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学 位 論 文 要 旨
Dissertation Summary

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論　文　名 : Properties modelled on minimal almost periodicity, and small subgroup
(Dissertation Title)　generating properties

A topological group is called *minimally almost periodic* (MinAP) if all its continuous homomorphisms to compact groups are trivial. This class of groups was introduced in 1940 by von Neumann due to the need of distinguishing elements of a group via almost periodic functions, and the theory itself originated in Bohr's work in Harmonic Analysis.

Several important subclasses of MinAP groups have received significant attention recently. A topological group G has the DW *property* (SSGP *property*) if for every neighbourhood U of the identity of G, the union of cyclic subgroups of U algebraically generates G (a dense subgroup of G, respectively). Property DW appeared implicitly in 1978 in a paper of Dierolf and Warken, and served as an inspiration for the SSGP property introduced by Gould in 2009. In 2016, Dikranjan and Shakhmatov invented a chain of properties $\mathtt{SSGP}(\alpha)$ for every ordinal α lying between SSGP and MinAP properties. To facilitate the presentation, we say that a topological group is $\mathtt{SSGP}(\infty)$ if it has property $\mathtt{SSGP}(\alpha)$ for some ordinal α. Then

$$\text{DW} \to \text{SSGP} \to \mathtt{SSGP}(\infty) \to \text{MinAP}, \qquad (\dagger)$$

and none of the implications in (†) are reversible, even for Abelian topological groups.

In this thesis we investigate topological groups having one of the properties from (†), making special emphasis on their algebraic structure. Furthermore, in order to clarify the fine structure of the gap between $\mathtt{SSGP}(\infty)$ and MinAP properties, we introduce and analyze a host of "in-between" new properties modelled on the classical MinAP property.

Below we give a summary of main results obtained in this thesis.

In Chapter 1 we collect notations and preliminary results.

In Chapter 2 we introduce the property MinAP($\mathcal{C}$) for every class $\mathcal{C}$ of topological groups. We say that a topological group is MinAP($\mathcal{C}$) if all its non-trivial homomorphisms to any group contained in the class $\mathcal{C}$ are discontinuous. Note that MinAP(Compact) coincides with the classic MinAP property. We investigate the class MinAP($\mathcal{C}$) for classes $\mathcal{C}$ of locally compact groups (LC), Lie groups, and groups without small subgroups (known as NSS groups).

This triad of properties is well-known for being part of the solution to Hilbert's 5th Problem achieved by Montgomery-Zippin and Yamabe: a topological group is Lie if and only if it is both locally compact and NSS.

The following diagram stratifies the gap between properies $\mathtt{SSGP}(\infty)$ and MinAP.

$$
\begin{array}{ccccc}
 & & \mathtt{SSGP}(\infty) & & \\
 & & \downarrow & & \\
 & \swarrow & \text{MinAP(LC)} + \text{MinAP(NSS)} & \searrow & \\
 & & \downarrow & & \\
\text{MinAP(LC)} & \longrightarrow & \text{MinAP(Lie)} & \longleftarrow & \text{MinAP(NSS)} \\
 & \searrow & \downarrow & \swarrow & \\
 & & \text{MinAP} & & \\
 & & \| & & \\
 & & \text{MinAP(Compact)} & &
\end{array}
$$

Figure 2.1: Implications between MinAP($\mathcal{C}$) properties, where LC denotes Locally compact.

We prove that none of the above implications are reversible in general (Theorem 2.3.5), yet some of them become reversible for Abelian topological groups. Namely, for Abelian topological groups, three properties MinAP(Locally compact), MinAP(Lie) and MinAP coincide (Corollary 2.3.6), and MinAP(NSS) property coincides with $\mathtt{SSGP}(\infty)$ (Corollary 2.5.2). As a consequence, Figure 2.1 considerably simplifies in Abelian topological groups as follows:

$$
\begin{array}{ccccc}
\text{MinAP(NSS)} & \longleftrightarrow & \mathtt{SSGP}(\infty) & \longleftrightarrow & \text{MinAP(Locally compact)} + \text{MinAP(NSS)} \\
\downarrow & & & & \downarrow \\
\text{MinAP(Lie)} & \longleftrightarrow & \text{MinAP} & \longleftrightarrow & \text{MinAP(Locally compact)}
\end{array}
$$

Figure 2.2: Simplified implications in Abelian topological groups.

The group of integers $\mathbb{Z}$ admits a MinAP group topology, but not an $\mathtt{SSGP}(\infty)$ group topology. Therefore, the equivalent properties at the upper level of Figure 2.2 do not imply any of the properties at its lower level.

In Chapter 3 we introduce the modification of MinAP property for a given property $\mathbf{P}$ of topological groups. We say that a topological group G is MinAP *mod* $\mathbf{P}$ if, for each continuous homomorphism $f : G \to K$ from G to a compact group K, the image $f[G]$ of G considered as a subgroup of K has property $\mathbf{P}$. When $\mathbf{P}$ is the property of being the trivial group, MinAP mod $\mathbf{P}$ property coincides with the classical minimal almost periodicity. If $\mathbf{P}$ and $\mathbf{Q}$ are properties of topological groups such that $\mathbf{P} \to \mathbf{Q}$, then MinAP mod $\mathbf{P} \to$ MinAP mod $\mathbf{Q}$. We investigate five properties $\mathbf{P}$ in this respect: one set-theoretic (finite), two algebraic (torsion and bounded torsion) and two topological (compact and connected). They relate as follows:

$$
\begin{array}{ccccc}
\text{MinAP} & \longrightarrow & \text{MinAP mod finite} & \longrightarrow & \text{MinAP mod compact} \\
\downarrow & & \downarrow & & \\
\text{MinAP mod connected} & & \text{MinAP mod bounded} & \longrightarrow & \text{MinAP mod torsion.}
\end{array}
$$

Figure 3.1: Implications between some MinAP mod $\mathbf{P}$ properties.

We prove that none of the above implications are reversible (Examples 3.3.2 and 3.6.8). When $\mathbf{P}$ is an invariant of continuous homomorphisms, we completely characterize the MinAP mod $\mathbf{P}$ property:

Theorem 3.6.1 (and Corollary 3.6.3). If **P** is a topological property invariant of continuous homomorphisms, then the following conditions are equivalent for every Abelian group G:

(i) G is MinAP mod **P**,

(ii) the image of G in its Bohr compactification under the canonical homomorphism has property **P**,

(iii) the quotient of G with respect to its von Neumann kernel equipped with its Bohr topology has property **P**.

We prove that *every* Abelian group admits a group topology which is MinAP mod **P** for **P** being one of the following properties: finite, bounded, compact, and torsion (Corollary 3.9.6). We also show that an Abelian group G admits a MinAP mod connected group topology if and only if it admits a MinAP group topology (Corollary 3.8.2). These results provide the description of the algebraic structure of MinAP mod **P** Abelian groups for *all* properties **P** in Figure 3.1.

In 2014, answering a long-standing question of Comfort and Protasov, Dikranjan and Shakhmatov described the algebraic structure of MinAP groups by proving that an Abelian group G admits a MinAP group topology if and only if G is connected with respect to its Markov-Zariski group topology; the latter happens precisely when, for every natural number n, the subgroup $nG = \{ng : g \in G\}$ of G is either trivial or infinite. After this, Comfort and Gould asked for a characterization of Abelian groups which admit an SSGP group topology. In 2016, Dikranjan and Shakhmatov obtained an "almost complete" description of the algebraic structure of SSGP *and* $\mathtt{SSGP}(\infty)$ groups leaving open a single remaining case resolved in Chapter 4. The complete solution is given in the following

Theorem 4.1.1. The following are equivalent for an Abelian group G:

(a) G admits an SSGP group topology,

(b) G admits an $\mathtt{SSGP}(\infty)$ group topology, and

(c) either G has infinite divisible rank $r_d(G) = \min\{r(nG) : n \text{ is a positive integer}\}$, or the quotient $H = G/t(G)$ of G by its torsion part $t(G)$ has finite 0-rank, and the quotient H/A has infinite rank for some (equivalently, every) free subgroup A of H such that H/A is torsion.

In Chapter 5 we construct group topologies with property DW on free groups of infinite rank (Theorem 5.1.2). When the rank is countably infinite, this topology can be chosen to be metric (Theorem 5.1.1). Whether the DW group topologies can be made metric in the uncountable case remains open. The existence of group topologies with property DW on finitely generated free groups remains unclear as well.

In Chapter 6 we obtain partial results on the algebraic structure of Abelian DW groups of finite 0-rank. The following theorem provides necessary conditions:

Theorem 6.5.1. Let G be an Abelian group of finite 0-rank $r_0(G)$. If G admits a DW group topology, then either one of the following holds:

(i) If $r_0(G) = 0$ (i.e. G is torsion), then every non-trivial p-component of G admits a *DW* group topology.

(ii) If $0 < r_0(G)$, then there exists prime number p such that the p-component of G has infinite divisible rank.

Let n be a positive integer. We show in Corollary 6.5.5 that $\mathbb{Q}^n$ has an SSGP group topology, yet the only subgroup of $\mathbb{Q}^n$ which admits a DW group topology is the trivial group.