r_0(G)} \oplus Z(p^\infty)^{r_p(G)}]$. Since $h \in H_p$ was arbitrary, we have shown that the inclusion

$$H_p \subseteq \pi_p \circ \phi[\mathbb{Q}^{r_0(G)} \oplus Z(p^\infty)^{r_p(G)}]$$

holds. The reverse inclusion clearly holds by the definition of $\pi_p \circ \phi$. □

Claim 71. For every prime $p \in \mathbb{P}$ the p-component H_p of H is a quotient group of $\mathbb{Q}^{r_0(G)} \oplus Z(p^\infty)^{r_p(G)}$

Proof. By our previous claim we have that the restriction of $\pi_p \circ \phi$ to the subgroup $\mathbb{Q}^{r_0(G)} \oplus Z(p^\infty)^{r_p(G)}$ is a surjective homomorphism, implying our claim. □

By Lemma A.1.4 we have that

$$r_{fg}(H_p) \leq r_{fg}(\mathbb{Q}^{r_0(G)} \oplus Z(p^\infty)^{r_p(G)}) \leq r_0(G) + r_p(G).$$

Since H_p is an Abelian p-group then by Lemmas A.1.2 and A.1.3 we have that $r_{fg}(H_p) = r_p(H_p) = r_p(H)$. Implying that

$$r_p(H) \leq r_0(G) + r_p(G)$$

as desired. □

Corollary A.2.2. *Let G be an Abelian group with finite free rank and finite p-ranks. Then $r_p(H) \leq r_0(G) + r_p(G)$ for every quotient group H of G.*

Proof. Suppose K is a subgroup of G and that $H = G/K$ is the quotient of G by K. Since G is naturally embeddable as a subgroup of its divisible hull D_G we can assume without loss of generality that $K \leq G \leq D_G$ where D_G is the divisible hull of G. By the third isomorphism theorem, the quotient group H is a subgroup of D_G/K. Since D_G is the divisible hull of G then $r_0(D_G) = r_0(G)$ and $r_p(D_G) = r_p(G)$ for every prime $p \in \mathbf{P}$. This implies that D_G is a divisible group of finite free rank and finite p-ranks. By our previous Lemma and Fact 1.2.2, we have that

$$r_p(H) \leq r_p(D_G/K) \leq r_0(D_G) + r_p(D_G) = r_0(G) + r_p(G)$$

holds as desired.

□

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