Anisotropic interpolation error analysis using a new geometric parameter and its applications

A Dissertation Submitted to the Graduate School of Science and Engineering of Ehime University for the Degree of Doctor of Philosophy in Mathematics

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Preface

This thesis is devoted to estimating the anisotropic interpolation error obtained using a new geometric parameter. We propose a new geometric condition for partitioning meshes, enabling anisotropic meshes to be used to estimate the optimal interpolation error. Anisotropic meshes are effective in problems in which, for example, the solution has anisotropic behaviour in some direction of the domain. Nevertheless, constructing accurate and efficient finite-element schemes to solve partial differential equations in various domains is challenging. Estimations of interpolation error are essential in ensuring the validity of the schemes; their accuracy sometimes depends on geometric conditions imposed on the meshing of the domain.

As far as is known, the first suggestion of a condition for a geometric mesh in obtaining error estimates of linear interpolation appeared in 1957 in [83, Section 3.8]. Called *Synge's condition*, it states that the maximum angle of each triangle in a mesh is smaller than some constant $< \pi$, see Section 2.2. In 1968, *Zlámal's condition* was proposed for ensuring the convergence of a finite-element approximation, [90]. Later, the *shape-regularity* condition was proposed that uses a ball containing an element. Many studies have imposed the shape-regularity condition on a family of meshes [19, 22, 25, 30, 31, 75]; specifically, in the shape-regular family of triangulations, the triangles or tetrahedra cannot become too flat. Furthermore, it is known that these four conditions, including both Zlámal's and the shape-regularity condition, are equivalent; see [20, Theorem 1] and Section 2.1. Because the shape-regularity condition has remarkable properties, this condition has remained a standard in finite-element error analysis. For example, a nondegenerate mesh is locally quasi-uniform in two or three dimensions. This property was used, for instance, in the analysis of the Scott–Zhang interpolation [80].

In 1976, several authors [13, 15, 39, 51] independently extended the result of [83] for two-dimensional meshes. In particular, the well-known paper of Babuška and Aziz [13] calls the mesh condition the *maximum-angle condition*. A certain family of triangulations subject to the maximum-angle condition allows the use of anisotropic finite-element meshes. Anisotropic meshes have different mesh sizes in different directions; imposing the shape-regularity assumption on triangulations is no longer valid for these meshes. If the maximum-angle condition is not valid, optimal interpolation properties are lost [13] and, in consequence, a finite-element solution may not converge to an exact solution. Later, in 1991, Křížek showed that the maximal-angle condition is equivalent to the circumscribed ball condition in two dimensions [57]. The associated family of mesh partitions is called *semiregular*; see also Sections 2.2 and 2.3. In 1992, Synge's condition was extended to tetrahedral elements [58]. However, the semiregular condition, which implies that simplicial interpolation error estimates preserve their optimal order of general Sobolev norms, has still not been obtained. Therefore, the main goal of this thesis is to derive the semiregular condition for three-dimensional meshes; see Sections 2.3 and 4.3. The key advance is to introduce a new geometric parameter H_T proposed in [47]; see also Section 4.1. Because this new geometric parameter is used in the interpolation error analysis, the coefficients in the error estimations are independent of the geometry of the simplices, and so the error estimations obtained may be applied to arbitrary meshes, including very "flat" or anisotropic simplices. Furthermore, while being sufficient to obtain optimal order interpolation error estimates, this geometric condition also appears to be simpler than Synge's (maximum-angle) condition. The quantity $\frac{H_T}{h_T}$ (see Chapter 4) is easily calculated numerically using finite-element methods. Therefore, the new condition may be useful, for example, in *a posteriori* error analyses. In a recent paper [50], the new condition was shown to be satisfied if and only if the maximum-angle condition holds. We expect the new mesh condition to become an alternative to the maximum-angle condition.

Anisotropic interpolation theory was developed in [10, 4, 23]. In some circumstances, the shape-regularity condition was not needed in obtaining optimal interpolation error estimates. The idea of Apel *et al.* was to construct a set of functionals satisfying conditions (5.5.2) in Section 5.5. Under the maximum-angle and coordinate system conditions, anisotropic interpolation error estimates were then deduced (e.g., see [4]). In contrast, Ishizaka *et al.* developed new estimations of the interpolation error within a general framework, derived the Raviart–Thomas interpolations on d-simplices, and proposed the new parameter H_T; see [47, 49, 46]. The heart of our analysis is to deduce a set of inequalities by a scaling argument; see Sections 5.3 and 9.4. Using these inequalities enables *delicate* interpolation error estimates to be obtained. Furthermore, imposing an additional mesh condition (Condition 3.3.1) yields anisotropic error estimates; see Theorems 5.6.1, 6.2.3, 7.2.1, 8.2.3, and 9.6.1.

This thesis gives three applications: the Crouzeix–Raviart approximation for the Poisson and Stokes equations, and the dual mixed approximation for the Poisson equation; see Chapter IV. For shape-regular mesh partitions, the Crouzeix–Raviart finite-element error estimates for the non-homogeneous Dirichlet Poisson and Stokes problems are known; see [21, 22, 30, 32, 41, 52, 76] and [63] for the modified Crouzeix–Raviart approximation of the Stokes problem. Applications to anisotropic finite-element methods are found in [4, 7, 8] for the second-order elliptic boundary-value problems, in [6, 59] for singularly perturbed problems, in [9, 5] for anisotropic phenomena in the

Stokes and Navier–Stokes problems, in [34, 60, 26] for anisotropic *a posteriori* error estimates, and in [23, 68] for fourth-order elliptic boundary-value problems.

The Crouzeix–Raviart finite-element space (Chapter 7) is nonconforming in $H_0^1(\Omega)$ and the Morley finite-element space, which is attractive for fourth-order problems, is nonconforming in $H_0^2(\Omega)$ (Chapter 8), where Ω is a polyhedral domain in $\mathbb{R}^d$, $d \in \{2, 3\}$. Therefore, an error between the exact solution and the nonconforming finite-element approximation solution with a H^1 or H^2-broken seminorm is divided into two parts (e.g., see [21, 30]). One is the interpolation error that measures how well the exact solution is approximated by the Crouzeix–Raviart (or Morley) finite-element functions; the other is a consistency error. For the former, the Crouzeix–Raviart interpolation error estimates (Theorem 7.3.1) or the Morley interpolation error estimates (Theorem 8.3.1) are used. For the latter, the standard scaling argument and the trace theorem are often used to obtain the error estimates. However, the situation is different without the shape-regularity condition. In this way, we could not derive the correct order for anisotropic meshes. To overcome this difficulty, we use the relation between the first-order Crouzeix–Raviart and the lowest-order Raviart–Thomas finite-element spaces (Lemma 11.6.1) as well as the lowest-order Raviart–Thomas interpolation error estimates for the anisotropic meshes (Theorem 9.7.3). This technique subsequently obtained the error estimates for the H^1 or H^2-broken seminorm of the anisotropic meshes for the second or fourth-order elliptic boundary value problems.

We present an error estimate for the first-order Raviart–Thomas finite-element approximation of the Poisson problem based on the dual mixed formulation (Chapter 12). The critical point is to show the discrete inf-sup condition (Lemma 12.3.2). To this end, we use the stability estimate of the Raviart–Thomas interpolation (Lemma 9.7.2) while imposing the new geometric condition (Condition 4.3.1).

In Chapter 13, we present the equivalence of the enriched piecewise-linear Crouzeix–Raviart finite-element method introduced by [45] and the first-order Raviart–Thomas finite-element method. In two dimensions, the work [12] represents pioneering research. Furthermore, Marini [69] found an expression relating the Raviart–Thomas and the Crouzeix–Raviart finite-element methods; specifically,

$$\bar{\sigma}_h^{RT}|_T = \nabla \bar{u}_h^{CR} - \frac{f_T^0}{2}(x - x_T) \quad \text{on } T,$$

$$\bar{u}_h^{RT}|_T = \Pi_T^0 \bar{u}_h^{CR} + \frac{f_T^0}{48} \sum_{i=1}^{3} |x_i - x_T|^2,$$

where T denotes a mesh element, x_i $(i = 1, 2, 3)$ the vertices of triangle T, x_T the barycentre of T such that $x_T := \frac{1}{3}(x_1 + x_2 + x_3)$, $(\bar{\sigma}_h^{RT}, \bar{u}_h^{RT})$ and $\bar{u}_h^{CR}$ denote respectively the Raviart–Thomas and Crouzeix–Raviart finite-element solutions with a given external piecewise-constant function f_T^0, and $\Pi_T^0 \bar{u}_h^{CR} = \frac{1}{|T|} \int_T \bar{u}_h^{CR} dx$. A proof in [45] was given recently stating that the enriched piecewise-linear Crouzeix–Raviart finite-element method is identical to the first-order Raviart–Thomas finite-element method for both the Poisson and Stokes problems in any number of dimensions. In the present paper, we extend Marini's results to three dimensions (Lemma 13.2.2).

In Chapter 14, we consider the Crouzeix–Raviart approximate problem for the Stokes equation. In particular, we introduce a well-balanced scheme (which is also called a *pressure robust scheme*) proposed in [63] and in [5] under the maximum-angle condition (Section 14.5). A well-balanced scheme can be desirable even if a body force is not curl-free, but has a relatively large curl-free component ([32, Remark 53.22]). The stability of the well-balanced scheme holds under the boundedness of the quantity $\frac{H_T}{h_T}$ on each element, e.g., under the new geometric condition (Condition 4.3.1). Therefore, this new parameter may be useful when creating mesh sequences satisfying the *a priori* estimate.

This thesis is organised into five parts. The first introduces the notation used in this thesis and reviews the mathematical tools required to apply finite-element methods. The second presents the new geometric parameter and introduces the geometric conditions for mesh partitions used in the error analysis of the finite-element methods. The third derives the interpolation error estimates, which are used on the anisotropic meshes. The fourth discusses the nonconforming approximation and the dual mixed formulation of Poisson problems on anisotropic meshes. The fifth is a compilation of appendices.

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Part I

Introduction

Chapter 1

Preliminalies

1.1 General Convention

Throughout this thesis, c denotes a constant independent of h (defined later) unless specified otherwise. Those values may change in each context.

1.2 Basic Notation

d	The space dimension, $d \in \{2, 3\}$
$\mathbb{R}^d$	d-dimensional real Euclidean space
$\mathbb{N}_0$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{R}_+$	The set of positive real numbers
$\lvert D\rvert := \mathrm{meas}(D)$	Lebesgue measure of $D \subset \mathbb{R}^d$
$v\vert_D$	Restriction of the function v to the set D
$\dim(V)$	Dimension of the vector space V
δ_{ij}	Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise
$(x_1, \ldots, x_d)^T$	Cartesian coordinates in $\mathbb{R}^d$

1.3 Vectors and Matrices

$(v_1, \ldots, v_d)^T$	Cartesian components of the vector v in $\mathbb{R}^d$
$x \cdot y$	Euclidean scalar product in $\mathbb{R}^d$: $x \cdot y := \sum_{i=1}^{d} x_i y_i$
$\lvert x\rvert_E$	Euclidean norm in $\mathbb{R}^d$: $\lvert x\rvert_E := (x \cdot x)^{1/2}$
$\mathbb{R}^{m \times n}$	Vector space $m \times n$ matrices with real-valued entries
$\mathcal{A}$, $\mathcal{B}$	Matrices

$\mathcal{A}_{ij}$ or $[\mathcal{A}]_{ij}$	Entry of $\mathcal{A}$ in the ith and the jth column
$\mathcal{A}^T$	Transpose of the matrix $\mathcal{A}$
$\mathrm{Tr}(\mathcal{A})$	Trace of $\mathcal{A}$: For $\mathcal{A} \in \mathbb{R}^{m\times n}$, $\mathrm{Tr}(\mathcal{A}) := \sum_{i=1}^{d} \mathcal{A}_{ii}$
$\det(\mathcal{A})$	Determinant of $\mathcal{A}$
$\mathrm{diag}(\mathcal{A})$	Diagonal of $\mathcal{A}$: For $\mathcal{A} \in \mathbb{R}^{m\times n}$, $\mathrm{diag}(\mathcal{A})_{ij} := \delta_{ij}\mathcal{A}_{ij}$, $1 \le i, j \le d$
$\mathcal{A}x$	Matrix-vector product: For $\mathcal{A} \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$, $(\mathcal{A}x)_i := \sum_{j=1}^{d} \mathcal{A}_{ij}x_j$ for $1 \le i \le d$
$\mathcal{A} : \mathcal{B}$	Double contraction: For $\mathcal{A} \in \mathbb{R}^{m\times n}$ and $\mathcal{B} \in \mathbb{R}^{m\times n}$, $\mathcal{A} : \mathcal{B} := \sum_{i=1}^{m}\sum_{j=1}^{n} \mathcal{A}_{ij}\mathcal{B}_{ij}$
$\lVert\mathcal{A}\rVert_2$	Operator norm of $\mathcal{A}$: For $\mathcal{A} \in \mathbb{R}^{d\times d}$, $\lVert\mathcal{A}\rVert_2 := \sup_{0\neq x\in\mathbb{R}^d} \frac{\lvert\mathcal{A}x\rvert_E}{\lvert x\rvert_E}$
$\lVert\mathcal{A}\rVert_{\max}$	Max norm of $\mathcal{A}$: For $\mathcal{A} \in \mathbb{R}^{d\times d}$, $\lVert\mathcal{A}\rVert_{\max} := \max_{1\le i,j\le d} \lvert\mathcal{A}_{ij}\rvert$
$O(d)$	$O(d)$ consists of all orthogonal matrices of determinant ± 1

In this thesis, we use the following facts.

For $\mathcal{A} \in \mathbb{R}^{m\times n}$, it holds that

$$\lVert\mathcal{A}\rVert_{\max} \le \lVert\mathcal{A}\rVert_2 \le \sqrt{mn}\lVert\mathcal{A}\rVert_{\max}, \tag{1.3.1}$$

e.g., see [38, p. 56]. For $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{m\times m}$, it holds that

$$\lVert\mathcal{A}\mathcal{B}\rVert_2 \le \lVert\mathcal{A}\rVert_2\lVert\mathcal{B}\rVert_2. \tag{1.3.2}$$

If $\mathcal{A}^T\mathcal{A}$ is a positive definite matrix in $\mathbb{R}^{d\times d}$, the spectral norm of the matrix $\mathcal{A}^T\mathcal{A}$ is the largest eigenvalue of $\mathcal{A}^T\mathcal{A}$; i.e.,

$$\lVert\mathcal{A}\rVert_2 = \big(\lambda_{\max}(\mathcal{A}^T\mathcal{A})\big)^{1/2} = \sigma_{\max}(\mathcal{A}), \tag{1.3.3}$$

where $\lambda_{\max}(\mathcal{A})$ and $\sigma_{\max}(\mathcal{A})$ are respectively the largest eigenvalues and singular values of $\mathcal{A}$.

If $\mathcal{A} \in O(d)$, because $\mathcal{A}^T = \mathcal{A}^{-1}$ and

$$\lvert\mathcal{A}x\rvert_E^2 = (\mathcal{A}x)^T(\mathcal{A}x) = x^T\mathcal{A}^T\mathcal{A}x = x^T\mathcal{A}^{-1}\mathcal{A}x = \lvert x\rvert_E^2,$$

it holds that

$$\lVert\mathcal{A}\rVert_2 = \sup_{0\neq x\in\mathbb{R}^d} \frac{\lvert\mathcal{A}x\rvert_E}{\lvert x\rvert_E} = \sup_{0\neq x\in\mathbb{R}^d} \frac{\lvert x\rvert_E}{\lvert x\rvert_E} = 1.$$

1.4 Function Spaces

In this thesis, we mainly use notations in [31, 37].

1.4.1 Lipschitz sets and domains

Definition 1.4.1 (Domains)**.** Let D be a domain of $\mathbb{R}^d$, that is, D is open, bounded, and connected. Remark that the conditions include "bounded".

Definition 1.4.2 (Lipschitz set and domain)**.** A open set D in $\mathbb{R}^d$ is said to be Lipschitz if for any $x \in \partial D$, there exists a neighbourhood V_x of x in $\mathbb{R}^d$, a rotation $R_x : \mathbb{R}^d \to \mathbb{R}^d$, and two real numbers $c_1 > 0$ and $c_2 > 0$ (c_1 and c_2 may depend on x) such that the following holds true:

(I) $V_x = x + R_x(B_{c_1} \times I_{c_2})$ with $B_{c_1} := B_{\mathbb{R}^{d-1}}(0, c_1)$, $I_{c_2} := (-c_2, c_2)$,

(II) There exists a Lipschitz function $\varphi_x : B_{c_1} \to \mathbb{R}$ such that $\varphi(0) = 0$, $\|\varphi\|_{L^\infty(B_{C_1})} \leq \frac{1}{2}c_2$, and

$$D \cap V_x = x + R_x \left(\{(y', y_d) \in B_{c_1} \times I_{c_2}]\ y_d < \varphi_x(y')\}\right), \tag{1.4.1a}$$
$$\partial D \cap V_x = x + R_x \left(\{(y', y_d) \in B_{c_1} \times I_{c_2}]\ y_d = \varphi_x(y')\}\right). \tag{1.4.1b}$$

We say that D is a Lipschitz domain if it is a domain and a Lipschitz set.

1.4.2 Sobolev Spaces

Let D be an open set in $\mathbb{R}^d$.

The symbol $\mathcal{C}^0(D)$ denotes the space of continuous functions defined in D and

$$\mathcal{C}^\ell(D) := \left\{\varphi \in \mathcal{C}^0(D);\ \partial^\alpha \varphi \in \mathcal{C}^0(D)\ \forall |\alpha| \leq \ell\right\}.$$

We also introduce the spaces

$$\begin{aligned}\mathcal{C}^\ell(\overline{D}) &:= \big\{\varphi \in \mathcal{C}^\ell(D);\ \partial^\alpha \varphi \text{ are bounded and uniformly continuous on } D \\ &\qquad 0 \leq \forall |\alpha| \leq \ell\big\},\\ \mathcal{C}^{\ell,1}(\overline{D}) &:= \big\{\varphi \in \mathcal{C}^\ell(\overline{D});\ \partial^\alpha \varphi \text{ are Lipschitz-continuous in } \overline{D},\ 0 \leq \forall |\alpha| \leq \ell\big\}.\end{aligned}$$

The spaces $\mathcal{C}^\ell(\overline{D})$ and $\mathcal{C}^{\ell,1}(\overline{D})$ are Banach spaces with norms

$$\begin{aligned}\|\varphi\|_{\mathcal{C}^\ell(\overline{D})} &:= \max_{0 \leq |\alpha| \leq \ell} \sup_{x \in D} |\partial^\alpha \varphi(x)|,\\ \|\varphi\|_{\mathcal{C}^{\ell,1}(\overline{D})} &:= \|\varphi\|_{\mathcal{C}^\ell(\overline{D})} + \max_{0 \leq |\alpha| \leq \ell} \sup_{x,y \in D, x \neq y} \frac{|\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)|}{|x-y|_E}.\end{aligned}$$

We denote by $\mathcal{C}_0^\infty(D)$ the space composed of the functions from D to $\mathbb{R}$ that are $\mathcal{C}^\infty$ and whose support in D is compact. The members of $\mathcal{C}_0^\infty(D)$ are called test functions.

Let p' be conjugate of p with $\dfrac{1}{p}+\dfrac{1}{p'}=1$. Let p^* be such that $\dfrac{1}{p^*}=\dfrac{1}{p}-\dfrac{1}{d}$, and $p^*=+\infty$ if $p=d$.

Let $\alpha := (\alpha_1,\dots,\alpha_d)^T \in \mathbb{N}_0^d$ be a multi-index. For the multi-index α, let

$$\partial^\alpha := \partial_x^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\dots\left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_d^{\alpha_d}} \quad \text{with} \quad |\alpha| := \sum_{i=1}^d \alpha_i.$$

Let ℓ be a nonnegative integer and $p\in\mathbb{R}$ with $1\le p\le\infty$. We define the Sobolev space

$$W^{\ell,p}(D) := \{\varphi\in L^p(D);\ \partial^\alpha\varphi\in L^p(D),\ 0\le|\alpha|\le\ell\},$$

equipped with the norms

$$\|\varphi\|_{W^{\ell,p}(D)} := \left(\sum_{0\le|\alpha|\le\ell}\|\partial^\alpha\varphi\|^p_{L^p(D)}\right)^{1/p} \quad \text{if } 1\le p<\infty,$$

$$\|\varphi\|_{W^{\ell,\infty}(D)} := \max_{0\le|\alpha|\le\ell}\left(\operatorname*{ess.sup}_{x\in D}|\partial^\alpha\varphi(x)|\right).$$

We use the semi-norms

$$|\varphi|_{W^{\ell,p}(D)} := \left(\sum_{|\alpha|=\ell}\|\partial^\alpha\varphi\|^p_{L^p(D)}\right)^{1/p} \quad \text{if } 1\le p<\infty,$$

$$|\varphi|_{W^{\ell,\infty}(D)} := \max_{|\alpha|=\ell}\left(\operatorname*{ess.sup}_{x\in D}|\partial^\alpha\varphi(x)|\right).$$

If $p=2$, we use the notation

$$H^\ell(D) := W^{\ell,2}(D), \quad L^2(D) := H^0(D).$$

The space $H^\ell(D)$ is a Hilbert space equipped with the scalar product

$$(\varphi,\psi)_{H^\ell(D)} := \sum_{|\alpha|\le\ell}(\partial^\alpha\varphi,\partial^\alpha\psi)_{L^2(D)},$$

where $(.,.) := (.,.)_{L^2(D)}$ denotes the L^2-inner product, which leads to the norm and semi-norm

$$\|\varphi\|_{H^\ell(D)} := \left(\sum_{|\alpha|\le\ell}\|\partial^\alpha\varphi\|^2_{L^2(D)}\right)^{1/2}, \quad |\varphi|_{H^\ell(D)} := \left(\sum_{|\alpha|=\ell}\|\partial^\alpha\varphi\|^2_{L^2(D)}\right)^{1/2},$$

where we also use $\|\cdot\| := \|\cdot\|_{L^2(D)}$. Furthermore, we define

$$W_0^{\ell,p}(D) := \overline{\mathcal{C}_0^\infty(D)}^{W^{\ell,p}(D)},$$

that is, $W_0^{\ell,p}(D)$ is the closure of $\mathcal{C}_0^\infty(D)$ for the norm $\|\cdot\|_{W^{\ell,p}(D)}$.

The dual space of $W^{\ell,p}(D)$ is defined $\mathcal{L}(W^{\ell,p}(D);\mathbb{R})$ and denoted by $W^{\ell,p}(D)'$. The space $W^{\ell,p}(D)'$ is a Banach space with norm

$$\|\chi\|_{W^{\ell,p}(D)'} := \sup_{v\in W^{\ell,p}(D)} \frac{|\chi(v)|}{\|v\|_{W^{\ell,p}(D)}} \quad \forall\chi \in W^{\ell,p}(D)'.$$

The symbol $W^{\ell,p}(D)^d$ denotes $\mathbb{R}^d$-valued functions whose components are in $W^{\ell,p}(D)$. For any $v = (v_1, \dots, v_d)^T \in W^{\ell,p}(D)^d$, the norm is defined by

$$\|v\|_{W^{\ell,p}(D)^d} := \left(\sum_{i=1}^d \|v_i\|^p_{W^{\ell,p}(D)}\right)^{1/p}.$$

1.4.3 Fractional-order Spaces

Let $s \in (0,1)$ and $p \in [1,\infty]$. We define

$$W^{s,p}(D) := \left\{\varphi \in L^p(D);\ |\varphi|_{W^{s,p}(D)} < \infty\right\},$$

where

$$|\varphi|_{W^{s,p}(D)} := \left(\int_D\int_D \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|_E^{sp+d}}dxdy\right)^{1/p}, \quad p<\infty,$$
$$|\varphi|_{W^{s,\infty}(D)} := \operatorname*{ess.sup}_{x,y\in D} \frac{|\varphi(x)-\varphi(y)|}{|x-y|_E^s}.$$

Setting $s > 1$, we define

$$W^{s,p}(D) := \{\varphi \in W^{m,p}(D); \partial^\alpha\varphi \in W^{\sigma,p}(D),\ \forall\alpha\ |\alpha| = m\},$$

where $m := \lfloor s\rfloor$ and $\sigma := s - m$. We denote $H^s(D) := W^{s,2}(D)$.

1.5 Finite-Element-Methods-Related Symbols

1.5.1 Symbols

$\mathcal{P}^k$	Vector space of polynomials in the variables $x_1, \ldots, x_d$ of global degree at most $k \in \mathbb{N}_0$
$N^{(d,k)}$	$N^{(d,k)} := \dim(\mathcal{P}^k) = \binom{d+k}{k}$
RT^k	The Raviart–Thomas polynomial space of order $k \in \mathbb{N}_0$ as $RT^k := (\mathcal{P}^k)^d + x\mathcal{P}^k$ for any $x \in \mathbb{R}^d$
$N^{(RT)}$	$N^{(RT)} := \dim RT^k$
T, T^s, $\widetilde{T}$, $\widehat{T}$, K	Closed simplices in $\mathbb{R}^d$
$\mathcal{P}^k(T)$, $RT^k(T)$	$\mathcal{P}^k(T)$ (or $RT^k(T)$) is spanned by the restriction to T of polynomials in $\mathcal{P}^k$ (or RT^k)

1.5.2 Meshes

Throughout this thesis, let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded polyhedral domain. Furthermore, we assume that Ω is convex if necessary. Let $\mathbb{T}_h = \{T\}$ be a simplicial mesh of $\overline{\Omega}$ made up of closed d-simplices, such as

$$\overline{\Omega} = \bigcup_{T \in \mathbb{T}_h} T,$$

with $h := \max_{T \in \mathbb{T}_h} h_T$, where $h_T := \operatorname{diam}(T)$. We also use a symbol ρ_T which means the radius of the largest ball inscribed in T. We assume that each face of any d-simplex T_1 in $\mathbb{T}_h$ is either a subset of the boundary $\partial\Omega$ or a face of another d-simplex T_2 in $\mathbb{T}_h$. That is, $\mathbb{T}_h$ is a simplicial mesh of $\overline{\Omega}$ without hanging nodes. Such mesh $\mathbb{T}_h$ is said to be conformal. Let $\{\mathbb{T}_h\}$ be a family of conformal meshes.

Let T be a simplex of $\mathbb{T}_h$ which is a convex full of $d+1$ vertices, $P_1, \ldots, P_{d+1}$, that do not belong to the same hyperplane. Let S_i be the face of a simplex T opposite to the vertex P_i. For $d = 3$, angles between faces of a tetrahedron are called *dihedral*, whereas angles between its edges are called *solid*.

1.5.3 Broken Sobolev Spaces

We adopt the concepts of mesh faces, averages and jumps, e.g., see [30, 74]. Let $\mathcal{F}_h^i$ be the set of interior faces and $\mathcal{F}_h^{\partial}$ the set of the faces on the boundary $\partial\Omega$. We set $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^{\partial}$. For any $F \in \mathcal{F}_h$, we define the unit normal n_F to F as follows: (i) If $F \in \mathcal{F}_h^i$ with $F = T_1 \cap T_2$, $T_1, T_2 \in \mathbb{T}_h$, let n_1 and n_2 be the outward unit normals of T_1 and T_2, respectively. Then, n_F is either of $\{n_1, n_2\}$; (ii) If $F \in \mathcal{F}_h^{\partial}$, n_F is the unit outward normal n to $\partial\Omega$. We also use the following set. For any $F \in \mathcal{F}_h$,

$$\mathbb{T}_F := \{T \in \mathbb{T}_h : \ F \subset T\}.$$

We consider $\mathbb{R}^q$-valued functions for some $q \in \mathbb{N}$. Let $p \in [1, \infty]$ and $s > 0$ be a positive real number. We define a broken (piecewise) Sobolev space as

$$W^{s,p}(\mathbb{T}_h; \mathbb{R}^q) := \{v \in L^p(\Omega; \mathbb{R}^q) : \ v|_T \in W^{s,p}(T; \mathbb{R}^q) \ \forall T \in \mathbb{T}_h\}$$

with the norms

$$\|v\|_{W^{s,p}(\mathbb{T}_h;\mathbb{R}^q)} := \left(\sum_{T \in \mathbb{T}_h} \|v\|^p_{W^{s,p}(T;\mathbb{R}^q)} \right)^{1/p} \qquad \text{if } p \in [1, \infty),$$
$$\|v\|_{W^{s,\infty}(\mathbb{T}_h;\mathbb{R}^q)} := \max_{T \in \mathbb{T}_h} \|v\|_{W^{s,\infty}(T;\mathbb{R}^q)}.$$

When $q = 1$, we denote $W^{s,p}(\mathbb{T}_h) := W^{s,p}(\mathbb{T}_h; \mathbb{R})$. When $p = 2$, we write $H^s(\mathbb{T}_h; \mathbb{R}^q) := W^{s,2}(\mathbb{T}_h; \mathbb{R}^q)$ and $H^s(\mathbb{T}_h) := W^{s,2}(\mathbb{T}_h; \mathbb{R})$. We use the norm

$$|\varphi|_{H^1(\mathbb{T}_h)} := \left(\sum_{T \in \mathbb{T}_h} \|\nabla \varphi\|^2_{L^2(T)^d} \right)^{1/2} \qquad \varphi \in H^1(\mathbb{T}_h).$$

Let $\varphi \in W^{s,p}(\mathbb{T}_h)$ with $s > \frac{1}{p}$ if $p \in (1, \infty)$, or $s \geq 1$ if $p = 1$. Suppose that $F \in \mathcal{F}^i_h$ with $F = T_1 \cap T_2$, $T_1, T_2 \in \mathbb{T}_h$. Set $\varphi_1 := \varphi|_{T_1}$ and $\varphi_2 := \varphi|_{T_2}$. The jump and the average of φ across F is then defined as

$$[\![\varphi]\!]_F := \varphi_1 - \varphi_2, \quad \{\{\varphi\}\}_F := \frac{1}{2}(\varphi_1 + \varphi_2).$$

For a boundary face $F \in \mathcal{F}^\partial_h$ with $F = \partial T \cap \partial\Omega$, $[\![\varphi]\!]_F := \varphi|_T$ and $\{\{\varphi\}\}_F := \varphi|_T$. For any $v \in W^{1,p}(\mathbb{T}_h; \mathbb{R}^d)$ with $p \in [1, \infty)$, we use the notation

$$[\![v \cdot n]\!]_F := v_1 \cdot n_F - v_2 \cdot n_F, \quad \{\{v\}\}_F := \frac{1}{2}(v_1 + v_2)$$

for the jump of the normal component of v. Whenever no confusion can arise, we simply write $[\![\varphi]\!]$, $\{\{\varphi\}\}$, $[\![v \cdot n]\!]$ and $\{\{v\}\}$, respectively. For $v \in W^{1,p}(\mathbb{T}_h; \mathbb{R}^d)$ and $\varphi \in W^{1,p}(\mathbb{T}_h)$ with $p \in [1, \infty)$, it holds that

$$[\![(v\varphi) \cdot n]\!]_F = \{\{v\}\}_F \cdot n_F [\![\varphi]\!]_F + [\![v \cdot n]\!]_F \{\{\varphi\}\}_F.$$

We define a broken gradient operator as follows. Let $p \in [1, \infty]$. For $\varphi \in W^{1,p}(\mathbb{T}_h)$, the broken gradient $\nabla_h : W^{1,p}(\mathbb{T}_h) \to L^p(\Omega)^d$ is defined by

$$(\nabla_h \varphi)|_T := \nabla(\varphi|_T) \quad \forall T \in \mathbb{T}_h,$$

and we define the broken $H(\mathrm{div}; T)$ space by

$$H(\mathrm{div}; \mathbb{T}_h) := \left\{ v \in L^2(\Omega)^d; \ v|_T \in H(\mathrm{div}; T) \ \forall T \in \mathbb{T}_h \right\},$$

and the broken divergence operator $\mathrm{div}_h : H(\mathrm{div}; \mathbb{T}_h) \to L^2(\Omega)$ such that, for all $v \in H(\mathrm{div}; \mathbb{T}_h)$,

$$(\mathrm{div}_h v)|_T := \mathrm{div}(v|_T) \quad \forall T \in \mathbb{T}_h.$$

1.5.4 Barycentric Coodinates

For a simplex $T \subset \mathbb{R}^d$, let $\{P_i\}_{i=1}^{d+1}$ be vertices of T and $(x_1^{(i)}, \dots, x_d^{(i)})^T$ coordinates of P_i. We set

$$\Delta := \det \begin{pmatrix} 1 & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1^{(d+1)} \\ \vdots & \vdots & \vdots \\ x_d^{(1)} & \cdots & x_d^{(d+1)} \end{pmatrix} > 0.$$

The barycentric coordinates $\{\lambda_i\}_{i=1}^{d+1} : \mathbb{R}^d \to \mathbb{R}$ of $P(x_a, \dots, x_d)$ with respect to $\{P_i\}_{i=1}^{d+1}$ are then defined as

$$\lambda_i(x) := \frac{1}{\Delta} \det \begin{pmatrix} 1 & \cdots & \overset{i}{\smile}\!\!\!\!1 & \cdots & 1 \\ x_1^{(1)} & \cdots & x_1 & \cdots & x_1^{(d+1)} \\ \vdots & & \vdots & & \vdots \\ x_d^{(1)} & \cdots & x_d & \cdots & x_d^{(d+1)} \end{pmatrix}.$$

The barycentric coordinates have the following properties:

$$\lambda_i(P_j) = \delta_{ij}, \quad \sum_{i=1}^{d+1} \lambda_i(x) = 1.$$

1.6 Useful Tools for Analysis

1.6.1 Jensen-type Inequality

In this thesis, we use the following Jensen-type inequality (see [30, Exercise 1.20]): Let $0 \le r \le s$ and $a_i \ge 0$, $i = 1, 2, \dots, n$ ($n \in \mathbb{N}$), be real numbers. We then have

$$\left(\sum_{i=1}^{n} a_i^s \right)^{1/s} \le \left(\sum_{i=1}^{n} a_i^r \right)^{1/r}. \tag{1.6.1}$$

1.6.2 Embedding Theorems

The following is well known as the Sobolev embedding theorem.

Theorem 1.6.1. *Let $d \geq 2$, $s > 0$, and $p \in [1, \infty]$. Let $D \subset \mathbb{R}^d$ be a bounded open subset of $\mathbb{R}^d$. If D is a Lipschitz set, we then have*

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(D) & \forall q \in [p, \frac{pd}{d-sp}] \text{ if } sp < d, \\ L^q(D) & \forall q \in [p, \infty), \text{ if } sp = d, \\ L^\infty(D) \cap \mathcal{C}^{0,\xi}(\overline{D}) & \xi = 1 - \frac{d}{sp} \text{ if } sp > d. \end{cases} \tag{1.6.2}$$

Furthermore,

$$W^{s,p}(D) \hookrightarrow L^\infty(D) \cap \mathcal{C}^0(\overline{D}) \quad (\text{case } s = d \text{ and } p = 1). \tag{1.6.3}$$

Proof. See, for example, [30, Corollary B.43, Theorem B.40] and [31, Theorem 2.31] and the references therein. □

The following is the embedding theorem related to operator from $W^{s,p}(D)$ into $L^q(S_r)$, where S_r is some plane r-dimensional piece belonging to D with dimensions $r < d$.

Theorem 1.6.2. *Let $p, q \in [1, +\infty]$ and $s \geq 1$ be an integer. Let $D \subset \mathbb{R}^d$ be a bounded open set having piecewise smooth boundaries. The following embeddings are then continuous:*

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(S_r) & \text{if } 1 \leq p < \frac{d}{s},\ r > d - sp \text{ and } q \leq \frac{pr}{d-sp}, \\ L^q(S_r) & \text{if } p = \frac{d}{s} \text{ for } q < +\infty. \end{cases} \tag{1.6.4}$$

Proof. See, for example, [61, Theorem 2.1 (p. 61)] and the references therein. □

Corollary 1.6.3. *Let $p \in [1, +\infty)$ and $s \geq 1$ be an integer. Let $D \subset \mathbb{R}^d$ be a bounded open set having piecewise smooth boundaries. The following embeddings are then continuous:*

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^p(S_r) & \text{if } sp > d - r \text{ for } p > 1, \\ L^p(S_r) & \text{if } s \geq d - r \text{ for } p = 1. \end{cases} \tag{1.6.5}$$

Proof. Setting $p = q$ in (1.6.4), we have the desired result. Also see [31, pp. 31, 32]. □

1.6.3 Trace Theorem

The following theorem is well-known and useful, e.g., see [31].

Theorem 1.6.4 (Trace)**.** *Let $p \in [1, \infty)$. Let $s > \frac{1}{p}$ if $p > 1$ or $s \geq 1$ if $p = 1$. Let D be a Lipschitz domain (e.g., see [31, Definition 3.2]) in $\mathbb{R}^d$. There exists a bounded linear operator $\gamma^g : W^{s,p}(D) \to L^p(\partial D)$ such that*

(I) $\gamma^g(\varphi) = \varphi|_{\partial D}$, whenever φ is smooth, e.g., $\varphi \in \mathcal{C}(\overline{D})$.

(II) The kernel of γ^g is $W_0^{s,p}(D)$.

(III) If $s = 1$ and $p = 1$, or if $s \in (\frac{1}{2}, \frac{3}{2})$ and $p = 2$, or if $s \in (\frac{1}{p}, 1]$ and $p \notin \{1, 2\}$, then $\gamma^g : W^{s,p}(D) \to W^{s-\frac{1}{p},p}(\partial D)$ is bounded and surjective, that is, there exists C^{γ^g} such that, for every functions $g \in W^{s-\frac{1}{p},p}(\partial D)$, one can find a function $\varphi_g \in W^{s,p}(D)$, called a lifting of g, such that

$$\gamma^g(\varphi_g) = g, \quad \|\varphi_g\|_{W^{s,p}(D)} \leq C^{\gamma^g} \ell_D^{\frac{1}{p}} \|g\|_{W^{s-\frac{1}{p},p}(\partial D)}, \tag{1.6.6}$$

where ℓ_D is a characteristic length of D, e.g., $\ell_D := \mathrm{diam}(D)$.

Proof. See [31, Theorem 3.10], and the references therein. □

We introduce the following remarks described in [31, Remarks 3.13 and 3.14].

Remark 1.6.5 ($W^{1,\infty}(D)$)**.** The trace theory in $W^{1,\infty}(D)$ is not trivial because $\mathcal{C}^\infty(D)$ is not dense in $L^\infty(D)$. The situation simplifies if D is quasiconvex because $W^{1,\infty}(D) = \mathcal{C}^{0,1}(D)$ in this case. Here, a set $D \subset \mathbb{R}^d$ is said to be quasiconvex if there exists $C \geq 1$ such that every pair of points $x, y \in D$ can be jointed by a curve R in D with $\mathrm{length}(R) \leq C|x - y|_E$.

Remark 1.6.6. If $\varphi \in W^{s,p}(D)$ with $p \in [1, \infty)$ and $s > 1 + \frac{1}{p}$ if $p > 1$ or $s \geq 2$ if $p = 1$, then $\nabla\varphi \in W^{s-1,p}(D)^d$, and we cam apply Theorem 1.6.4 componentwise, that is, $\gamma^g(\nabla\varphi) \in W^{s-1-\frac{1}{p},p}(\partial D)$.

Theorem 1.6.7 (Trace on low-dimensional manifolds)**.** *Let $p \in [1, \infty)$ and let D be a Lipschitz domain in $\mathbb{R}^d$. Let M be a smooth, or polyhedral, manifold of dimension r in $\overline{D}$, $r \in \{0 : d\}$. Then, there exists a bounded trace operator from $W^{s,p}(D)$ to $L^p(M)$, provided $sp > d - r$, or $s \geq d - r$ if $p = 1$.*

Proof. See [31, Theorem 3.15]. □

Theorem 1.6.8 (Normal derivative)**.** *Let $p \in (1, \infty)$ and $s - \frac{1}{p} \in (1, 2)$. Let D be a domain in $\mathbb{R}^d$ with a boundary of class $\mathcal{C}^{k,1}$ with $k = 1$ if $s \leq 2$, and $k = 2$ otherwise. There exists a bounded linear map $\gamma^{\partial_n} : W^{s,p}(D) \to W^{s-1-\frac{1}{p},p}(\partial D)$ so that $\gamma^{\partial_n}(\varphi) := (n \cdot \nabla)\varphi|_{\partial D}$ for all $\varphi \in \mathcal{C}^1(\overline{D})$, and letting $\gamma_1 := (\gamma^g, \gamma^{\partial_n}) : W^{s,p}(D) \to W^{s-\frac{1}{p},p}(\partial D) \times W^{s-1-\frac{1}{p},p}(\partial D)$,* *(I) The map γ_1 is bounded and surjective.*

(II) The kernel of γ_1 is $W_0^{s,p}(D)$.

Proof. See [31, Theorem 3.16], and the references therein. □

1.6.4 Bramble–Hilbert–type Lemma

The Bramble–Hilbert–type lemma (e.g., see [29, 22]) plays a major role in interpolation error analysis. We mainly use the following estimates on anisotropic meshes proposed in [4, Lemma 2.1].

Lemma 1.6.9. *Let $D \subset \mathbb{R}^d$ be a connected open set that is star-shaped with respect to balls B. Let γ be a multi-index with $m := |\gamma|$ and $\varphi \in L^1(D)$ be a function with $\partial^\gamma \varphi \in W^{\ell-m,p}(D)$, where $\ell \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \le m \le \ell$, $p \in [1, \infty]$. It then holds that*

$$\|\partial^\gamma(\varphi - Q^{(\ell)}\varphi)\|_{W^{\ell-m,p}(D)} \le C^{BH} |\partial^\gamma \varphi|_{W^{\ell-m,p}(D)}, \tag{1.6.7}$$

where C^{BH} depends only on d, ℓ, $\operatorname{diam} D$, and $\operatorname{diam} B$, and $Q^{(\ell)}\varphi$ is defined as

$$(Q^{(\ell)}\varphi)(x) := \sum_{|\delta| \le \ell-1} \int_B \eta(y)(\partial^\delta \varphi)(y) \frac{(x-y)^\delta}{\delta!} dy \in \mathcal{P}^{\ell-1}, \tag{1.6.8}$$

where $\eta \in \mathcal{C}_0^\infty(B)$ is a given function with $\int_B \eta dx = 1$.

To give local interpolation error estimates on isotropic meshes, we use the inequalities given in [28, Theorem 1.1] and [29, 22, 87] which are variants of the Bramble–Hilbert lemma.

Lemma 1.6.10. *Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \le p \le \infty$. There exists a polynomial $\eta \in \mathcal{P}^{m-1}$ such that*

$$|\varphi - \eta|_{W^{k,p}(D)} \le C^{BH}(d, m) \operatorname{diam}(D)^{m-k} |\varphi|_{W^{m,p}(D)}, \quad k = 0, 1, \dots, m. \tag{1.6.9}$$

Proof. The proof is found in [28, Theorem 1.1]. □

Remark 1.6.11. In [22, Lemma 4.3.8], the Bramble–Hilbert lemma is given as follows. Let B be a ball in $D \subset \mathbb{R}^d$ such that D is star-shaped with respect to B and its radius $r > \frac{1}{2} r_{\max}$, where $r_{\max} := \sup\{r :$ D is star-shaped with respect to a ball of radius $r\}$. Let $\varphi \in W^{m,p}(D)$ with $m \in \mathbb{N}$ and $1 \le p \le \infty$. There exists a polynomial $\eta \in \mathcal{P}^{m-1}$ such that

$$|\varphi - \eta|_{W^{k,p}(D)} \le C^{BH}(d,m,\gamma)\,\mathrm{diam}(D)^{m-k}|\varphi|_{W^{m,p}(D)}, \quad k = 0,1,\dots,m. \tag{1.6.10}$$

Here, γ is called the chunkiness parameter of D, which is defined by

$$\gamma := \frac{\mathrm{diam}(D)}{r_{\max}}.$$

The main drawback is that the constant $C^{BH}(d,m,\gamma)$ depends on the chunkiness parameter. Meanwhile, the constant $C^{BH}(d,m)$ of the estimate (1.6.9) does not depend on the geometric parameter γ.

Remark 1.6.12. For general Sobolev spaces $W^{m,p}(\Omega)$, the upper bounds on the constant $C^{BH}(d,m)$ are not given, as far as we know. However, when $p = 2$, the following result has been obtained by Verf*ü*rth [87].

Let $D \subset \mathbb{R}^d$ be a bounded convex domain. Let $\varphi \in H^m(D)$ with $m \in \mathbb{N}$. There exists a polynomial $\eta \in \mathcal{P}^{m-1}$ such that

$$|\varphi - \eta|_{H^k(D)} \le C^{BH}(d,k,m)\,\mathrm{diam}(D)^{m-k}|\varphi|_{H^m(D)}, \quad k = 0,1,\dots,m-1. \tag{1.6.11}$$

Verf*ü*rth has given upper bounds on the constants in the estimates such that

$$C^{BH}(d,k,m) \le \pi^{k-m} \binom{d+k-1}{k}^{1/2} \frac{\{(m-k)!\}^{1/2}}{\{\left[\frac{m-k}{d}\right]!\}^{d/2}},$$

where $[x]$ denotes the largest integer less than or equal to x.

As an example, let us consider the case $d = 3$, $k = 1$, and $m = 2$. We then have

$$C^{BH}(3,1,2) \le \frac{\sqrt{3}}{\pi},$$

thus on the standard reference element $\widehat{T}$ introduced in Section 3.1.2, we obtain

$$|\hat{\varphi} - \hat{\eta}|_{H^1(\widehat{T})} \le \frac{\sqrt{6}}{\pi}|\hat{\varphi}|_{H^2(\widehat{T})} \quad \forall \hat{\varphi} \in H^2(\widehat{T}),$$

becase $\mathrm{diam}(\widehat{T}) = \sqrt{2}$.

1.6.5 Poincaré inequality

Theorem 1.6.13 (Poincaré inequality). *Let* $D \subset \mathbb{R}^d$ *be a convex domain with diameter* $\mathrm{diam}(D)$. *It then holds that, for* $\varphi \in H^1(D)$ *with* $\int_D \varphi dx = 0$,

$$\|\varphi\|_{L^2(D)} \leq \frac{\mathrm{diam}(D)}{\pi} |\varphi|_{H^1(D)}. \tag{1.6.12}$$

Proof. The proof is found in [70, Theorem 3.2], also see [73]. □

Remark 1.6.14. The coefficient $\frac{1}{\pi}$ of (1.6.12) may be improved. The best constant in eqrefpoincare is described in [31, Remark 3.25].

Part II

Geometric Conditions

Chapter 2

Motivation

2.1 Regularity Conditions

Let $\widehat{T} \subset \mathbb{R}^d$ and $T \subset \mathbb{R}^d$ be a reference element and a simplex, respectively. Let these two elements be affine equivalent. Let us consider two finite elements $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ and $\{T, P, \Sigma\}$ with associated normed vector spaces $V(\widehat{T})$ and $V(T)$. The transformation Φ_T takes the form

$$\Phi_T : \widehat{T} \ni \hat{x} \mapsto \Phi_T(\hat{x}) := \mathcal{B}_T \hat{x} + b_T \in T,$$

where $\mathcal{B}_T \in \mathbb{R}^{d\times d}$ is an invertible matrix and $b_T \in \mathbb{R}^d$. Let $I_T : V(T) := W^{2,p}(T) \to P := \mathcal{P}^1(T)$ with $p \in [1, \infty]$ be an interpolation on T with $I_T p = p$ for any $p \in \mathcal{P}^1(T)$. According to the classical theory (e.g., see [25, 30]), there exists a positive constant c, independent of h_T, such that

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq c \left(\|\mathcal{B}_T\|_2 \|\mathcal{B}_T^{-1}\|_2 \right) \|\mathcal{B}_T\|_2 |\varphi|_{W^{2,p}(T)}.$$

Here, the quantity $\|\mathcal{B}_T\|_2 \|\mathcal{B}_T^{-1}\|_2$ is called the *Euclidean condition number* of $\mathcal{B}_T$. By standard estimates (e.g., see [30, Lemma 1.100]), we have

$$\|\mathcal{B}_T\|_2 \|\mathcal{B}_T^{-1}\|_2 \leq c \frac{h_T}{\rho_T}, \quad \|\mathcal{B}_T\|_2 \leq c h_T.$$

It thus holds that

$$|\varphi - I_T\varphi|_{W^{1,p}(T)} \leq c \frac{h_T}{\rho_T} h_T |\varphi|_{W^{2,p}(T)}.$$

As geometric conditions to obtain global interpolation error estimate and to prove that this estimate converges to zero as $h \to 0$, the *shape-regularity condition* is widely used and well known. This condition states as follows.

Condition 2.1.1 (Shape-regularity condition)**.** There exists a constant $\gamma_1 > 0$ such that

$$\rho_T \geq \gamma_1 h_T \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{2.1.1}$$

Under the condition, that is, when the quantity $\frac{h_T}{\rho_T}$ is controlled, it holds that

$$|\varphi - I_h \varphi|_{W^{1,p}(\Omega)} \leq ch|\varphi|_{W^{2,p}(\Omega)},$$

where $I_h \varphi$ is the standard global linear interpolation of φ on $\mathbb{T}_h$.

Furthermore, geometric conditions equivalent to the shape-regularity condition are known; that is, the following three conditions are equivalent to the shape-regularity one. The proof is found in [20, Theorem 1].

Condition 2.1.2 (Zlámal's condition)**.** There exists a constant $\gamma_2 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$, any simplex $T \in \mathbb{T}_h$ and any dihedral angle ψ and for $d = 3$, also any solid angle θ of T, we have

$$\psi \geq \gamma_2, \quad \theta \geq \gamma_2. \tag{2.1.2}$$

Condition 2.1.3. There exists a constant $\gamma_3 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T| \geq \gamma_3 h_T^d. \tag{2.1.3}$$

Condition 2.1.4. There exists a constant $\gamma_4 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T| \geq \gamma_4 |B^T|, \tag{2.1.4}$$

where $B^T \supset T$ is the circumscribed ball of T.

Note 2.1.5. If Condition 2.1.1 or 2.1.2 or 2.1.3 or 2.1.4 holds, a family of simplicial partitions is called *regular*.

Note 2.1.6. Condition 2.1.2 was presented by Zlámal [90] in 1968. The condition is called the *minimum-angle condition* and guarantees the convergence of finite element methods for linear elliptic problems on $\mathbb{R}^2$. Zlámal's condition can be generalised into $\mathbb{R}^n$ for any $n \in \{2, 3, \ldots\}$. Later, the shape-regularity condition (the inscribed ball condition) was introduced; see, for example, [25]. Triangles or tetrahedra cannot be too flat in a shape-regular family of triangulations.

Note 2.1.7. Condition 2.1.3 seems to be simpler than Condition 2.1.1 or Condition 2.1.2. Therefore, it may be useful to analyse theoretical finite element methods and implement finite element codes to keep nondegenerate mesh partitions.

Remark 2.1.8. What happens if the shape-regularity condition is violated, that is, the triangle becomes too flat as $h_T \to 0$? One of the answer is the quantity $\frac{h_T^d}{|T|} h_T$, equivalently $\frac{h_T}{\rho_T} h_T$ may diverge. As an example, let $T \subset \mathbb{R}^2$ be the simplex with vertices $P_1 := (0,0)^T$, $P_2 := (2s,0)^T$ and $P_3 := (s, s^\varepsilon)^T$ for $0 < s \ll 1$, $s \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$. We then have

$$\frac{h_T^2}{|T|} h_T = \frac{8s^3}{s^{1+\varepsilon}} = 8s^{2-\varepsilon}.$$

If $\varepsilon \geq 2$, the quantity $\frac{h_T^2}{|T|} h_T$ diverges as $h_T \to 0$. Even if $\varepsilon < 2$, the convergence order $h_T^{2-\varepsilon}$ is worse than h_T which is optimal order under imposing the shape-regular meshes. Therefore, It depends on ε whether interpolation error estimates converge or not. On general meshes, it may not easy to control "degree of crushing of elements."

Remark 2.1.9. We give the another example. Let $T \subset \mathbb{R}^2$ be the simplex with vertices $P_1 := (0,0)^T$, $P_2 := (s,0)^T$ and $P_3 := (0, s^\varepsilon)^T$ for $0 < s \ll 1$, $s \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$. We then have

$$\frac{h_T^2}{|T|} h_T = \frac{(s^2 + s^{2\varepsilon})^{3/2}}{\frac{1}{2} s^{1+\varepsilon}} \leq c s^{2-\varepsilon}.$$

As with Remark 2.1.8, if $\varepsilon \geq 2$, the quantity $\frac{h_T^2}{|T|} h_T$ diverges as $h_T \to 0$. However, in some cases, it is known that it is not necessary for Condition 2.1.1 to obtain

$$|\varphi - I_h \varphi|_{W^{1,p}(\Omega)} \leq ch |\varphi|_{W^{2,p}(\Omega)}.$$

In this case, we observe that the Euclidean condition number $\|\mathcal{B}_T\|_2 \|\mathcal{B}_T^{-1}\|_2$ of $\mathcal{B}_T$ may be overestimated.

2.2 Semi-regularity Conditions for $d = 2$

In 1957, Synge [83, Section 3.8] proposed the following condition.

Condition 2.2.1 (Synge's condition)**.** There exists $\frac{\pi}{3} \leq \gamma_5 < \pi$ such that, for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\theta_{T,\max} \leq \gamma_5, \tag{2.2.1}$$

where $\theta_{T,\max}$ is the maximal angle of T.

Under Condition 2.2.1, Synge proved an optimal interpolation error estimate as follows.

$$\|\varphi - I_h\varphi\|_{W^{1,p}(\Omega)} \leq ch|\varphi|_{W^{2,p}(\Omega)} \quad \text{for } p = \infty.$$

The inequality (2.2.1) is called *Synge's condition* or the *maximum-angle condition.* In 1976, several author's [13, 15, 39, 51] independently proved the convergence of finite element for $p < \infty$. We observe that the minimal angle may tend to zero as $h \to 0$ under this condition. If Synge's condition does not hold, optimal order of the interpolation for linear triangular elements is lost as when no imposing Zlámal's condition, see e.g. [13, p. 223], and Remark 2.1.8.

In [57], Křížek proposed the following circumscribed ball condition for $d = 2$ which is equivalent to Synge's condition.

Condition 2.2.2. There exists $\gamma_6 > 0$ such that, for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\frac{R_2}{h_T} \leq \gamma_6, \tag{2.2.2}$$

where R_2 is the radius of the circumscribed ball of $T \subset \mathbb{R}^2$.

Note 2.2.3. If Condition 2.2.1 or 2.2.2 holds, the associated families of partitions are called *semi-regular.*

Assume that Condition 2.1.3 holds, that is, there exists a constant $\gamma_3 > 0$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$, we have

$$|T| \geq \gamma_3 h_T^2.$$

Let $T \subset \mathbb{R}^2$ be the triangle with vertices P_1, P_2 and P_3 such that the maximum angle $\theta_{T,\max}$ of T is $\angle P_2P_1P_3$. We then have $h_T = |P_2P_3|$ and

$$\frac{R_2}{h_T} = \frac{|P_2P_3|}{2h_T \sin\theta_{T,\max}} = \frac{|P_1P_2||P_1P_3|}{2|P_1P_2||P_1P_3|\sin\theta_{T,\max}} \leq c\frac{h_T^2}{|T|} \leq \frac{c}{\gamma_3} =: \gamma_6.$$

This implies that each regular family is semi-regular. However, the converse implication does not hold.

Remark 2.2.4. Consider the same example as Remark 2.1.9. We easily calculate the quantity $\frac{R_2}{h_T}$ as follows.

$$\frac{R_2}{h_T} = \frac{\frac{1}{2}\sqrt{s^2 + s^{2\varepsilon}}}{\sqrt{s^2 + s^{2\varepsilon}}} = \frac{1}{2}.$$

This example implies that Synge's condition makes it possible to use anisotropic meshes.

2.3 Semi-regularity Conditions for $d = 3$

Synge's condition (2.2.1) is extended to the case of tetrahedra in [58].

Condition 2.3.1. There exists a constant $0 < \gamma_7 < \pi$ such that for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$,

$$\theta_{T,\max} \leq \gamma_7, \tag{2.3.1a}$$

$$\psi_{T,\max} \leq \gamma_7, \tag{2.3.1b}$$

where $\theta_{T,\max}$ is the maximum angle of all triangular faces of the tetrahedron T and $\psi_{T,\max}$ is the maximum dihedral angle of T.

Anisotropic interpolation theory has also been developed [10, 4, 23]. Under Synge's and coordinate system conditions, anisotropic interpolation error estimates can then be deduced (e.g., see [4]).

On the other hand, is there a semi-regularity condition which equivalent to Synge's condition (2.3.1) for $d = 3$? As far as we know, an extension of (2.2.2) to $\mathbb{R}^3$ so that optimal interpolation error estimates in Sobolev norms on simplicial finite elements hold is still open, e.g., see [20]. In this part, we propose a new geometric condition to become an alternative to Synge's condition for $d = 3$.

Chapter 3

Settings for the analysis of anisotropic interpolation theory

3.1 Reference and Mesh Elements

This section introduces the Jacobian matrix proposed in [56] for the three-dimensional case and that proposed in [67] for the two-dimensional case. Also see [47, Section 3].

Let us first define a diagonal matrix $\widehat{\mathcal{A}}^{(d)}$ as

$$\widehat{\mathcal{A}}^{(d)} := \mathrm{diag}(h_1, \dots, h_d), \quad h_i \in \mathbb{R}_+ \quad \forall i. \tag{3.1.1}$$

We easily obtain the inverse matrix of $\widehat{A}^{(d)}$:

$$(\widehat{\mathcal{A}}^{(d)})^{-1} = \mathrm{diag}(h_1^{-1}, \dots, h_d^{-1}). \tag{3.1.2}$$

3.1.1 Two-dimensional case

Let $\widehat{T} \subset \mathbb{R}^2$ be the reference triangle with vertices $\widehat{P}_1 := (0,0)^T$, $\widehat{P}_2 := (1,0)^T$, and $\widehat{P}_3 := (0,1)^T$.

Let $\widetilde{\mathfrak{T}}^{(2)}$ be the family of triangles

$$\widetilde{T} = \widehat{\mathcal{A}}^{(2)}(\widehat{T})$$

with vertices $\widetilde{P}_1 := (0,0)^T$, $\widetilde{P}_2 := (h_1, 0)^T$, and $\widetilde{P}_3 := (0, h_2)^T$.

We next define the regular matrix $\widetilde{\mathcal{A}} \in \mathbb{R}^{2\times 2}$ by

$$\widetilde{\mathcal{A}} := \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix}, \tag{3.1.3}$$

with parameters

$$s^2 + t^2 = 1, \quad t > 0.$$

For $\widetilde{T} \in \widetilde{\mathfrak{T}}^{(2)}$, let $\mathfrak{T}^{(2)}$ be the family of triangles

$$T^s = \widetilde{\mathcal{A}}(\widetilde{T})$$

with vertices $P_1^s := (0,0)^T$, $P_2^s := (h_1, 0)^T$, and $P_3^s := (h_2 s, h_2 t)^T$. Then, $h_1 = |P_1^s - P_2^s| > 0$ and $h_2 = |P_1^s - P_3^s| > 0$.

We easily obtain the inverse matrix of $\widetilde{\mathcal{A}}$:

$$\widetilde{\mathcal{A}}^{-1} = \begin{pmatrix} 1 & -\frac{s}{t} \\ 0 & \frac{1}{t} \end{pmatrix}. \tag{3.1.4}$$

3.1.2 Three-dimensional cases

In the three-dimensional case, we need to consider the following two cases (i) and (ii); also see Condition 3.2.2.

Let $\widehat{T}_1$ and $\widehat{T}_2$ be reference tetrahedra with the following vertices:

(i) $\widehat{T}_1$ has the vertices $\widehat{P}_1 := (0,0,0)^T$, $\widehat{P}_2 := (1,0,0)^T$, $\widehat{P}_3 := (0,1,0)^T$, and $\widehat{P}_4 := (0,0,1)^T$;

(ii) $\widehat{T}_2$ has the vertices $\widehat{P}_1 := (0,0,0)^T$, $\widehat{P}_2 := (1,0,0)^T$, $\widehat{P}_3 := (1,1,0)^T$, and $\widehat{P}_4 := (0,0,1)^T$.

Let $\widetilde{\mathfrak{T}}_i^{(3)}$, $i = 1, 2$, be the family of triangles

$$\widetilde{T}_i = \widehat{\mathcal{A}}^{(3)}(\widehat{T}_i), \quad i = 1, 2,$$

with vertices

(i) $\widetilde{P}_1 := (0,0,0)^T$, $\widetilde{P}_2 := (h_1,0,0)^T$, $\widetilde{P}_3 := (0,h_2,0)^T$, and $\widetilde{P}_4 := (0,0,h_3)^T$;

(ii) $\widetilde{P}_1 := (0,0,0)^T$, $\widetilde{x}_2 := (h_1,0,0)^T$, $\widetilde{P}_3 := (h_1,h_2,0)^T$, and $\widetilde{P}_4 := (0,0,h_3)^T$.

We next define the regular matrices $\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2 \in \mathbb{R}^{3\times 3}$ as

$$\widetilde{\mathcal{A}}_1 := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \ \widetilde{\mathcal{A}}_2 := \begin{pmatrix} 1 & -s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix} \tag{3.1.5}$$

with parameters

$$\begin{cases} s_1^2 + t_1^2 = 1, \ s_1 > 0, \ t_1 > 0, \ h_2 s_1 \le h_1/2, \\ s_{21}^2 + s_{22}^2 + t_2^2 = 1, \ t_2 > 0, \ h_3 s_{21} \le h_1/2. \end{cases}$$

For $\widetilde{T}_i \in \widetilde{\mathfrak{T}}_i^{(3)}$, $i = 1, 2$, let $\mathfrak{T}_i^{(3)}$, $i = 1, 2$, be the family of tetrahedra

$$T_i^s = \widetilde{\mathcal{A}}_i(\widetilde{T}_i), \quad i = 1, 2,$$

with vertices

$$\begin{aligned} &P_1^s := (0,0,0)^T, \ P_2^s := (h_1, 0, 0)^T, \ P_4^s := (h_3 s_{21}, h_3 s_{22}, h_3 t_2)^T, \\ &\begin{cases} P_3^s := (h_2 s_1, h_2 t_1, 0)^T & \text{for case (i)}, \\ P_3^s := (h_1 - h_2 s_1, h_2 t_1, 0)^T & \text{for case (ii)}. \end{cases} \end{aligned}$$

Then, $h_1 = |P_1^s - P_2^s| > 0$, $h_3 = |P_1^s - P_4^s| > 0$, and

$$h_2 = \begin{cases} |P_1^s - P_3^s| > 0 & \text{for case (i)}, \\ |P_2^s - P_3^s| > 0 & \text{for case (ii)}. \end{cases}$$

We easily obtain the inverse matrices of $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$:

$$\widetilde{\mathcal{A}}_1^{-1} = \begin{pmatrix} 1 & -\frac{s_1}{t_1} & \frac{s_1 s_{22} - t_1 s_{21}}{t_1 t_2} \\ 0 & \frac{1}{t_1} & -\frac{s_{22}}{t_1 t_2} \\ 0 & 0 & \frac{1}{t_2} \end{pmatrix}, \ \widetilde{\mathcal{A}}_2^{-1} = \begin{pmatrix} 1 & \frac{s_1}{t_1} & \frac{-s_1 s_{22} - t_1 s_{21}}{t_1 t_2} \\ 0 & \frac{1}{t_1} & -\frac{s_{22}}{t_1 t_2} \\ 0 & 0 & \frac{1}{t_2} \end{pmatrix}. \tag{3.1.6}$$

Note 3.1.1. Throughout this thesis, a symbol $\widetilde{\mathcal{A}}$ is either $\widetilde{\mathcal{A}}_1$ or $\widetilde{\mathcal{A}}_2$. When there is no ambiguity, the script 1 or 2 is dropped.

3.2 Standard Elements

In the following, we impose conditions for $T^s \in \mathfrak{T}^{(2)}$ in the two-dimensional case and $T^s \in \mathfrak{T}_1^{(3)} \cup \mathfrak{T}_2^{(3)} =: \mathfrak{T}^{(3)}$ in the three-dimensional case.

Condition 3.2.1 (Case in which $d = 2$)**.** Let $T^s \in \mathfrak{T}^{(2)}$ with the vertices P_i^s $(i = 1, \ldots, 3)$ introduced in Section 3.1.1. We assume that $\overline{P_2^s P_3^s}$ is the longest edge of T^s; that is, $h_{T^s} := |P_2^s - P_3^s|$. Recall that $h_1 = |P_1^s - P_2^s|$ and $h_2 = |P_1^s - P_3^s|$. We then assume that $h_2 \le h_1$. Note that $h_1 = \mathcal{O}(h_{T^s})$.

Condition 3.2.2 (Case in which $d = 3$)**.** Let $T^s \in \mathfrak{T}^{(3)}$ with the vertices P_i^s ($i = 1, \ldots, 4$) introduced in Section 3.1.2. Let E_i ($1 \leq i \leq 6$) be the edges of T^s. We denote the edge of T^s that has the minimum length by $E_{\min}$; that is, $|E_{\min}| = \min_{1 \leq i \leq 6} |E_i|$. We set $h_2 := |E_{\min}|$ and assume that

$$\text{the end points of } E_{\min} \text{ are either } \{P_1^s, P_3^s\} \text{ or } \{P_2^s, P_3^s\}.$$

Among the four edges that share an end point with $E_{\min}$, we consider the longest edge $E_{\max}^{(\min)}$. Let P_1^s and P_2^s be the end points of edge $E_{\max}^{(\min)}$. We thus have that

$$h_1 = |E_{\max}^{(\min)}| = |P_1^s - P_2^s|.$$

Consider cutting $\mathbb{R}^3$ using the plane that contains the midpoint of edge $E_{\max}^{(\min)}$ and is perpendicular to the vector $P_1^s - P_2^s$. Then, there are two cases:

(Type i) P_3^s and P_4^s belong to the same half-space;

(Type ii) P_3^s and P_4^s belong to different half-spaces.

In each case, we set

(Type i) P_1^s and P_3^s as the end points of $E_{\min}$, that is, $h_2 = |P_1^s - P_3^s|$;

(Type ii) P_2^s and P_3^s as the end points of $E_{\min}$, that is, $h_2 = |P_2^s - P_3^s|$.

Finally, we set that $h_3 = |P_1^s - P_4^s|$. Note that we implicitly assume that P_1^s and P_4^s belong to the same half-space. Additionally, note that $h_1 = \mathcal{O}(h_{T^s})$.

Each d-simplex is congruent to the unique $T^s \in \mathfrak{T}^{(d)}$ satisfying Condition 3.2.1 or Condition 3.2.2. T^s is therefore called the *standard element* of the d-simplex. See Figures 3.1 and 3.2.

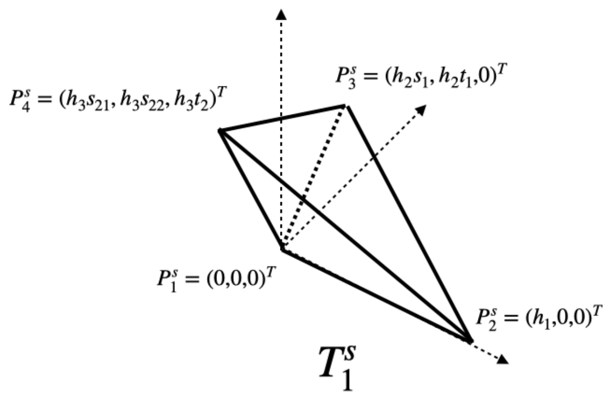

Fig. 3.1: Standard Element (Type i)

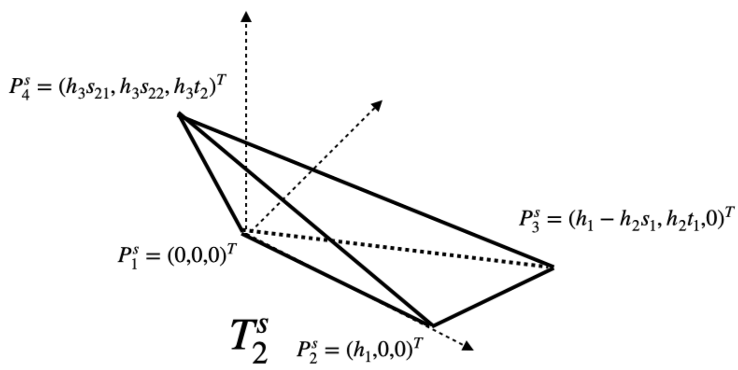

Fig. 3.2: Standard Element (Type ii)

Remark 3.2.3. Let $\mathbb{T}_h$ be a conformal mesh (Section 1.5). We assume that any simplex $T \in \mathbb{T}_h$ is transformed into $T_1^s \in \mathfrak{T}^{(2)}$ such that Condition 3.2.1 is satisfied in the two-dimensional case or $T_i^s \in \mathfrak{T}_i^{(3)}$, $i = 1, 2$, such that Condition 3.2.2 is satisfied in the three-dimensional case through appropriate rotation, translation, and mirror imaging. Note that none of the lengths of the edges of a simplex or the measure of the simplex is changed by the transformation.

Note 3.2.4. Note that the length of all edges of a simplex and measure of the simplex does not change by the transformation. It then holds that

$$h_i \leq h_{T^s} = h_T, \quad i = 1, \dots, d. \tag{3.2.1}$$

3.3 Additional Condition

The following condition is used for obtaining optimal interpolation error estimates.

Condition 3.3.1. In anisotropic interpolation error analysis, we impose the following geometric condition for the simplex T^s:

(I) If $d = 2$, there are no additional conditions;

(II) If $d = 3$, there must exist a positive constant M independent of h_{T^s} such that $|s_{22}| \leq M \frac{h_2 t_1}{h_3}$. Note that if $s_{22} \neq 0$, this condition means that the order of h_3 with respect to h_{T^s} coincides with the order of h_2, and if $s_{22} = 0$, the order of h_3 may be different from that of h_2.

3.4 Affine Mappings and Piola Transforms

In the present thesis, we adopt the following affine mappings and Piola transformations.

Definition 3.4.1 (Affine mappings)**.** Let $\widetilde{T}, \widehat{T} \subset \mathbb{R}^d$ be the simplices defined in Sections 3.1.1 and 3.1.2. That is to say,

$$\widetilde{T} = \widehat{\Phi}(\widehat{T}), \quad T^s = \widetilde{\Phi}(\widetilde{T}) \quad \text{with} \quad \tilde{x} := \widehat{\Phi}(\hat{x}) := \widehat{\mathcal{A}}^{(d)} \hat{x}, \quad x^s := \widetilde{\Phi}(\tilde{x}) := \widetilde{\mathcal{A}} \tilde{x}.$$

We then define an affine mapping $\Phi^s : \widehat{T} \to T^s$ as

$$\Phi^s := \widetilde{\Phi} \circ \widehat{\Phi} : \widehat{T} \to T^s, \; x^s := \Phi^s(\hat{x}) := \mathcal{A}^s \hat{x}, \quad \mathcal{A}^s := \widetilde{\mathcal{A}} \widehat{\mathcal{A}}^{(d)}. \tag{3.4.1}$$

Furthermore, let Φ_{T^s} be an affine mapping defined as

$$\Phi_{T^s} : T^s \ni x^s \mapsto \mathcal{A}_T x^s + b_T \in T, \tag{3.4.2}$$

where $b_T \in \mathbb{R}^d$ and $\mathcal{A}_T \in O(d)$ is a rotation and mirror imaging matrix. We define an affine mapping $\Phi : \widehat{T} \to T$ as

$$\Phi := \Phi_{T^s} \circ \Phi^s : \widehat{T} \to T, \ x := \Phi(\hat{x}) = (\Phi_{T^s} \circ \Phi^s)(\hat{x}) = \mathcal{A}\hat{x} + b_T, \tag{3.4.3}$$

where $\mathcal{A} := \mathcal{A}_T \mathcal{A}^s$.

Definition 3.4.2 (Piola transforms)**.** Let $T \in \mathbb{T}_h$. Let T^s, $\widetilde{T}$, and $\widehat{T} \subset \mathbb{R}^d$ be the simplices defined in Sections 3.1.1 and 3.1.2. Let Φ, Φ_{T^s}, $\widetilde{\Phi}$, and $\widehat{\Phi}$ be the affine mappings defined in Definition 3.4.1; that is,

$$\widetilde{T} = \widehat{\Phi}(\widehat{T}), \quad T^s = \widetilde{\Phi}(\widetilde{T}), \quad T = \Phi_{T^s}(T^s), \quad \Phi = \Phi_{T^s} \circ \widetilde{\Phi} \circ \widehat{\Phi}.$$

We set $V(\widehat{T}) := \mathcal{C}(\widehat{T})^d$. The Piola transformation $\Psi^s := \widetilde{\Psi} \circ \widehat{\Psi} : V(\widehat{T}) \to V(T^s)$ is defined as

$$\begin{aligned} \Psi^s : V(\widehat{T}) &\to V(T^s) \\ \hat{v} &\mapsto v^s(x^s) := \widetilde{\Psi} \circ \widehat{\Psi}(\hat{v})(x^s) = \frac{1}{\det(\mathcal{A}^s)} \mathcal{A}^s \hat{v}(\hat{x}) \end{aligned} \tag{3.4.4}$$

with two Piola transformations:

$$\begin{aligned} \widehat{\Psi} : V(\widehat{T}) &\to V(\widetilde{T}) \\ \hat{v} &\mapsto \tilde{v}(\tilde{x}) := \widehat{\Psi}(\hat{v})(\tilde{x}) := \frac{1}{\det(\widehat{\mathcal{A}}^{(d)})} \widehat{\mathcal{A}}^{(d)} \hat{v}(\hat{x}), \\ \widetilde{\Psi} : V(\widetilde{T}) &\to V(T) \\ \tilde{v} &\mapsto v^s(x^s) := \widetilde{\Psi}(\tilde{v})(x^s) := \frac{1}{\det(\widetilde{\mathcal{A}})} \widetilde{\mathcal{A}} \tilde{v}(\tilde{x}). \end{aligned}$$

Let $\Psi_{T^s} : V(T^s) \to V(T)$ be the Piola transformation defined as

$$\begin{aligned} \Psi_{T^s} : V(T^s) &\to V(T) \\ v^s &\mapsto v(x) := \Psi_{T^s}(v^s)(x) := \frac{1}{\det(\mathcal{A}_T)} \mathcal{A}_T v^s(x^s). \end{aligned} \tag{3.4.5}$$

We define the Piola transformation $\Psi : V(\widehat{T}) \to V(T)$ as

$$\begin{aligned} \Psi : V(\widehat{T}) &\to V(T) \\ \hat{v} &\mapsto v(x) := \Psi_{T^s} \circ \Psi^s(\hat{v})(x) := \frac{1}{\det(\mathcal{A})} \mathcal{A} \hat{v}(\hat{x}). \end{aligned} \tag{3.4.6}$$

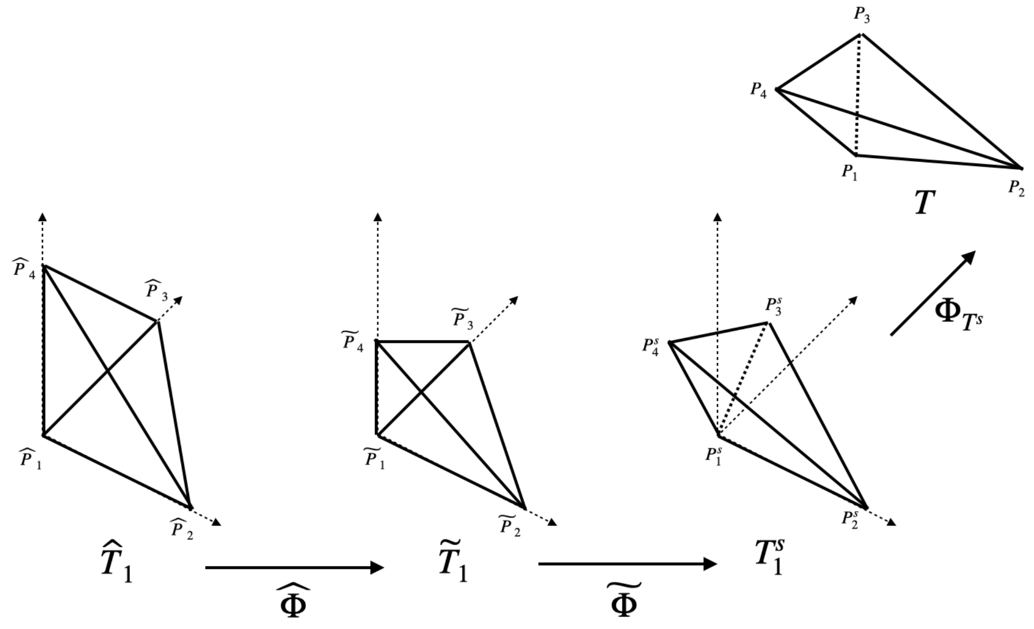

Fig. 3.3: Affine Map (Type i)

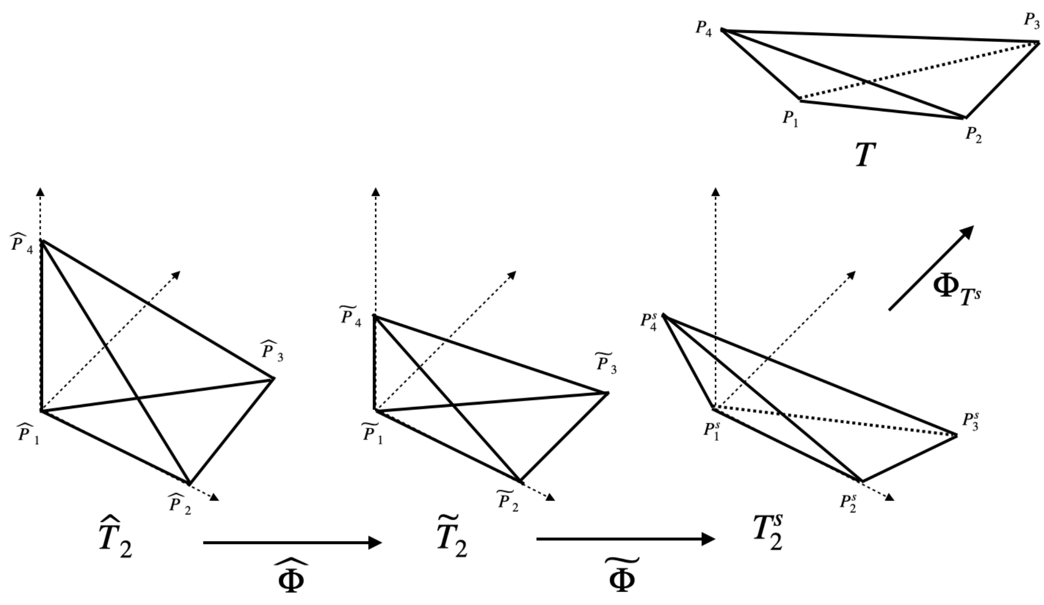

Fig. 3.4: Affine Map (Type ii)

Lemma 3.4.3 (Property of the Piola transformations 1). *If $\hat{v} \in C^1(\widehat{T})^d$, then $v := \Psi\hat{v} \in C^1(T)^d$ and there holds*

$$J_x v = \frac{1}{\det(\mathcal{A})}\mathcal{A}\widehat{J}_{\hat{x}}\hat{v}\mathcal{A}^{-1}, \tag{3.4.7}$$

$$\operatorname{div} v = \frac{1}{\det(\mathcal{A})}\widehat{\operatorname{div}}\hat{v}, \tag{3.4.8}$$

where $J_x v$ and $\widehat{J}_{\hat{x}}\hat{v}$ denote the Jacobian matrixes of v and $\hat{v}$, respectively.

Proof. From the definition of the Piola transformation (3.4.6), we have

$$\begin{aligned} J_x v(x) &= \frac{1}{\det(\mathcal{A})}\mathcal{A}J_x(\hat{v}\circ\Phi^{-1})(x) = \frac{1}{\det(\mathcal{A})}\mathcal{A}\widehat{J}_{\hat{x}}\hat{v}(\hat{x})J_x\Phi^{-1}(x) \\ &= \frac{1}{\det(\mathcal{A})}\mathcal{A}\widehat{J}_{\hat{x}}\hat{v}(\hat{x})\mathcal{A}^{-1}. \end{aligned}$$

Because the property of the trace, we get

$$\begin{aligned} \operatorname{div} v = \operatorname{Tr}(J_x v) &= \frac{1}{\det(\mathcal{A})}\operatorname{Tr}(\mathcal{A}\widehat{J}_{\hat{x}}\hat{v}\mathcal{A}^{-1}) \\ &= \frac{1}{\det(\mathcal{A})}\operatorname{Tr}(\widehat{J}_{\hat{x}}\hat{v}) = \frac{1}{\det(\mathcal{A})}\widehat{\operatorname{div}}\hat{v}. \end{aligned}$$

□

Lemma 3.4.4 (Property of the Piola transformations 2). *Let $p \in [1,\infty)$. For $\hat{\varphi} \in W^{1,p'}(\widehat{T})$, $\hat{v} \in W^{1,p}(\widehat{T})^d$ with $\varphi := \hat{\varphi}\circ\Phi^{-1}$ and $v := \Psi(\hat{v})$, it holds that*

$$\int_T \operatorname{div} v\varphi dx = \int_{\widehat{T}} \widehat{\operatorname{div}}\hat{v}\hat{\varphi}d\hat{x}, \tag{3.4.9}$$

$$\int_T (v\cdot\nabla_x)\varphi dx = \int_{\widehat{T}} (\hat{v}\cdot\widehat{\nabla}_{\hat{x}})\hat{\varphi}d\hat{x}, \tag{3.4.10}$$

$$\int_{\partial T} (v\cdot n)\varphi ds = \int_{\partial\widehat{T}} (\hat{v}\cdot\hat{n})\hat{\varphi}d\hat{s}. \tag{3.4.11}$$

Proof. Because $\det(\mathcal{A})$ is positive, by a change a variable,

$$\int_T \operatorname{div} v\varphi dx = \frac{1}{\det(\mathcal{A})}\int_{\widehat{T}} \widehat{\operatorname{div}}\hat{v}\hat{\varphi}|\det(\mathcal{A})|d\hat{x},$$

which leads to (3.4.9).

Because

$$\nabla_x \varphi = \mathcal{A}^{-T} \widehat{\nabla}_{\hat{x}} \hat{\varphi},$$

we have

$$\begin{aligned}\int_T (v \cdot \nabla_x)\varphi dx &= \frac{1}{\det(\mathcal{A})} \int_{\widehat{T}} (\mathcal{A}\hat{v} \cdot \mathcal{A}^{-T}\widehat{\nabla}_{\hat{x}})\hat{\varphi} |\det(\mathcal{A})| d\hat{x} \\ &= \int_{\widehat{T}} [(\mathcal{A}\hat{v})^T \mathcal{A}^{-T} \widehat{\nabla}_{\hat{x}}]\hat{\varphi} d\hat{x} = \int_{\widehat{T}} (\hat{v} \cdot \widehat{\nabla}_{\hat{x}})\hat{\varphi} d\hat{x},\end{aligned}$$

which is (3.4.10).

From (3.4.9) and (3.4.10), applying the Gauss–Green formula yields

$$\begin{aligned}\int_{\partial T} (v \cdot n)\varphi ds &= \int_T (\operatorname{div} v)\varphi dx + \int_T (v \cdot \nabla_x)\varphi dx \\ &= \int_{\widehat{T}} (\widehat{\operatorname{div}} \hat{v})\hat{\varphi} d\hat{x} + \int_{\widehat{T}} (\hat{v} \cdot \widehat{\nabla}_{\hat{x}})\hat{\varphi} d\hat{x} = \int_{\partial \widehat{T}} (\hat{v} \cdot \hat{n})\hat{\varphi} d\hat{s},\end{aligned}$$

which is (3.4.11). □

3.5 Additional Notation

For convenience, we introduce two definitions.

Definition 3.5.1. We define a parameter $\mathscr{H}_i$, $i = 1, \ldots, d$, as

$$\begin{cases} \mathscr{H}_1 := h_1, \quad \mathscr{H}_2 := h_2 t \quad \text{if } d = 2, \\ \mathscr{H}_1 := h_1, \quad \mathscr{H}_2 := h_2 t_1, \quad \mathscr{H}_3 := h_3 t_2 \quad \text{if } d = 3. \end{cases}$$

For a multi-index $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$, we use the following notation:

$$\mathscr{H}^\beta := \mathscr{H}_1^{\beta_1} \cdots \mathscr{H}_d^{\beta_d}, \quad \mathscr{H}^{-\beta} := \mathscr{H}_1^{-\beta_1} \cdots \mathscr{H}_d^{-\beta_d}.$$

We also define $h^\beta := h_1^{\beta_1} \cdots h_d^{\beta_d}$ and $h^{-\beta} := h_1^{-\beta_1} \cdots h_d^{-\beta_d}$.

Definition 3.5.2. We define vectors $r_n \in \mathbb{R}^d$, $n = 1, \ldots, d$, as follows. If $d = 2$,

$$r_1 := (1, 0)^T, \quad r_2 := (s, t)^T,$$

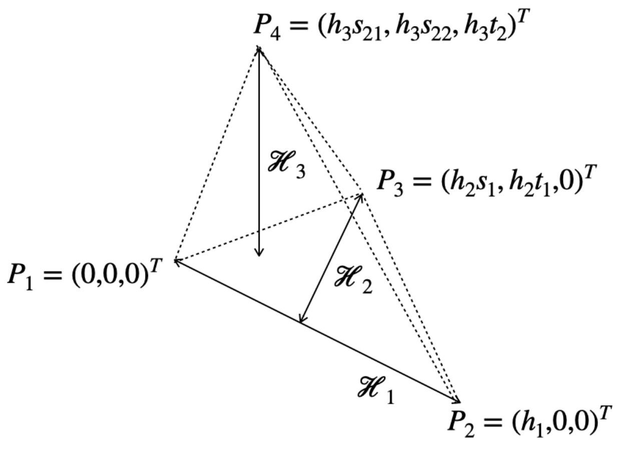

Fig. 3.5: New parameter $\mathscr{H}_i$, $i = 1, 2, 3$

and if $d = 3$,

$$r_1 := (1,0,0)^T, \quad r_3 := (s_{21}, s_{22}, t_2)^T,$$
$$\begin{cases} r_2 := (s_1, t_1, 0)^T & \text{for case (i)}, \\ r_2 := (-s_1, t_1, 0)^T & \text{for case (ii)}. \end{cases}$$

For a sufficiently smooth function φ and vector function $v := (v_1, \ldots, v_d)^T$, we define the directional derivative as, for $i \in \{1 : d\}$,

$$\frac{\partial \varphi}{\partial r_i} := [(\mathcal{A}_T r_i) \cdot \nabla_x]\varphi = \sum_{j_0=1}^{d} (\mathcal{A}_T r_i)_{j_0} \frac{\partial \varphi}{\partial x_{j_0}},$$
$$\frac{\partial v}{\partial r_i} := \left(\frac{\partial v_1}{\partial r_i}, \ldots, \frac{\partial v_d}{\partial r_i} \right)^T = ([(\mathcal{A}_T r_i) \cdot \nabla_x]v_1, \ldots, [(\mathcal{A}_T r_i) \cdot \nabla_x]v_d)^T,$$

where $\mathcal{A}_T \in O(d)$ is the orthogonal matrix defined in Definition 3.4.1. For a multi-index $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$, we use the notation

$$\partial_r^\beta \varphi := \frac{\partial^{|\beta|} \varphi}{\partial r_1^{\beta_1} \ldots \partial r_d^{\beta_d}}.$$

Furthermore, for $\varphi^s = \varphi \circ \Phi_{T^s}$ and $v^s = \Psi_{T^s}^{-1} v$, we define the directional derivative as, for $i \in \{1 : d\}$,

$$\frac{\partial \varphi^s}{\partial {r_i}^s} := [r_i \cdot \nabla_{x^s}]\varphi^s = \sum_{j_0=1}^{d} (r_i)_{j_0} \frac{\partial \varphi^s}{\partial x^s_{j_0}},$$
$$\frac{\partial v^s}{\partial r^s_i} := \left(\frac{\partial v^s_1}{\partial r^s_i}, \dots, \frac{\partial v^s_d}{\partial r^s_i} \right)^T := ([r_i \cdot \nabla_{x^s}]v^s_1, \dots, [r_i \cdot \nabla_{x^s}]v^s_d)^T.$$

For a multi-index $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$, we use the notation

$$\partial^\beta_{r^s} \varphi^s := \frac{\partial^{|\beta|} \varphi^s}{(\partial r^s_1)^{\beta_1} \dots (\partial r^s_d)^{\beta_d}}.$$

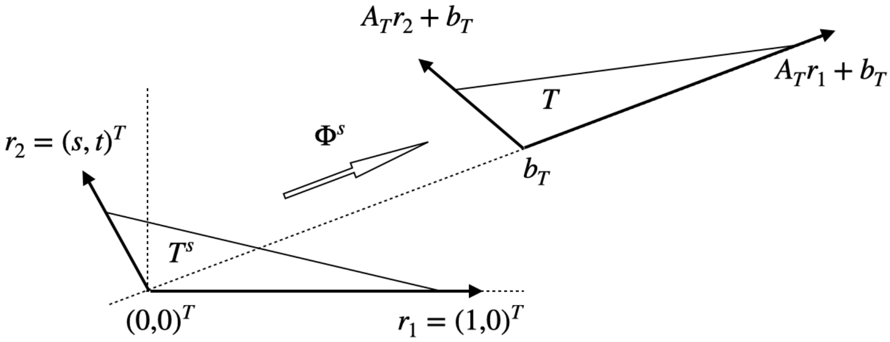

Fig. 3.6: Affine mapping Φ^s and vectors r_i, $i = 1, 2$

Remark 3.5.3. The vectors r_i, $i \in \{1, \dots, d\}$ defined in Definition 3.5.2 are unit vectors. Indeed, if $d = 2$,

$$|r_1|_E = 1, \quad |r_2|_E = \sqrt{s^2 + t^2} = 1,$$

if $d = 3$,

$$|r_1|_E = 1, \quad |r_2|_E = \sqrt{s_1^2 + t_1^2} = 1, \quad |r_3|_E = \sqrt{s_{21}^2 + s_{22}^2 + t_2^2} = 1.$$

Note 3.5.4. Recall that

$$|s| \leq 1, \; h_2 \leq h_1 \quad \text{if } d = 2,$$
$$|s_1| \leq 1, \; |s_{21}| \leq 1, \; h_2 \leq h_3 \leq h_1 \quad \text{if } d = 3.$$

When $d = 3$, if Condition 3.3.1 is imposed, there exists a positive constant M independent of h_T such that $|s_{22}| \leq M \frac{h_2 t_1}{h_3}$. We thus have, if $d = 2$,

$$h_1 |[\widetilde{\mathcal{A}}]_{j1}| \leq \mathscr{H}_j, \quad h_2 |[\widetilde{\mathcal{A}}]_{j2}| \leq \mathscr{H}_j, \quad j = 1, 2,$$

and, if $d = 3$, for $\widetilde{A} \in \{\widetilde{A}_1, \widetilde{A}_2\}$ and $j = 1, 2, 3$,

$$h_1 |[\widetilde{\mathcal{A}}]_{j1}| \leq \mathscr{H}_j, \quad h_2 |[\widetilde{\mathcal{A}}]_{j2}| \leq \mathscr{H}_j, \quad h_3 |[\widetilde{\mathcal{A}}]_{j3}| \leq \max\{1, M\} \mathscr{H}_j, \quad j = 1, 2, 3.$$

3.6 Euclidean Condition Number

Lemma 3.6.1. *It holds that*

$$\|\widehat{\mathcal{A}}^{(d)}\|_2 \leq h_{T^s}, \quad \|\widehat{\mathcal{A}}^{(d)}\|_2 \|(\widehat{\mathcal{A}}^{(d)})^{-1}\|_2 = \frac{\max\{h_1, \cdots, h_d\}}{\min\{h_1, \cdots, h_d\}}, \tag{3.6.1a}$$

$$\|\widetilde{\mathcal{A}}\|_2 \leq \begin{cases} \sqrt{2} & \text{if } d = 2, \\ 2 & \text{if } d = 3, \end{cases} \qquad \|\widetilde{\mathcal{A}}\|_2 \|\widetilde{\mathcal{A}}^{-1}\|_2 \leq \begin{cases} \frac{h_1 h_2}{|T^s|} = \frac{H_{T^s}}{h_{T^s}} & \text{if } d = 2, \\ \frac{2}{3} \frac{h_1 h_2 h_3}{|T^s|} = \frac{2}{3} \frac{H_{T^s}}{h_{T^s}} & \text{if } d = 3, \end{cases} \tag{3.6.1b}$$

$$\|\mathcal{A}_T\|_2 = 1, \quad \|\mathcal{A}_T^{-1}\|_2 = 1. \tag{3.6.1c}$$

where a parameter H_{T^s} is defined later (Definition 4.1.2). Furthermore, we have

$$|\det(\mathcal{A}^s)| = |\det(\widetilde{\mathcal{A}})| |\det(\widehat{\mathcal{A}}^{(d)})| = \frac{|T^s|}{|\widetilde{T}|} \frac{|\widetilde{T}|}{|\widehat{T}|} = d! |T^s|, \quad |\det(\mathcal{A}_T)| = 1. \tag{3.6.2}$$

Proof. We fist show the equality (3.6.2). Because we obtain

$$\int_{T^s} dx = \int_{\widehat{T}} |\det(\mathcal{A}^s)| d\hat{x}, \; \int_{T^s} dx = \int_{\widetilde{T}} |\det(\widetilde{\mathcal{A}})| d\tilde{x}, \; \int_{\widetilde{T}} d\tilde{x} = \int_{\widehat{T}} |\det(\widehat{\mathcal{A}}^{(d)})| d\hat{x},$$

we conclude (3.6.2).

We next show the equality (3.6.1a). From

$$(\widehat{\mathcal{A}}^{(d)})^T \widehat{\mathcal{A}}^{(d)} = \operatorname{diag}(h_1^2, \ldots, h_d^2), \quad (\widehat{\mathcal{A}}^{(d)})^{-1} (\widehat{\mathcal{A}}^{(d)})^{-T} = \operatorname{diag}(h_1^{-2}, \ldots, h_d^{-2}),$$

we have

$$\|\widehat{\mathcal{A}}^{(d)}\|_2 = \lambda_{\max}((\widehat{\mathcal{A}}^{(d)})^T \widehat{\mathcal{A}}^{(d)})^{1/2} = \max\{h_1, \cdots, h_d\} \le h_{T^s},$$

and

$$\begin{aligned}\|\widehat{\mathcal{A}}^{(d)}\|_2 \|(\widehat{\mathcal{A}}^{(d)})^{-1}\|_2 &= \lambda_{\max}((\widehat{\mathcal{A}}^{(d)})^T \widehat{\mathcal{A}}^{(d)})^{1/2} \lambda_{\max}((\widehat{\mathcal{A}}^{(d)})^{-1} (\widehat{\mathcal{A}}^{(d)})^{-T})^{1/2} \\ &= \frac{\max\{h_1, \cdots, h_d\}}{\min\{h_1, \cdots, h_d\}},\end{aligned}$$

which leads to (3.6.1a).

We next show the equality (3.6.1b). We consider for each dimension, $d = 2, 3$.

Two-dimensional case

From

$$\widetilde{\mathcal{A}}^T \widetilde{\mathcal{A}} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}, \quad \widetilde{\mathcal{A}}^{-1} \widetilde{\mathcal{A}}^{-T} = \frac{1}{t^2} \begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix},$$

we have

$$\|\widetilde{\mathcal{A}}\|_2 = \lambda_{\max}(\widetilde{\mathcal{A}}^T \widetilde{\mathcal{A}})^{1/2} \le (1 + |s|)^{1/2} \le \sqrt{2},$$

and

$$\|\widetilde{\mathcal{A}}\|_2 \|\widetilde{\mathcal{A}}^{-1}\|_2 = \lambda_{\max}(\widetilde{\mathcal{A}}^T \widetilde{\mathcal{A}})^{1/2} \lambda_{\max}(\widetilde{\mathcal{A}}^{-1} \widetilde{\mathcal{A}}^{-T})^{1/2} \le \frac{2}{t} = \frac{h_1 h_2}{|T^s|},$$

which leads to (3.6.1b) for $d = 2$. We here used the fact that $|T^s| = \frac{1}{2} h_1 h_2 t$.

Three-dimensional case

The matrices $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ introduced in (3.1.5) can be decomposed as $\widetilde{\mathcal{A}}_1 = \widetilde{\mathcal{M}}_0 \widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{A}}_2 = \widetilde{\mathcal{M}}_0 \widetilde{\mathcal{M}}_2$ with

$$\widetilde{\mathcal{M}}_0 := \begin{pmatrix} 1 & 0 & s_{21} \\ 0 & 1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \ \widetilde{\mathcal{M}}_1 := \begin{pmatrix} 1 & s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \widetilde{\mathcal{M}}_2 := \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of $\widetilde{\mathcal{M}}_2^T \widetilde{\mathcal{M}}_2$ coincide with those of $\widetilde{\mathcal{M}}_1^T \widetilde{\mathcal{M}}_1$, and we may therefore suppose without loss of generality that we have Case (i).

We have the inequalities

$$\begin{aligned}\|\widetilde{\mathcal{A}}_1\|_2 &= \lambda_{\max}(\widetilde{\mathcal{A}}_1^T\widetilde{\mathcal{A}}_1)^{1/2} \le \lambda_{\max}(\widetilde{\mathcal{M}}_0^T\widetilde{\mathcal{M}}_0)^{1/2}\lambda_{\max}(\widetilde{\mathcal{M}}_1^T\widetilde{\mathcal{M}}_1)^{1/2} \\ &\le \left(1+\sqrt{s_{21}^2+s_{22}^2}\right)^{1/2}(1+|s_1|)^{1/2} \le 2,\end{aligned}$$

and

$$\begin{aligned}\|\widetilde{\mathcal{A}}_1\|_2\|\widetilde{\mathcal{A}}_1^{-1}\|_2 &= \lambda_{\max}(\widetilde{\mathcal{A}}_1^T\widetilde{\mathcal{A}}_1)^{1/2}\lambda_{\max}(\widetilde{\mathcal{A}}_1^{-1}\widetilde{\mathcal{A}}_1^{-T})^{1/2} \\ &\le \frac{\left(1+\sqrt{s_{21}^2+s_{22}^2}\right)(1+|s_1|)}{t_1t_2} \le \frac{4}{t_1t_2} = \frac{2}{3}\frac{h_1h_2h_3}{|T^s|},\end{aligned}$$

where we used the fact that $|T^s| = \frac{1}{6}h_1h_2h_3t_1t_2$.

Because the length of all edges of a simplex and measure of the simplex are not changed by a rotation and mirror imaging matrix and $\mathcal{A}_T, \mathcal{A}_T^{-1} \in O(d)$,

$$|\det(\mathcal{A}_T)| = \frac{|T|}{|T^s|} = 1, \quad \|\mathcal{A}_T\|_2 = 1, \quad \|\mathcal{A}_T^{-1}\|_2 = 1,$$

which is (3.6.1c) and (3.6.2). □

Remark 3.6.2. It holds that

$$\|\widetilde{\mathcal{A}}^{-1}\|_2 \le \begin{cases} C_2^{tilde}\frac{h_1h_2}{|T^s|} = C_2^{tilde}\frac{H_{T^s}}{h_{T^s}} & \text{if } d=2, \\ \frac{C_3^{tilde}}{2}\frac{h_1h_2h_3}{|T^s|} = \frac{C_3^{tilde}}{2}\frac{H_{T^s}}{h_{T^s}} & \text{if } d=3, \end{cases} \tag{3.6.3}$$

where $C_2^{tilde} := \max\{t, |s|, 1\}$ and $C_3^{tilde} := \max\{t_1t_2, s_1t_2, t_2, s_1|s_{22}|+t_1|s_{21}|, |s_{22}|, t_1\}$.

Proof. Recall that if $d = 2$,

$$\widetilde{\mathcal{A}}^{-1} = \frac{1}{t}\begin{pmatrix} t & -s \\ 0 & 1 \end{pmatrix},$$

and if $d = 3$,

$$\begin{aligned}\widetilde{\mathcal{A}}_1^{-1} &= \frac{1}{t_1t_2}\begin{pmatrix} t_1t_2 & -s_1t_2 & s_1s_{22}-t_1s_{21} \\ 0 & t_2 & -s_{22} \\ 0 & 0 & t_1 \end{pmatrix}, \\ \widetilde{\mathcal{A}}_2^{-1} &= \frac{1}{t_1t_2}\begin{pmatrix} t_1t_2 & s_1t_2 & -s_1s_{22}-t_1s_{21} \\ 0 & t_2 & -s_{22} \\ 0 & 0 & t_1 \end{pmatrix}.\end{aligned}$$

We consider for each dimension, $d = 2, 3$.

Two-dimensional case

From (1.3.1), we have

$$\|\mathcal{A}^{-1}\|_2 \le 2\|\mathcal{A}^{-1}\|_{\max} \le 2\max\{t, |s|, 1\}\frac{1}{t} = \max\{t, |s|, 1\}\frac{h_1 h_2}{|T^s|}.$$

Three-dimensional case

Let $\mathcal{A}^{-1} \in \{\mathcal{A}_1^{-1}, \mathcal{A}_2^{-1}\}$. From (1.3.1), we have

$$\begin{aligned}
\|\mathcal{A}^{-1}\|_2 &\le 3\|\mathcal{A}^{-1}\|_{\max} \\
&\le 3\max\{t_1 t_2, s_1 t_2, t_2, s_1|s_{22}| + t_1|s_{21}|, |s_{22}|, t_1\}\frac{1}{t_1 t_2} \\
&= \frac{1}{2}\max\{t_1 t_2, s_1 t_2, t_2, s_1|s_{22}| + t_1|s_{21}|, |s_{22}|, t_1\}\frac{h_1 h_2 h_3}{|T^s|}.
\end{aligned}$$

□

Remark 3.6.3. As described in Section 2.1, using $\mathcal{A}^s = \widetilde{\mathcal{A}}\widehat{\mathcal{A}}^{(d)}$ instead of $\mathcal{B}_T$, Lemma 3.6.1 and the standard argument, the interpolation error estimate is rewritten as

$$\begin{aligned}
|\varphi - I_{T^s}\varphi|_{W^{1,p}(T^s)} &\le c\left(\|\mathcal{A}^s\|_2\|(\mathcal{A}^s)^{-1}\|_2\right)\|\mathcal{A}^s\|_2|\varphi|_{W^{2,p}(T^s)} \\
&\le c\frac{\max\{h_1, \cdots, h_d\}}{\min\{h_1, \cdots, h_d\}}\frac{H_{T^s}}{h_{T^s}}h_{T^s}|\varphi|_{W^{2,p}(T^s)}.
\end{aligned}$$

In general, the optimal order is $\mathcal{O}(h_{T^s})$ for a linear interpolation operator I_{T^s}. Additional two factors

$$\frac{\max\{h_1, \cdots, h_d\}}{\min\{h_1, \cdots, h_d\}}, \quad \frac{H_{T^s}}{h_{T^s}}$$

may worsen the convergence order on anisotropic meshes. However, imposing an additional assumption (see Theorem 5.5.1) and a geometric condition (see Condition 4.3.1), the above two factors can disappear explicitly.

Chapter 4

New Semi-regularity Condition

4.1 New Parameters

We introduce two new parameters, which are proposed in [47].

Definition 4.1.1 (New parameter H_T)**.** For $T \in \mathbb{T}_h$, we denote by L_i edges of the simplex T. We define the new parameter H_T as

$$H_T := \frac{h_T^2}{|T|} \min_{1 \le i \le 3} |L_i| \quad \text{if } d = 2, \quad H_T := \frac{h_T^2}{|T|} \min_{1 \le i,j \le 6, i \neq j} |L_i||L_j| \quad \text{if } d = 3. \tag{4.1.1}$$

Definition 4.1.2 (New parameter H_{T^s})**.** The parameter H_{T^s} is defined as

$$H_{T^s} := \frac{\prod_{i=1}^d h_i}{|T^s|} h_{T^s}. \tag{4.1.2}$$

Remark 4.1.3. We set

$$H(h) := \max_{T \in \mathbb{T}_h} H_T.$$

If the maximum-angle condition is violated, the parameter $H(h)$ may diverge as $h \to 0$ on anisotropic meshes. Therefore, imposing the maximum-angle condition for mesh partitions guarantees the convergence of finite element methods [11]. Reference [13] studied cases in which the finite element solution may not converge to the exact solution.

4.2 Properties of New Parameters

We first show relation between h_{T^s} and H_{T^s}.

Lemma 4.2.1. *It holds that*

$$\begin{cases} h_{T^s} \le \frac{1}{2} H_{T^s} & \text{if } d = 2, \\ h_{T^s} < \frac{1}{6} H_{T^s} & \text{if } d = 3. \end{cases} \tag{4.2.1}$$

Proof. We consider for each dimension, $d = 2, 3$.

Two-dimensional case

By construct of the standard element in the two-dimensional case, the angle $\theta_{\max} := \angle P_2 P_1 P_3$ is the maximum angle of T^s. We then have $\frac{\pi}{3} < \theta_{\max} < \pi$, that is, $0 < \sin\theta_{\max} \le 1$. Therefore, it holds that

$$H_{T^s} = \frac{h_1 h_2}{|T^s|} h_{T^s} = \frac{2}{\sin\theta_{\max}} h_{T^s} \ge 2 h_{T^s}.$$

We here used the fact that $|T^s| = \frac{1}{2} h_1 h_2 \sin\theta_{\max}$.

Three-dimensional case

We denote by ϕ_{T^s} the angle between the base $\triangle P_1 P_2 P_3$ of T^s and the segment $\overline{P_1 P_4}$. Recall that there are two types' standard element, (Type i) or (Type ii). We denote by θ_{T^s}

(Type i) the angle between the segments $\overline{P_1 P_2}$ and $\overline{P_1 P_3}$, that is, $\theta_{T^s} := \angle P_2 P_1 P_3$, or

(Type ii) the angle between the segments $\overline{P_2 P_1}$ and $\overline{P_2 P_3}$, that is, $\theta_{T^s} := \angle P_1 P_2 P_3$.

We set $t_1 := \sin\theta_{T^s}$ and $t_2 := \sin\phi_{T^s}$. By construct of the standard element in the three-dimensional case, the angle $\angle P_1 P_3 P_2$ is the maximum angle of the base $\triangle P_1 P_2 P_3$ of T^s. Therefore, we have $0 < \theta_{T^s} < \frac{\pi}{2}$. Because $0 < \phi_{T^s} < \pi$, it holds that

$$H_{T^s} = \frac{h_1 h_2 h_3}{|T^s|} h_{T^s} = \frac{6}{\sin\theta_{T^s} \sin\phi_{T^s}} h_{T^s} > 6 h_{T^s}.$$

We here used the fact that $|T^s| = \frac{1}{6} h_1 h_2 h_3 \sin\theta_{T^s} \sin\phi_{T^s}$. $\square$

According to the following lemma, the parameters H_{T^s} and H_T are equivalent.

Lemma 4.2.2. *It holds that*

$$\frac{1}{2}H_T < H_{T^s} < 2H_T. \tag{4.2.2}$$

Furthermore, in the two-dimensional case, H_T is equivalent to the circumradius R_2 of T.

Proof. We consider for each dimension, $d = 2, 3$.

Two-dimensional case

Let L_i $(i = 1, 2, 3)$ denote edges of the triangle T with $|L_1| \le |L_2| \le |L_3|$. It obviously holds that $h_2 = |L_1|$ and $h_{T^s} = |L_3| = h_T$. Because $h_2 \le h_1 < 2h_{T^s}$ and $h_{T^s} < h_1 + h_2 \le 2h_1$ for the triangle $\triangle P_1P_2P_3$, it holds that

$$\frac{1}{2}h_T = \frac{1}{2}h_{T^s} < h_1 = |L_2| < 2h_{T^s} = 2h_T.$$

We thus have

$$\frac{1}{2}H_T = \frac{1}{2}\frac{|L_1|}{|T|}h_T^2 < H_{T^s} = \frac{h_1h_2}{|T^s|}h_{T^s} < 2\frac{|L_1|}{|T|}h_T^2 = 2H_T.$$

Furthermore, it holds that

$$2R_2 = 2\frac{|L_1||L_2||L_3|}{4|T|} < H_T = \frac{|L_1|}{|T|}h_T^2 < 8\frac{|L_1||L_2||L_3|}{4|T|} = 8R_2.$$

Three-dimensional case

Let L_i $(i = 1, \dots, 6)$ denote edges of the triangle T with $|L_1| \le |L_2| \le \cdots \le |L_6|$. It obviously holds that $h_2 = |L_1|$ and $h_{T^s} = |L_6| = h_T$. Recall that there are two types' standard element, (Type i) or (Type ii).

(Type i) We set $h_4 := |P_3 - P_4|$, $h_5 := |P_2 - P_4|$ and $h_6 := |P_2 - P_3|$. Because $h_1 = |E_{\max}^{(\min)}| = |P_1 - P_2|$ is the longest edge among the four edges that share an end point with E_1, it holds that

$$h_2 \le \min\{h_3, h_4, h_6\} \le \max\{h_3, h_4, h_6\} \le h_1. \tag{4.2.3}$$

Because P_1 and P_4 belong to the same half-space for the triangle $\triangle P_1P_2P_4$, it holds that

$$\begin{cases} h_3 \le h_5 \le h_1 = h_{T^s} & \text{or} \\ h_3 \le h_1 \le h_5 = h_{T^s}. \end{cases}$$

We thus have

$$\begin{cases} h_3 \le h_5 \le h_1 = h_{T^s} & \text{or} \\ h_3 \le h_1 \le h_{T^s} < 2h_1, & \frac{1}{2}h_{T^s} < h_1 \le h_{T^s}. \end{cases}$$

Because $h_3 \le h_5$, the length of the edge L_2 is equal to the one of h_3, h_4 or h_6.

Assume that $|L_2| = h_3$. We then have

$$\frac{1}{2}H_T = \frac{1}{2}\frac{|L_1||L_2|}{|T|}h_T^2 < H_{T^s} = \frac{h_1 h_2 h_3}{|T^s|}h_{T^s} \le \frac{|L_1||L_2|}{|T|}h_T^2 = H_T(< 2H_T).$$

Assume that $|L_2| = h_4$. We consider the triangle $\triangle P_1P_3P_4$. From the assumption, we have $h_2 \le h_4 \le h_3$ and $\frac{1}{2}h_3 < h_4 \le h_3$. We then obtain

$$\frac{1}{2}H_T = \frac{1}{2}\frac{|L_1||L_2|}{|T|}h_T^2 < H_{T^s} = \frac{h_1 h_2 h_3}{|T^s|}h_{T^s} < 2\frac{|L_1||L_2|}{|T|}h_T^2 = 2H_T.$$

Assume that $|L_2| = h_6$. We consider the triangle $\triangle P_1P_2P_3$. Because P_1 and P_3 belong to the same half-space for the triangle $\triangle P_1P_2P_3$, it holds that $h_2 \le h_6 \le h_1$ and $\frac{1}{2}h_1 < h_6 \le h_1$. From (4.2.3), we have

$$\frac{1}{2}h_3 \le \frac{1}{2}h_1 < h_6 \le h_1.$$

Because $h_6 \le h_3$, we then obtain

$$\frac{1}{2}H_T = \frac{1}{2}\frac{|L_1||L_2|}{|T|}h_T^2 < H_{T^s} = \frac{h_1 h_2 h_3}{|T^s|}h_{T^s} < 2\frac{|L_1||L_2|}{|T|}h_T^2 = 2H_T.$$

(Type ii) We set $h_4 := |P_3 - P_4|$, $h_5 := |P_2 - P_4|$, and $h_6 := |P_1 - P_3|$. Because $h_1 = |E_{\max}^{(\min)}| = |P_1 - P_2|$ is the longest edge among the four edges that share an end point with L_1, it holds that

$$h_2 \le \min\{h_4, h_5, h_6\} \le \max\{h_4, h_5, h_6\} \le h_1. \tag{4.2.4}$$

Because P_1 and P_4 belong to the same half-space for the triangle $\triangle P_1P_2P_4$ and (4.2.4), it holds that

$$h_3 \le h_5 \le h_1.$$

This implies that $h_1 = h_{T^s}$. Therefore, the length of the edge L_2 is equal to the one of h_3, h_4, or h_6.

Assume that $|L_2| = h_3$. We then have

$$\left(\frac{1}{2}H_T <\right) H_T = \frac{|L_1||L_2|}{|T|}h_T^2 = H_{T^s} = \frac{h_1 h_2 h_3}{|T^s|}h_{T^s}$$
$$= \frac{|L_1||L_2|}{|T|}h_T^2 = H_T(< 2H_T).$$

Assume that $|L_2| = h_4$. For the triangle $\triangle P_2P_3P_4$, we have

$$h_2 \leq h_4 \leq h_5 < 2h_4.$$

Because $h_3 \leq h_5$ and $h_4 \leq h_3$, it holds that

$$\left(\frac{1}{2}H_T <\right) H_T = \frac{|L_1||L_2|}{|T|}h_T^2 \leq H_{T^s} = \frac{h_1 h_2 h_3}{|T^s|}h_{T^s}$$
$$< 2\frac{|L_1||L_2|}{|T|}h_T^2 = 2H_T.$$

Assume that $|L_2| = h_6$. We have $h_1 < h_2 + h_6 < 2h_6$ for the triangle $\triangle P_1P_2P_3$. Therefore, since $h_6 \leq h_3 \leq h_1$, we obtain

$$\left(\frac{1}{2}H_T <\right) H_T = \frac{|L_1||L_2|}{|T|}h_T^2 \leq H_{T^s} = \frac{h_1 h_2 h_3}{|T^s|}h_{T^s}$$
$$< 2\frac{|L_1||L_2|}{|T|}h_T^2 = 2H_T.$$

□

4.3 New Geometric Mesh Condition equivalent to the Maximum-angle Condition

As mentioned in Section 2.3, as far as we know, there is not a semi-regularity condition which equivalent to Synge's condition (2.3.1) for $d = 3$. In what following, we present the new geometric condition which is proposed in [47]. This condition includes the case of $d = 2$.

Condition 4.3.1. A family of meshes $\{\mathbb{T}_h\}$ has a semi-regular property if there exists $\gamma_8 > 0$ such that

$$\frac{H_T}{h_T} \leq \gamma_8 \quad \forall \mathbb{T}_h \in \{\mathbb{T}_h\}, \quad \forall T \in \mathbb{T}_h. \tag{4.3.1}$$

Equivalently, there exists $\gamma_9 > 0$ such that

$$\frac{H_{T^s}}{h_{T^s}} \leq \gamma_9. \tag{4.3.2}$$

In the following theorem, we show that the new semi-regularity condition is equivalent to the maximum-angle condition.

Theorem 4.3.2. *For a family of conformal meshes* $\{\mathbb{T}_h\}$*, there exists* $\gamma_8 > 0$ *such that* (4.3.1) *if and only if*

(d=2) *there exists* $\frac{\pi}{3} \leq \gamma_5 < \pi$ *such that* (2.2.1) *for any* $\mathbb{T}_h \in \{\mathbb{T}_h\}$ *and any simplex* $T \in \mathbb{T}_h$*;*

(d=3) *there exists a constant* $0 < \gamma_7 < \pi$ *such that* (2.3.1) *for any* $\mathbb{T}_h \in \{\mathbb{T}_h\}$ *and any simplex* $T \in \mathbb{T}_h$*.*

Equivalently, there exists $\gamma_9 > 0$ *such that* (4.3.2) *if and only if*

(d=2) *there exists* $\frac{\pi}{3} \leq \gamma_{10} < \pi$ *such that*

$$\theta_{T^s,\max} \leq \gamma_{10}, \tag{4.3.3}$$

where $\theta_{T^s,\max}$ *is the maximum angle of* T^s*, and*

(d=3) *there exists a constant* $0 < \gamma_{11} < \pi$ *such that*

$$\theta_{T^s,\max} \leq \gamma_{11}, \quad \psi_{T^s,\max} \leq \gamma_{11}, \tag{4.3.4}$$

where $\theta_{T^s,\max}$ *is the maximum angle of all triangular faces of the tetrahedron* T^s *and* $\psi_{T^s,\max}$ *is the maximum dihedral angle of* T^s*.*

Proof. We consider for each dimension, $d = 2, 3$.

Two-dimensional case

We use the previous research in [57], that is, there exists $\frac{\pi}{3} \leq \gamma_5 < \pi$ such that (2.2.1) for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$ if and only if there exists $\gamma_6 > 0$ such that, for any $\mathbb{T}_h \in \{\mathbb{T}_h\}$ and any simplex $T \in \mathbb{T}_h$

$$\frac{R_2}{h_T} \leq \gamma_6.$$

Combining this result with the fact that H_T is equivalent to the circumradius R_2 of T (Lemma 4.2.2), we have the desired conclusion.

Three-dimensional case

The proof can be found in [50], also see Appendix A. □

4.4 Good elements or not for $d = 2, 3$?

In this subsection, we consider good elements on meshes. In this paper, we define "good elements" on meshes as there existing $\gamma_0 > 0$ satisfying (4.3.1). We treat a "Right-angled triangle", "Blade" and "Dagger" for $d = 2$, and "Spire", "Spear", "Spindle", "Spike", "Splinter" and "Sliver" for $d = 3$, which are introduced in [23]. We give the quantities $h_{\max}/h_{\min}$ and H_T/h_T for those elements.

4.4.1 Isotropic mesh

If the geometric condition (2.1.3) is satisfied, it holds that

$$\frac{H_T}{h_T} \leq \frac{h_T^d}{|T|} \leq \frac{1}{\gamma_3}, \quad \frac{h_{\max}}{h_{\min}} \leq c\frac{h_{T^s}^d}{|T^s|} = c\frac{h_T^d}{|T|} \leq \frac{c}{\gamma_3}.$$

In this case, elements satisfying the geometric condition (2.1.3) are "good."

4.4.2 Anisotropic mesh: two-dimensional case

Let $S \subset \mathbb{R}^2$ be a triangle. Let $0 < t \ll 1$, $t \in \mathbb{R}$ and $\varepsilon, \delta, \gamma \in \mathbb{R}$.

Example 4.4.1 (Right-angled triangle)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $P_1 := (0,0)^T$, $P_2 := (t,0)^T$ and $P_3 := (0,t^\varepsilon)^T$ with $1 < \varepsilon$. We then have $h_1 = t$ and $h_2 = t^\varepsilon$; i.e.,

$$\frac{h_{\max}}{h_{\min}} \leq t^{1-\varepsilon} \to \infty \quad \text{as } t \to 0, \quad \frac{H_S}{h_S} = 2.$$

In this case, the element S is "good."

Example 4.4.2 (Dagger)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $P_1 := (0,0)^T$, $P_2 := (t,0)^T$ and $P_3 := (t^\delta, t^\varepsilon)^T$ with $1 < \varepsilon < \delta$. We then have $h_1 = \sqrt{(t-t^\delta)^2 + t^{2\varepsilon}}$ and $h_2 = \sqrt{t^{2\delta} + t^{2\varepsilon}}$; i.e.,

$$\begin{aligned} \frac{h_{\max}}{h_{\min}} &= \frac{\sqrt{(t-t^\delta)^2 + t^{2\varepsilon}}}{\sqrt{t^{2\delta} + t^{2\varepsilon}}} \leq ct^{1-\varepsilon} \to \infty \quad \text{as } t \to 0, \\ \frac{H_S}{h_S} &= \frac{\sqrt{(t-t^\delta)^2 + t^{2\varepsilon}}\sqrt{t^{2\delta} + t^{2\varepsilon}}}{\frac{1}{2}t^{1+\varepsilon}} \leq c. \end{aligned}$$

In this case, the element S is "good."

Remark 4.4.3. In the above examples, $h_2 \approx \mathscr{H}_2$ holds. That is, the good element $S \subset \mathbb{R}^2$ may satisfy conditions such as $h_2 \approx \mathscr{H}_2$.

Example 4.4.4 (Blade)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $P_1 := (0,0)^T$, $P_2 := (2t,0)^T$ and $P_3 := (t,t^\varepsilon)^T$ with $1 < \varepsilon$. We then have $h_1 = h_2 = \sqrt{t^2 + t^{2\varepsilon}}$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = 1, \quad \frac{H_S}{h_S} = \frac{t^2 + t^{2\varepsilon}}{t^{1+\varepsilon}} \to \infty \quad \text{as } t \to 0.$$

In this case, the element S is "not good."

Example 4.4.5 (Dagger)**.** Let $S \subset \mathbb{R}^2$ be the simplex with vertices $P_1 := (0,0)^T$, $P_2 := (t,0)^T$ and $P_3 := (t^\delta, t^\varepsilon)^T$ with $1 < \delta < \varepsilon$. We then have $h_1 = \sqrt{(t-t^\delta)^2 + t^{2\varepsilon}}$ and $h_2 = \sqrt{t^{2\delta} + t^{2\varepsilon}}$; i.e.,

$$\begin{aligned} \frac{h_{\max}}{h_{\min}} &= \frac{\sqrt{(t-t^\delta)^2 + t^{2\varepsilon}}}{\sqrt{t^{2\delta} + t^{2\varepsilon}}} \le c t^{1-\delta} \to \infty \quad \text{as } t \to 0, \\ \frac{H_S}{h_S} &= \frac{\sqrt{(t-t^\delta)^2 + t^{2\varepsilon}}\sqrt{t^{2\delta} + t^{2\varepsilon}}}{\frac{1}{2}t^{1+\varepsilon}} \le c t^{\delta-\varepsilon} \to \infty \quad \text{as } t \to 0. \end{aligned}$$

In this case, the element S is "not good."

4.4.3 Anisotropic mesh: three-dimensional case

Example 4.4.6. Let $T \subset \mathbb{R}^3$ be a tetrahedron. Let S be the base of T; i.e., $S = \triangle P_1 P_2 P_3$. Recall that

$$\frac{H_T}{h_T} = \frac{h_1 h_2 h_3}{|T|} = \frac{h_1 h_2}{\frac{1}{2} h_1 h_2 t_1} \frac{h_3}{\frac{1}{3} h_3 t_2} \le \frac{H_S}{h_S} \frac{\alpha_3}{\frac{1}{3}\mathscr{H}_3}. \tag{4.4.1}$$

If the triangle S is "not good" such as in Examples 4.4.4 and 4.4.5, the quantity (4.4.1) may diverge. In the following, we consider the case that the triangle S is "good".

Assume that there exists a positive constant M such that $\frac{H_S}{h_S} \le M$. For simplicity, we set $P_1 := (0,0,0)^T$, $P_2 := (2t,0,0)^T$, and $P_3 := (2t - \sqrt{4t^2 - t^{2\gamma}}, t^\gamma, 0)^T$ with $1 < \gamma$. Then,

$$h_1 = 2t, \quad h_2 = \sqrt{\frac{4t^{2\gamma}}{2 + \sqrt{4 - t^{2\gamma - 2}}}},$$

and because $h_{\max} \approx ct$,

$$\frac{h_{\max}}{h_{\min}} \le \frac{ct}{\alpha_2} \le c t^{1-\gamma} \to \infty \quad \text{as } t \to 0.$$

If we set $P_4 := (t, 0, t^{\varepsilon})^T$ with $1 < \varepsilon$, the triangle $\triangle P_1 P_2 P_4$ is the blade (Example 4.4.4). Then,

$$h_3 = \sqrt{t^2 + t^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{t^{2t+\gamma}}{t^{1+\gamma+\varepsilon}} \leq c t^{1-\varepsilon} \to \infty \quad \text{as } t \to 0.$$

In this case, the element T is "not good."

If we set $P_4 := (t^{\delta}, 0, t^{\varepsilon})^T$ with $1 < \delta < \varepsilon < \gamma$, the triangle $\triangle P_1 P_2 P_4$ is the dagger (Example 4.4.5). Then,

$$h_3 = \sqrt{t^{2\delta} + t^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{t^{1+\gamma+\delta}}{t^{1+\gamma+\varepsilon}} \leq c t^{\delta-\varepsilon} \to \infty \quad \text{as } t \to 0.$$

In this case, the element T is "not good."

If we set $P_4 := (t^{\delta}, 0, t^{\varepsilon})^T$ with $1 < \varepsilon < \delta < \gamma$, the triangle $\triangle P_1 P_2 P_4$ is the dagger (Example 4.4.2). Then,

$$h_3 = \sqrt{t^{2\delta} + t^{2\varepsilon}}.$$

We thus have

$$\frac{H_T}{h_T} \leq c \frac{t^{1+\gamma+\varepsilon}}{t^{1+\gamma+\varepsilon}} \leq c.$$

In this case, the element T is "good" and $h_3 \approx h_3 t_2 = \mathscr{H}_3$ holds.

In [23], the spire has a cycle of three daggers among its four triangles. The splinter has four daggers. The spear and spike have two daggers and two blades as triangles. The spindle has four blades as triangles.

Remark 4.4.7. The above examples reveal that the good element $T \subset \mathbb{R}^3$ may satisfy conditions such as $h_2 \approx \mathscr{H}_2$ and $h_3 \approx \mathscr{H}_3$.

Example 4.4.8. Using an element T called *Sliver*, we compare the three quantities $\frac{h_T^3}{|T|}$, $\frac{H_T}{h_T}$, and $\frac{R_3}{h_T}$, where the formulation of the circumradius R_3 of a tetrahedron T is as follows, e.g., see [44]. Let a, b and c be the lengths of

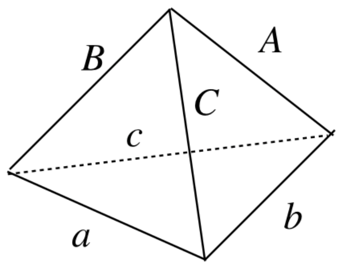

Fig. 4.1: R_3

the three edges of T and A, B, C the length of the opposite edges of a, b, c, respectively. Then,

$$R_3 = \frac{\sqrt{(aA + bB + cC)(aA + bB - cC)(aA - bB + cC)(-aA + bB + cC)}}{24|T|},$$

see Fig. 4.1.

Let $T \subset \mathbb{R}^3$ be the simplex with vertices $P_1 := (t^{\varepsilon_2}, 0, 0)^T$, $P_2 := (-t^{\varepsilon_2}, 0, 0)^T$, $P_3 := (0, -t, t^{\varepsilon_1})^T$, and $P_4 := (0, t, t^{\varepsilon_1})^T$ $(\varepsilon_1, \varepsilon_2 > 1)$, where $t := \frac{1}{N}$, $N \in \mathbb{N}$. Let L_i $(1 \le i \le 6)$ be the edges of T with $h_{\min} = L_1 \le L_2 \le \cdots \le L_6 = h_T$. Recall that $h_{\max} \approx h_T$ and

$$\frac{h_{\max}}{h_{\min}} \le c\frac{L_6}{L_1}, \quad \frac{H_T}{h_T} = \frac{L_1 L_2}{|T|} h_T.$$

Table 4.1: $h_T^3/|T|$, H_T/h_T and R_3/h_T ($\varepsilon_1 = 1.5$, $\varepsilon_2 = 1.0$)

N	t	L_6/L_1	$h_T^3/\lvert T\rvert$	H_T/h_T	R_3/h_T
32	3.1250e-02	1.4033	6.7882e+01	3.4471e+01	5.0195e-01
64	1.5625e-02	1.4087	9.6000e+01	4.8375e+01	5.0098e-01
128	7.8125e-03	1.4115	1.3576e+02	6.8147e+01	5.0049e-01

In Table 4.1, the angle between $\triangle P_1P_2P_3$ and $\triangle P_1P_2P_4$ tends to π as $t \to 0$, and the simplex T is "not good." In Table 4.2, the angle between $\triangle P_1P_3P_4$ and $\triangle P_2P_3P_4$ tends to 0 as $t \to 0$, the simplex T is "good." In Table 4.3, from the numerical result, the simplex T is "not good."

Table 4.2: $h_T^3/|T|$, H_T/h_T and R_3/h_T ($\varepsilon_1 = 1.0$, $\varepsilon_2 = 1.5$)

N	t	L_6/L_1	$h_T^3/\|T\|$	H_T/h_T	R_3/h_T
32	3.1250e-02	5.6569	6.7882e+01	8.5513	5.0006e-01
64	1.5625e-02	8.0000	9.6000e+01	8.5184	5.0002e-01
128	7.8125e-03	1.1314e+01	1.3576e+02	8.5018	5.0000e-01

Table 4.3: $h_T^3/|T|$, H_T/h_T and R_3/h_T ($\varepsilon_1 = 1.5$, $\varepsilon_2 = 1.5$)

N	t	L_6/L_1	$h_T^3/\|T\|$	H_T/h_T	R_3/h_T
32	3.1250e-02	5.6569	3.8400e+02	3.4986e+01	1.4170
64	1.5625e-02	8.0000	7.6800e+02	4.8744e+01	2.0010
128	7.8125e-03	1.1314e+01	1.5360e+03	6.8411e+01	2.8288

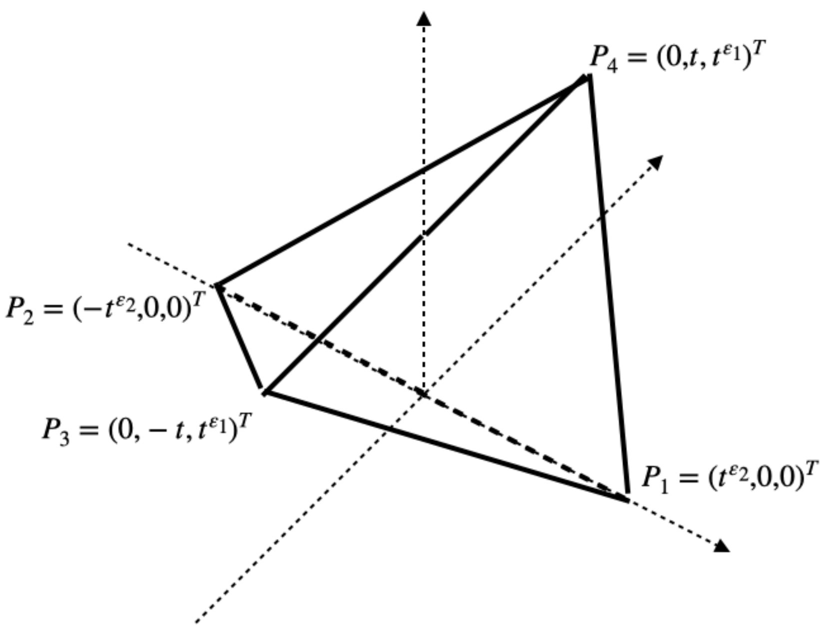

Fig. 4.2: Sliver

Part III

Anisotropic Interpolation Error Estimates

Chapter 5

Interpolation of Smooth Functions

5.1 Finite Element Generation

We follow the procedure described in [30, Section 1.4.1 and 1.2.1]; also see [47, Section 3.5].

For the reference element $\widehat{T}$ defined in Sections 3.1.1 and 3.1.2, let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be a fixed reference finite element, where $\widehat{P}$ is a vector space of functions $\hat{p} : \widehat{T} \to \mathbb{R}^n$ for some positive integer n (typically $n = 1$ or $n = d$) and $\widehat{\Sigma}$ is a set of n_0 linear forms $\{\hat{\chi}_1, \ldots, \hat{\chi}_{n_0}\}$ such that

$$\widehat{P} \ni \hat{p} \mapsto (\hat{\chi}_1(\hat{p}), \ldots, \hat{\chi}_{n_0}(\hat{p}))^T \in \mathbb{R}^{n_0}$$

is bijective; i.e., $\widehat{\Sigma}$ is a basis for $\mathcal{L}(\widehat{P}; \mathbb{R})$. Further, we denote by $\{\hat{\theta}_1, \ldots, \hat{\theta}_{n_0}\}$ in $\widehat{P}$ the local ($\mathbb{R}^n$-valued) shape functions such that

$$\hat{\chi}_i(\hat{\theta}_j) = \delta_{ij}, \quad 1 \leq i, j \leq n_0.$$

Let $V(\widehat{T})$ be a normed vector space of functions $\hat{\varphi} : \widehat{T} \to \mathbb{R}^n$ such that $\widehat{P} \subset V(\widehat{T})$ and the linear forms $\{\hat{\chi}_1, \ldots, \hat{\chi}_{n_0}\}$ can be extended to $V(\widehat{T})'$. The local interpolation operator $I_{\widehat{T}}$ is then defined by

$$I_{\widehat{T}} : V(\widehat{T}) \ni \hat{\varphi} \mapsto \sum_{i=1}^{n_0} \hat{\chi}_i(\hat{\varphi}) \hat{\theta}_i \in \widehat{P}. \tag{5.1.1}$$

It obviously holds that, for any $\hat{\varphi} \in V(\widehat{T})$,

$$\hat{\chi}_i(I_{\widehat{T}} \hat{\varphi}) = \hat{\chi}_i(\hat{\varphi}) \quad i = 1, \ldots, n_0. \tag{5.1.2}$$

Proposition 5.1.1. *$\widehat{P}$ is invariant under $I_{\widehat{T}}$, that is,*

$$I_{\widehat{T}}\hat{p} = \hat{p} \quad \forall \hat{p} \in \widehat{P}. \tag{5.1.3}$$

Proof. The proof is found in [30, Proposition 1.30]. □

Let Φ^s, $\widetilde{\Phi}$, and $\widehat{\Phi}$ be the affine mappings defined in (3.4.1). For $T^s = \widetilde{\Phi}(\widetilde{T}) = \widetilde{\Phi} \circ \widehat{\Phi}(\widehat{T})$, we first define a Banach space $V(T^s)$ of $\mathbb{R}^n$-valued functions that is the counterpart of $V(\widehat{T})$ and define a linear bijection mapping by

$$\psi^s := \psi_{\widehat{T}} \circ \psi_{\widetilde{T}} : V(T^s) \ni \varphi^s \mapsto \hat{\varphi} := \psi^s(\varphi^s) := \varphi^s \circ \Phi^s \in V(\widehat{T}),$$

with two linear bijection mappings:

$$\psi_{\widetilde{T}} : V(T^s) \ni \varphi^s \mapsto \tilde{\varphi} := \psi_{\widetilde{T}}(\varphi^s) := \varphi^s \circ \widetilde{\Phi} \in V(\widetilde{T}),$$
$$\psi_{\widehat{T}} : V(\widetilde{T}) \ni \tilde{\varphi} \mapsto \hat{\varphi} := \psi_{\widehat{T}}(\tilde{\varphi}) := \tilde{\varphi} \circ \widehat{\Phi} \in V(\widehat{T}).$$

Furthermore, the triple $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$ is defined by

$$\begin{cases} \widetilde{T} = \widehat{\Phi}(\widehat{T}); \\ \widetilde{P} = \{\psi_{\widehat{T}}^{-1}(\hat{p});\ \hat{p} \in \widehat{P}\}; \\ \widetilde{\Sigma} = \{\{\tilde{\chi}_i\}_{1 \le i \le n_0};\ \tilde{\chi}_i = \hat{\chi}_i(\psi_{\widehat{T}}(\tilde{p})), \forall \tilde{p} \in \widetilde{P}, \hat{\chi}_i \in \widehat{\Sigma}\}, \end{cases}$$

while the triple $\{T^s, P^s, \Sigma^s\}$ is defined by

$$\begin{cases} T^s = \widetilde{\Phi}(\widetilde{T}); \\ P^s = \{\psi_{\widetilde{T}}^{-1}(\tilde{p});\ \tilde{p} \in \widetilde{P}\}; \\ \Sigma^s = \{\{\chi_i^s\}_{1 \le i \le n_0};\ \chi_i^s = \tilde{\chi}_i(\psi_{\widetilde{T}}(p^s)), \forall p^s \in P^s, \tilde{\chi}_i \in \widetilde{\Sigma}\}. \end{cases}$$

The triples $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$ and $\{T^s, P^s, \Sigma^s\}$ are then finite elements. The local shape functions are $\tilde{\theta}_i = \psi_{\widehat{T}}^{-1}(\hat{\theta}_i)$ and $\theta_i^s = \psi_{\widetilde{T}}^{-1}(\tilde{\theta}_i)$, $1 \le i \le n_0$, and the associated local interpolation operators are respectively defined by

$$I_{\widetilde{T}} : V(\widetilde{T}) \ni \tilde{\varphi} \mapsto I_{\widetilde{T}}\tilde{\varphi} := \sum_{i=1}^{n_0} \tilde{\chi}_i(\tilde{\varphi})\tilde{\theta}_i \in \widetilde{P}, \tag{5.1.4}$$

$$I_{T^s} : V(T^s) \ni \varphi^s \mapsto I_{T^s}\varphi^s := \sum_{i=1}^{n_0} \chi_i^s(\varphi^s)\theta_i^s \in P^s. \tag{5.1.5}$$

For any $T \in \mathbb{T}_h$, let Φ_{T^s} be the affine mapping defined in Definition 3.4.1. We define a Banach space $V(T)$ of $\mathbb{R}^n$-valued functions that is the counterpart of $V(T^s)$ and define a linear bijection mapping by

$$\psi_{T^s} : V(T) \ni \varphi \mapsto \varphi^s := \psi_{T^s}(\varphi) := \varphi \circ \Phi_{T^s} \in V(T^s).$$

For the finite element $\{T^s, P^s, \Sigma^s\}$, we define the triple $\{T, P, \Sigma\}$ as

$$\begin{cases} T = \Phi_T(T^s); \\ P = \{\psi_{T^s}^{-1}(p^s);\ p^s \in P^s\}; \\ \Sigma = \{\{\chi_i\}_{1 \le i \le n_0};\ \chi_i = \chi_i^s(\psi_{T^s}(p)),\ \forall p \in P,\ \chi_i^s \in \Sigma^s\}. \end{cases}$$

The triple $\{T, P, \Sigma\}$ is then a finite element. The local shape function is $\theta_i = \psi_{T^s}^{-1}(\theta_i^s)$, $1 \le i \le n_0$, and the associated local interpolation operators are respectively defined by

$$I_T : V(T) \ni \varphi \mapsto I_T \varphi := \sum_{i=1}^{n_0} \chi_i(\varphi) \theta_i \in P. \tag{5.1.6}$$

Proposition 5.1.2. *The diagrams*

$$\begin{array}{ccccccc} V(T) & \xrightarrow{\psi_{T^s}} & V(T^s) & \xrightarrow{\psi_{\widetilde{T}}} & V(\widetilde{T}) & \xrightarrow{\psi_{\widehat{T}}} & V(\widehat{T}) \\ {\scriptstyle I_T}\downarrow & & {\scriptstyle I_{T^s}}\downarrow & & {\scriptstyle I_{\widetilde{T}}}\downarrow & & \downarrow{\scriptstyle I_{\widehat{T}}} \\ P & \xrightarrow[\psi_{T^s}]{} & P^s & \xrightarrow[\psi_{\widetilde{T}}]{} & \widetilde{P} & \xrightarrow[\psi_{\widehat{T}}]{} & \widehat{P} \end{array}$$

commute.

Proof. For example, see [30, Proposition 1.62]. □

Example 5.1.3. Let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element.

(I) For the Lagrange finite element of degree k, we set $V(\widehat{T}) := \mathcal{C}^0(\widehat{T})$.

(II) For the Hermite finite element, we set $V(\widehat{T}) := \mathcal{C}^1(\widehat{T})$.

(III) For the Crouzeix–Raviart finite element with $k = 1$, we set $V(\widehat{T}) := W^{1,1}(\widehat{T})$.

5.2 Remarks on the Anisotropic Interpolation

The proof of Theorem 2 in [47] included a mistake, and we need to modify its statement to correct this error.

In [47], we showed the two lemmata.

Lemma 4 ([47])**.** *Let $1 \leq p \leq \infty$ and $k \geq 0$. Let ℓ be such that $0 \leq \ell \leq k$. Let $\hat{\varphi} \in W^{m,p}(\widehat{T})$ and $\hat{\psi} \in W^{\ell+1,p}(\widehat{T})$. It then holds that, for all $m \in \{0, \dots, \ell+1\}$,*

$$\frac{|\tilde{\varphi}|_{W^{m,p}(\widetilde{T})}}{|\tilde{\psi}|_{W^{\ell+1,p}(\widetilde{T})}} \leq \max_{1 \leq i \leq d}\{h_i^{\ell+1-m}\} \left(\sum_{|\beta|=m} (h^{-\beta})^p \|\partial^\beta \hat{\varphi}\|^p_{L^p(\widehat{T})} \right)^{1/p} \times \left(\sum_{|\beta|=m} (h^{-\beta})^p |\partial^\beta \hat{\psi}|^p_{W^{\ell+1-m,p}(\widehat{T})} \right)^{-1/p}, \quad (4.1)$$

with $\tilde{\varphi} := \hat{\varphi} \circ \widehat{\Phi}^{-1}$ and $\tilde{\psi} := \hat{\psi} \circ \widehat{\Phi}^{-1}$.

Proof. Let β, γ and δ be multi-indices with $|\beta| = m$, $|\gamma| = \ell + 1$ and $|\delta| = \ell + 1 - m$.

We first have, from $\hat{x}_j = h_j^{-1}\tilde{x}_j$, that

$$\partial^\beta \tilde{\varphi} = h_1^{-\beta_1} \cdots h_d^{-\beta_d} \partial^\beta \hat{\varphi} = h^{-\beta} \partial^\beta \hat{\varphi}.$$

If $1 \leq p < \infty$, through a change in variable, we obtain

$$|\tilde{\varphi}|^p_{W^{m,p}(\widetilde{T})} = \sum_{|\beta|=m} \|\partial^\beta \tilde{\varphi}\|^p_{L^p(\widetilde{T})} = |\det(\widehat{\mathcal{A}}^{(d)})| \sum_{|\beta|=m} (h^{-\beta})^p \|\partial^\beta \hat{\varphi}\|^p_{L^p(\widehat{T})}.$$

We similarly have

$$\begin{aligned}
&|\tilde{\psi}|^p_{W^{\ell+1,p}(\widetilde{T})} \\
&= \sum_{|\gamma|=\ell+1} \|\partial^\gamma \tilde{\psi}\|^p_{L^p(\widetilde{T})} \\
&= \sum_{|\delta|=\ell+1-m} \sum_{|\beta|=m} \|\partial^\delta \partial^\beta \tilde{\psi}\|^p_{L^p(\widetilde{T})} \\
&= |\det(\widehat{\mathcal{A}}^{(d)})| \sum_{|\delta|=\ell+1-m} \sum_{|\beta|=m} (h^{-\delta-\beta})^p \|\partial^\delta \partial^\beta \hat{\psi}\|^p_{L^p(\widehat{T})} \\
&\geq |\det(\widehat{\mathcal{A}}^{(d)})| \min_{1 \leq i \leq d}\{h_i^{-|\delta|p}\} \sum_{|\delta|=\ell+1-m} \sum_{|\beta|=m} (h^{-\beta})^p \|\partial^\delta \partial^\beta \hat{\psi}\|^p_{L^p(\widehat{T})}.
\end{aligned}$$

When $p = \infty$, a proof can be made by an analogous argument. □

Lemma 5 ([47])**.** *Let $1 \leq p \leq \infty$ and $k \geq 0$. Let ℓ be such that $0 \leq \ell \leq k$. Let $\hat{\varphi} \in W^{m,p}(\widetilde{T})$ and $\hat{\psi} \in W^{\ell+1,p}(\widetilde{\widetilde{T}})$. It then holds that, for all*

$m \in \{0, \dots, \ell+1\}$,

$$\frac{|\varphi^s|_{W^{m,p}(T^s)}}{|\psi^s|_{W^{\ell+1,p}(T^s)}} \le C^{A,d} \left(\frac{H_{T^s}}{h_{T^s}} \right)^m \frac{|\tilde{\varphi}|_{W^{m,p}(\widetilde{T})}}{|\tilde{\psi}|_{W^{\ell+1,p}(\widetilde{T})}}, \tag{4.2}$$

with $\varphi^s := \tilde{\varphi} \circ \widetilde{\Phi}^{-1}$ *and* $\psi^s := \tilde{\psi} \circ \widetilde{\Phi}^{-1}$. *Here,* $C^{A,2} := \sqrt{2}^{\ell+1-m} C^{sc}$, *and* $C^{A,3} := \frac{2^{\ell+1}}{3^m} C^{sc}$, *where* C^{sc} *is a constant independent of* T^s *and* $\widetilde{T}$.

Proof. Using the standard estimates in [30, Lemma 1.101], we easily get

$$\frac{|\varphi^s|_{W^{m,p}(T^s)}}{|\psi^s|_{W^{\ell+1,p}(T^s)}} \le C^{sc} \left(\|\widetilde{\mathcal{A}}\|_2 \|\widetilde{\mathcal{A}}^{-1}\|_2 \right)^m \|\widetilde{\mathcal{A}}\|_2^{\ell+1-m} \frac{|\tilde{\varphi}|_{W^{m,p}(\widetilde{T})}}{|\tilde{\psi}|_{W^{\ell+1,p}(\widetilde{T})}}. \tag{4.3}$$

The inequality (4.2) follows from (4.3) and (3.6.1). □

Using the above lemmata and the standard Bramble–Hilbert–type lemma on the reference elements (see, for example, (1.6.9) and (1.6.10)), the following theorem was proved.

Incorrect statement

Let $1 \le p \le \infty$ and assume that there exists a nonnegative integer k such that

$$\mathcal{P}^k \subset \widehat{P} \subset W^{k+1,p}(\widehat{T}) \subset V(\widehat{T}).$$

Let ℓ $(0 \le \ell \le k)$ be such that $W^{\ell+1,p}(\widehat{T}) \subset V(\widehat{T})$ with continuous embedding. It then holds that, for any $m \in \{0, \dots, \ell+1\}$ and any $\varphi^s \in W^{\ell+1,p}(T^s)$,

$$|\varphi^s - I_{T^s}\varphi^s|_{W^{m,p}(T^s)} \le C^I \left(\frac{H_{T^s}}{h_{T^s}} \right)^m h_{T^s}^{\ell+1-m} |\varphi^s|_{W^{\ell+1,p}(T^s)}, \tag{4.10}$$

where C^I is a positive constant independent of h_{T^s}.

As explained the above, there exist mistakes in the proof of this theorem, and its statement is not valid in the original form. To clarify the following description, we explain the errors in the proof. Let $\widehat{T} \subset \mathbb{R}^2$ be the reference element defined in Section 3.1.1. We set $k = m = \ell = 1$, $p = 2$. For $\hat{\varphi} \in H^2(\widehat{T})$, we set $\tilde{\varphi} := \hat{\varphi} \circ \widehat{\Phi}^{-1}$ and $\varphi := \tilde{\varphi} \circ \widetilde{\Phi}^{-1}$. In the proof, we tried to show the existence of positive constants $C_E^{(1)}$ and $C_E^{(2)}$, independent of h_T, and a polynomial $\hat{\eta} := \tilde{\eta} \circ \widehat{\Phi} \in \mathcal{P}^1$ such that

$$\frac{|\tilde{\varphi} - I_{\widetilde{T}}\tilde{\varphi}|_{H^1(\widetilde{T})}}{|\tilde{\varphi}|_{H^2(\widetilde{T})}} \le \frac{|\tilde{\varphi} - \tilde{\eta}|_{H^1(\widetilde{T})}}{|\tilde{\varphi}|_{H^2(\widetilde{T})}} + \frac{|I_{\widetilde{T}}(\tilde{\eta} - \tilde{\varphi})|_{H^1(\widetilde{T})}}{|\tilde{\varphi}|_{H^2(\widetilde{T})}},$$

and, using Lemma 4,

$$\frac{|\tilde{\varphi}-\tilde{\eta}|_{H^1(\widetilde{T})}}{|\tilde{\varphi}|_{H^2(\widetilde{T})}} \le h_{T^s} \frac{\sum_{i=1}^2 h_i^{-2} \|\partial_{\hat{x}_i}(\hat{\varphi}-\hat{\eta})\|_{L^2(\widehat{T})}}{\sum_{i=1}^2 h_i^{-2} |\partial_{\hat{x}_i}\hat{\varphi}|_{H^1(\widehat{T})}} \le C_E^{(1)} h_{T^s},$$

$$\frac{|I_{\widetilde{T}}(\tilde{\eta}-\tilde{\varphi})|_{H^1(\widetilde{T})}}{|\tilde{\varphi}|_{H^2(\widetilde{T})}} \le h_{T^s} \frac{\sum_{i=1}^2 h_i^{-2} \|\partial_{\hat{x}_i} I_{\widehat{T}}(\hat{\varphi}-\hat{\eta})\|_{L^2(\widehat{T})}}{\sum_{i=1}^2 h_i^{-2} |\partial_{\hat{x}_i}\hat{\varphi}|_{H^1(\widehat{T})}} \le C_E^{(2)} h_{T^s},$$

or equivalently

$$\begin{aligned} & h_1^{-2}\|\partial_{x_1}(\hat{\varphi}-\hat{\eta})\|_{L^2(\widehat{T})}^2 + h_2^{-2}\|\partial_{x_2}(\hat{\varphi}-\hat{\eta})\|_{L^2(\widehat{T})}^2 \\ & \quad \le C_E^{(1)} \left(h_1^{-2}|\partial_{x_1}\hat{\varphi}|_{H^1(\widehat{T})}^2 + h_2^{-2}|\partial_{x_2}\hat{\varphi}|_{H^1(\widehat{T})}^2 \right), \end{aligned} \tag{5.2.1}$$

$$\begin{aligned} & h_1^{-2}\|\partial_{x_1} I_{\widehat{T}}(\hat{\varphi}-\hat{\eta})\|_{L^2(\widehat{T})}^2 + h_2^{-2}\|\partial_{x_2} I_{\widehat{T}}(\hat{\varphi}-\hat{\eta})\|_{L^2(\widehat{T})}^2 \\ & \quad \le C_E^{(2)} \left(h_1^{-2}|\partial_{x_1}\hat{\varphi}|_{H^1(\widehat{T})}^2 + h_2^{-2}|\partial_{x_2}\hat{\varphi}|_{H^1(\widehat{T})}^2 \right). \end{aligned} \tag{5.2.2}$$

To this end, we employed Lemma 1.6.10, a variant of the Bramble–Hilbert lemma, which claims that there exists a polynomial $\eta_\beta \in \mathcal{P}^1$ such that

$$\begin{aligned} \|\partial_{x_1}(\hat{\varphi}-\eta_\beta)\|_{L^2(\widehat{T})}^2 + \|\partial_{x_2}(\hat{\varphi}-\eta_\beta)\|_{L^2(\widehat{T})}^2 &\le \|\hat{\varphi}-\eta_\beta\|_{H^2(\widehat{T})}^2 \\ &\le C_E^{(3)} |\hat{\varphi}|_{H^2(\widehat{T})}^2, \end{aligned} \tag{5.2.3}$$

where the constant $C_E^{(3)}$ depends only on $\widehat{T}$. However, in general, we cannot deduce (5.2.1) from (5.2.3) for arbitrary h_1 and h_2. Furthermore, from the definition of the interpolation operator, we have, for $j = 1, 2$,

$$\begin{aligned} \|\partial_{x_j} I_{\widehat{T}}(\hat{\varphi}-\hat{\eta})\|_{L^2(\widehat{T})} &\le \sum_{i=1}^{n_0} |\hat{\chi}_i(\hat{\varphi}-\hat{\eta})| \|\partial_{x_j}\hat{\theta}_i\|_{L^2(\widehat{T})} \\ &\le \sum_{i=1}^{n_0} \|\hat{\chi}_i\|_{H^2(\widehat{T})'} \|\hat{\varphi}-\hat{\eta}\|_{H^2(\widehat{T})} \|\partial_{x_j}\hat{\theta}_i\|_{L^2(\widehat{T})} \\ &\le n_0 \max_{1\le i\le n_0} \left(\|\hat{\chi}_i\|_{H^2(\widehat{T})'} \|\partial_{x_j}\hat{\theta}_i\|_{L^2(\widehat{T})} \right) \|\hat{\varphi}-\hat{\eta}\|_{H^2(\widehat{T})}. \end{aligned} \tag{5.2.4}$$

Setting $\hat{\eta} := \eta_\beta$ in (5.2.4) yields

$$\|\partial_{x_j} I_{\widehat{T}}(\hat{\varphi}-\eta_\beta)\|_{L^2(\widehat{T})} \le C_E^{(4)} |\hat{\varphi}|_{H^2(\widehat{T})}. \tag{5.2.5}$$

However, in general, we cannot deduce (5.2.2) from (5.2.5) for arbitrary h_1 and h_2. That is why the proof of [47, Theorem 2] is incorrect.

We consider this problem. Using Lemma 4, we have

$$\frac{|\tilde{\varphi} - I_{\widetilde{T}}\tilde{\varphi}|_{H^1(\widetilde{T})}}{|\tilde{\varphi}|_{H^2(\widetilde{T})}} \leq h_{T^s} \frac{\sum_{i=1}^{2} h_i^{-2} \|\partial_{\hat{x}_i}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})}}{\sum_{i=1}^{2} h_i^{-2} |\partial_{\hat{x}_i}\hat{\varphi}|_{H^1(\widehat{T})}}. \tag{5.2.6}$$

From Lemma 1.6.9, there exist two polynomials $\hat{\eta}_1, \hat{\eta}_2 \in \mathcal{P}^1$ such that, for $i = 1, 2$,

$$\|\partial_{x_i}(\hat{\varphi} - \hat{\eta}_i)\|_{H^1(\widehat{T})} \leq C_i^{BH} |\partial_{x_i}\hat{\varphi}|_{H^1(\widehat{T})}. \tag{5.2.7}$$

Using the triangle inequality, we have

$$\|\partial_{\hat{x}_i}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})} \leq \|\partial_{\hat{x}_i}(\hat{\varphi} - \hat{\eta}_i)\|_{L^2(\widehat{T})} + \|\partial_{\hat{x}_i}(\hat{\eta}_i - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})}. \tag{5.2.8}$$

If it holds that

$$\|\partial_{\hat{x}_i}(\hat{\eta}_i - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})} \leq c \|\partial_{\hat{x}_i}(\hat{\eta}_i - \hat{\varphi})\|_{H^1(\widehat{T})}, \tag{5.2.9}$$

the inequality (5.2.8) together with the Sobolev embedding theorem $H^1(\widehat{T}) \hookrightarrow L^2(\widehat{T})$, (5.2.7) and (5.2.9) can be estimated as

$$\|\partial_{\hat{x}_i}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})} \leq c \|\partial_{\hat{x}_i}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{H^1(\widehat{T})} \leq c |\partial_{x_i}\hat{\varphi}|_{H^1(\widehat{T})}.$$

However, in general, this proof does not valid. Because

$$\begin{aligned} \|\partial_{\hat{x}_i}(\hat{\eta}_i - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})} &= \|\partial_{\hat{x}_i}(I_{\widehat{T}}\hat{\eta}_i - I_{\widehat{T}}\hat{\varphi})\|_{L^2(\widehat{T})} \\ &\leq c \|\hat{\eta}_i - \hat{\varphi}\|_{H^2(\widehat{T})}, \end{aligned}$$

and thus,

$$\inf_{\hat{\eta}_i \in \mathcal{P}^1} \|\hat{\eta}_i - \hat{\varphi}\|_{H^2(\widehat{T})} \leq c |\hat{\varphi}|_{H^2(\widehat{T})}$$

in which the quantity h_i is not included. Therefore, the quantity h_i^{-2} in (5.2.6) remains.

To overcome this problem, we give two theorems (Theorem 5.4.1, Theorem 5.6.1) to replace [47, Theorem 2].

5.3 Scaling Argument

This section gives estimates related to a scaling argument corresponding to [30, Lemma 1.101].

Note 5.3.1. Recall that

$$|s| \le 1, \quad h_2 \le h_1 \quad \text{if } d=2,$$
$$|s_1| \le 1,\ |s_{21}| \le 1, \quad h_2 \le h_3 \le h_1 \quad \text{if } d=3.$$

When $d = 3$, if Condition 3.3.1 is imposed, there exists a positive constant M independent of h_T such that $|s_{22}| \le M\frac{h_2 t_1}{h_3}$. We thus have, if $d = 2$,

$$h_1|[\widetilde{\mathcal{A}}]_{j1}| \le \mathscr{H}_j, \quad h_2|[\widetilde{\mathcal{A}}]_{j2}| \le \mathscr{H}_j, \quad j=1,2,$$

and, if $d = 3$, for $\widetilde{\mathcal{A}} \in \{\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2\}$ and $j = 1, 2, 3$,

$$h_1|[\widetilde{\mathcal{A}}]_{j1}| \le \mathscr{H}_j, \quad h_2|[\widetilde{\mathcal{A}}]_{j2}| \le \mathscr{H}_j, \quad h_3|[\widetilde{\mathcal{A}}]_{j3}| \le \max\{1, M\}\mathscr{H}_j, \quad j=1,2,3.$$

Note 5.3.2. We use the following calculations in (5.3.2). For any multi-indices β and γ, we have

$$\partial_{\hat{x}}^{\beta+\gamma} = \frac{\partial^{|\beta|+|\gamma|}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d} \partial \hat{x}_1^{\gamma_1} \cdots \partial \hat{x}_d^{\gamma_d}}$$

$$= \underbrace{\sum_{i_1^{(1)}, i_1^{(0,1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_1^{(1)} 1} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} \cdots \sum_{i_{\beta_1}^{(1)}, i_{\beta_1}^{(0,1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_{\beta_1}^{(1)} 1} [\mathcal{A}_T]_{i_{\beta_1}^{(0,1)} i_{\beta_1}^{(1)}}}_{\beta_1 \text{times}} \cdots$$

$$\underbrace{\sum_{i_1^{(d)}, i_1^{(0,d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_1^{(d)} d} [\mathcal{A}_T]_{i_1^{(0,d)} i_1^{(d)}} \cdots \sum_{i_{\beta_d}^{(d)}, i_{\beta_d}^{(0,d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_{\beta_d}^{(d)} d} [\mathcal{A}_T]_{i_{\beta_d}^{(0,d)} i_{\beta_d}^{(d)}}}_{\beta_d \text{times}}$$

$$\underbrace{\sum_{j_1^{(1)}, j_1^{(0,1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{j_1^{(1)} 1} [\mathcal{A}_T]_{j_1^{(0,1)} j_1^{(1)}} \cdots \sum_{j_{\gamma_1}^{(1)}, j_{\gamma_1}^{(0,1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{j_{\gamma_1}^{(1)} 1} [\mathcal{A}_T]_{j_{\gamma_1}^{(0,1)} j_{\gamma_1}^{(1)}}}_{\gamma_1 \text{times}} \cdots$$

$$\underbrace{\sum_{j_1^{(d)}, j_1^{(0,d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{j_1^{(d)} d} [\mathcal{A}_T]_{j_1^{(0,d)} j_1^{(d)}} \cdots \sum_{j_{\gamma_d}^{(d)}, j_{\gamma_d}^{(0,d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{j_{\gamma_d}^{(d)} d} [\mathcal{A}_T]_{j_{\gamma_d}^{(0,d)} j_{\gamma_d}^{(d)}}}_{\gamma_d \text{times}}$$

$$\underbrace{\frac{\partial^{\beta_1}}{\partial x_{i_1^{(0,1)}} \cdots \partial x_{i_{\beta_1}^{(0,1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_d}}{\partial x_{i_1^{(0,d)}} \cdots \partial x_{i_{\beta_d}^{(0,d)}}}}_{\beta_d \text{times}} \underbrace{\frac{\partial^{\gamma_1}}{\partial x_{j_1^{(0,1)}} \cdots \partial x_{j_{\gamma_1}^{(0,1)}}}}_{\gamma_1 \text{times}} \cdots \underbrace{\frac{\partial^{\gamma_d}}{\partial x_{j_1^{(0,d)}} \cdots \partial x_{j_{\gamma_d}^{(0,d)}}}}_{\gamma_d \text{times}}.$$

Let $\hat{\varphi} \in \mathcal{C}^{\ell}(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \widehat{\Phi}^{-1}$, $\varphi^s = \tilde{\varphi} \circ \widetilde{\Phi}^{-1}$ and $\varphi = \varphi^s \circ \Phi_{T^s}^{-1}$. Then, for $1 \le i \le d$,

$$\begin{aligned}
\left|\frac{\partial \hat{\varphi}}{\partial \hat{x}_i}\right| &= \left| \sum_{i_1^{(1)}=1}^{d} \sum_{i_1^{(0,1)}=1}^{d} h_i [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} \frac{\partial \varphi}{\partial x_{i_1^{(0,1)}}} \right| \\
&= h_i \left| \sum_{i_1^{(1)}=1}^{d} \sum_{i_1^{(0,1)}=1}^{d} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} (r_i)_{i_1^{(1)}} \frac{\partial \varphi}{\partial x_{i_1^{(0,1)}}} \right| = h_i \left| \frac{\partial \varphi}{\partial r_i} \right| \\
&\le h_i \|\widetilde{\mathcal{A}}\|_{\max} \|\mathcal{A}_T\|_{\max} \sum_{i_1^{(1)}=1}^{d} \sum_{i_1^{(0,1)}=1}^{d} \left| \frac{\partial \varphi}{\partial x_{i_1^{(0,1)}}} \right|,
\end{aligned}$$

and for $1 \le i, j \le d$,

$$\begin{aligned}
\left|\frac{\partial^2 \hat{\varphi}}{\partial \hat{x}_i \partial \hat{x}_j}\right| &= \Bigg| \sum_{i_1^{(1)}, j_1^{(1)}=1}^{d} \sum_{i_1^{(0,1)}, j_1^{(0,1)}=1}^{d} h_i h_j [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} [\widetilde{\mathcal{A}}]_{j_1^{(1)} j} \\
&\qquad [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} [\mathcal{A}_T]_{j_1^{(0,1)} j_1^{(1)}} \frac{\partial^2 \varphi}{\partial x_{i_1^{(0,1)}} \partial x_{j_1^{(0,1)}}} \Bigg| = \alpha_i \alpha_j \left| \frac{\partial^2 \varphi}{\partial r_i \partial r_j} \right| \\
&\le h_i h_j \sum_{j_1^{(1)}=1}^{d} |[\widetilde{\mathcal{A}}]_{j_1^{(1)} j}| \left| \sum_{j_1^{(0,1)}=1}^{d} [\mathcal{A}_T]_{j_1^{(0,1)} j_1^{(1)}} \frac{\partial^2 \varphi}{\partial r_i \partial x_{j_1^{(0,1)}}} \right| \\
&\le h_i h_j \|\widetilde{\mathcal{A}}\|_{\max} \|\mathcal{A}_T\|_{\max} \sum_{j_1^{(0,1)}=1}^{d} \left| \frac{\partial^2 \varphi}{\partial r_i \partial x_{j_1^{(0,1)}}} \right| \\
&\le h_i h_j \|\widetilde{\mathcal{A}}\|_{\max}^2 \|\mathcal{A}_T\|_{\max}^2 \sum_{i_1^{(0,1)}, j_1^{(0,1)}=1}^{d} \left| \frac{\partial^2 \varphi}{\partial x_{i_1^{(0,1)}} \partial x_{j_1^{(0,1)}}} \right|.
\end{aligned}$$

Note 5.3.3. We use the following calculations in (5.3.3). For any multi-indices β and γ, we have

$$\begin{aligned}
&\partial_{\hat{x}}^{\beta+\gamma} = \frac{\partial^{|\beta|+|\gamma|}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d} \partial \hat{x}_1^{\gamma_1} \cdots \partial \hat{x}_d^{\gamma_d}} \\
&= \underbrace{\sum_{i_1^{(1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_1^{(1)} 1} \cdots \sum_{i_{\beta_1}^{(1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_{\beta_1}^{(1)} 1}}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_1^{(d)} d} \cdots \sum_{i_{\beta_d}^{(d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_{\beta_d}^{(d)} d}}_{\beta_d \text{times}}
\end{aligned}$$

$$\underbrace{\sum_{j_1^{(1)}=1}^{d} h_1[\widetilde{\mathcal{A}}]_{j_1^{(1)}1} \cdots \sum_{j_{\gamma_1}^{(1)}=1}^{d} h_1[\widetilde{\mathcal{A}}]_{j_{\gamma_1}^{(1)}1}}_{\gamma_1 \text{times}} \cdots \underbrace{\sum_{j_1^{(d)}=1}^{d} h_d[\widetilde{\mathcal{A}}]_{j_1^{(d)}d} \cdots \sum_{j_{\gamma_d}^{(d)}=1}^{d} h_d[\widetilde{\mathcal{A}}]_{j_{\gamma_d}^{(d)}d}}_{\gamma_d \text{times}}$$

$$\frac{\partial^{\beta_1}}{\underbrace{\partial x^s_{i_1^{1)}} \cdots \partial x^s_{i_{\beta_1}^{(1)}}}_{\beta_1 \text{times}}} \cdots \frac{\partial^{\beta_d}}{\underbrace{\partial x^s_{i_1^{(d)}} \cdots \partial x^s_{i_{\beta_d}^{(d)}}}_{\beta_d \text{times}}} \frac{\partial^{\gamma_1}}{\underbrace{\partial x^s_{j_1^{(1)}} \cdots \partial x^s_{j_{\gamma_1}^{(1)}}}_{\gamma_1 \text{times}}} \cdots \frac{\partial^{\gamma_d}}{\underbrace{\partial x^s_{j_1^{(d)}} \cdots \partial x^s_{j_{\gamma_d}^{(d)}}}_{\gamma_d \text{times}}}.$$

Let $\hat{\varphi} \in \mathcal{C}^{\ell}(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \widehat{\Phi}^{-1}$ and $\varphi^s = \tilde{\varphi} \circ \widetilde{\Phi}^{-1}$. Then, for $1 \le i \le d$,

$$\left| \frac{\partial \hat{\varphi}}{\partial \hat{x}_i} \right| \le \sum_{i_1^{(1)}=1}^{d} h_i \left| [\widetilde{\mathcal{A}}]_{i_1^{(1)}i} \right| \left| \frac{\partial \varphi^s}{\partial x^s_{i_1^{(1)}}} \right| \le \begin{cases} h_i \|\widetilde{\mathcal{A}}\|_{\max} \sum_{i_1^{(1)}=1}^{d} \left| \frac{\partial \varphi^s}{\partial x^s_{i_1^{(1)}}} \right| & \text{or,} \\ c \sum_{i_1^{(1)}=1}^{d} \mathscr{H}_{i_1^{(1)}} \left| \frac{\partial \varphi^s}{\partial x^s_{i_1^{(1)}}} \right|, \end{cases}$$

and for $1 \le i, j \le d$,

$$\left| \frac{\partial^2 \hat{\varphi}}{\partial \hat{x}_i \partial \hat{x}_j} \right| = \left| \sum_{i_1^{(1)}, j_1^{(1)}=1}^{d} h_i h_j [\widetilde{\mathcal{A}}]_{i_1^{(1)}i} [\widetilde{\mathcal{A}}]_{j_1^{(1)}j} \frac{\partial^2 \varphi^s}{\partial x^s_{i_1^{(1)}} \partial x^s_{j_1^{(1)}}} \right|$$

$$\le \begin{cases} h_i h_j \|\widetilde{\mathcal{A}}\|^2_{\max} \sum_{i_1^{(1)}, j_1^{(1)}=1}^{d} \left| \frac{\partial^2 \varphi^s}{\partial x^s_{i_1^{(1)}} \partial x^s_{j_1^{(1)}}} \right| & \text{or,} \\ h_j \sum_{j_1^{(1)}=1}^{d} |[\widetilde{\mathcal{A}}]_{j_1^{(1)}j}| \left| \sum_{i_1^{(1)}=1}^{d} h_i [\widetilde{\mathcal{A}}]_{i_1^{(1)}i} \frac{\partial^2 \varphi^s}{\partial x^s_{i_1^{(1)}} \partial x^s_{j_1^{(1)}}} \right| & \\ \quad \le c h_j \|\widetilde{\mathcal{A}}\|_{\max} \sum_{j_1^{(1)}=1}^{d} \sum_{i_1^{(1)}=1}^{d} \mathscr{H}_{i_1^{(1)}} \left| \frac{\partial^2 \varphi^s}{\partial x^s_{i_1^{(1)}} \partial x^s_{j_1^{(1)}}} \right| & \text{or,} \\ c \sum_{i_1^{(1)}=1}^{d} \sum_{j_1^{(1)}=1}^{d} \mathscr{H}_{i_1^{(1)}} \mathscr{H}_{j_1^{(1)}} \left| \frac{\partial^2 \varphi^s}{\partial x^s_{i_1^{(1)}} \partial x^s_{j_1^{(1)}}} \right|. \end{cases}$$

Lemma 5.3.4. *Let $m, \ell \in \mathbb{N}_0$ with $\ell \ge m$. Let $\beta := (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$ and $\gamma := (\gamma_1, \dots, \gamma_d) \in \mathbb{N}_0^d$ be multi-indices with $|\beta| = m$ and $|\gamma| = \ell - m$. Then, for any $\hat{\varphi} \in W^{m,p}(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \widehat{\Phi}^{-1}$ and $\varphi^s = \tilde{\varphi} \circ \widetilde{\Phi}^{-1}$, it holds that*

$$|\varphi^s|_{W^{m,p}(T^s)} \le c |\det(\mathcal{A}^s)|^{\frac{1}{p}} \|\widetilde{\mathcal{A}}^{-1}\|_2^m \left(\sum_{|\beta|=m} (h^{-\beta})^p \|\partial^{\beta}_{\hat{x}} \hat{\varphi}\|^p_{L^p(\widehat{T})} \right)^{1/p} \quad \textit{if } p \in [1, \infty), \tag{5.3.1a}$$

$$|\varphi^s|_{W^{m,\infty}(T^s)} \le c \|\widetilde{\mathcal{A}}^{-1}\|_2^m \max_{|\beta|=m} \left(h^{-\beta} \|\partial^{\beta}_{\hat{x}} \hat{\varphi}\|_{L^\infty(\widehat{T})} \right) \quad \textit{if } p = \infty. \tag{5.3.1b}$$

Let $p \in [0,\infty]$. Furthermore, for any $\hat{\varphi} \in W^{\ell,p}(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \widehat{\Phi}^{-1}$, $\varphi^s = \tilde{\varphi} \circ \widetilde{\Phi}^{-1}$ and $\varphi = \varphi^s \circ \Phi_{T^s}^{-1}$, it holds that

$$\|\partial_{\hat{x}}^{\beta} \partial_{\hat{x}}^{\gamma} \hat{\varphi}\|_{L^p(\widehat{T})} \leq c |\det(\mathcal{A}^s)|^{-\frac{1}{p}} \|\widetilde{\mathcal{A}}\|_2^m h^{\beta} \sum_{|\epsilon|=|\gamma|} h^{\varepsilon} |\partial_r^{\epsilon} \varphi|_{W^{m,p}(T)}. \tag{5.3.2}$$

In particular, if Condition 3.3.1 is imposed, then for any $\hat{\varphi} \in W^{\ell,p}(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \widehat{\Phi}^{-1}$ and $\varphi^s = \tilde{\varphi} \circ \widetilde{\Phi}^{-1}$, it holds that

$$\|\partial_{\hat{x}}^{\beta} \partial_{\hat{x}}^{\gamma} \hat{\varphi}\|_{L^p(\widehat{T})} \leq c |\det(\mathcal{A}^s)|^{-\frac{1}{p}} \|\widetilde{\mathcal{A}}\|_2^m h^{\beta} \sum_{|\epsilon|=|\gamma|} \mathscr{H}^{\varepsilon} |\partial^{\varepsilon} \varphi^s|_{W^{m,p}(T^s)}. \tag{5.3.3}$$

Here, for $p = \infty$ and any positive real x, $x^{-\frac{1}{p}} = 1$.

Proof. We divide the proof into three parts.

Proof of (5.3.1)

Let $p \in [1,\infty)$. Because the space $\mathcal{C}^m(\widehat{T})$ is dense in the space $W^{m,p}(\widehat{T})$, we show (5.3.1) for $\hat{\varphi} \in \mathcal{C}^m(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \widehat{\Phi}^{-1}$ and $\varphi^s = \tilde{\varphi} \circ \widetilde{\Phi}^{-1}$. From $\hat{x}_j = h_j^{-1} \tilde{x}_j$, we have that, for any multi-index β,

$$\partial^{\beta} \tilde{\varphi} = h_1^{-\beta_1} \cdots h_d^{-\beta_d} \partial^{\beta} \hat{\varphi} = h^{-\beta} \partial^{\beta} \hat{\varphi}. \tag{5.3.4}$$

Through a change in variable, we obtain

$$|\tilde{\varphi}|_{W^{m,p}(\widetilde{T})}^p = \sum_{|\beta|=m} \|\partial^{\beta} \tilde{\varphi}\|_{L^p(\widetilde{T})}^p = |\det(\widehat{\mathcal{A}}^{(d)})| \sum_{|\beta|=m} (h^{-\beta})^p \|\partial^{\beta} \hat{\varphi}\|_{L^p(\widehat{T})}^p. \tag{5.3.5}$$

From the standard estimate in [30, Lemma 1.101], we have

$$|\varphi^s|_{W^{m,p}(T^s)} \leq C_1^{SA} |\det(\widetilde{\mathcal{A}})|^{\frac{1}{p}} \|\widetilde{\mathcal{A}}^{-1}\|_2^m |\tilde{\varphi}|_{W^{m,p}(\widetilde{T})}. \tag{5.3.6}$$

Inequality (5.3.1a) follows from (5.3.5) and (5.3.6) with (3.6.2).

We consider the case that $p = \infty$. A function $\hat{\varphi} \in W^{m,\infty}(\widehat{T})$ belongs to the space $W^{m,p}(\widehat{T})$ for any $p \in [1,\infty)$. It therefore holds that $\tilde{\varphi} \in W^{m,p}(\widetilde{T})$

for any $p \in [1, \infty)$ and, from (1.6.1),

$$\begin{aligned}
\|\partial^\gamma \tilde{\varphi}\|_{L^p(\widetilde{T})} &\leq |\tilde{\varphi}|_{W^{|\gamma|,p}(\widetilde{T})} \\
&= |\det(\widehat{\mathcal{A}}^{(d)})|^{\frac{1}{p}} \left(\sum_{|\beta|=|\gamma|} (h^{-\beta})^p \|\partial^\beta \hat{\varphi}\|^p_{L^p(\widehat{T})} \right)^{1/p} \\
&\leq \left(\sup_{1\leq p} |\det(\widehat{\mathcal{A}}^{(d)})|^{\frac{1}{p}} \right) \sum_{|\beta|=|\gamma|} h^{-\beta} \|\partial^\beta \hat{\varphi}\|_{L^p(\widehat{T})} \\
&\leq c \left(\sup_{1\leq p} |\det(\widehat{\mathcal{A}}^{(d)})|^{\frac{1}{p}} \right) \sum_{|\beta|=|\gamma|} h^{-\beta} \|\partial^\beta \hat{\varphi}\|_{L^\infty(\widehat{T})} < \infty \qquad (5.3.7)
\end{aligned}$$

for multi-index $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq m$. This implies that the function $\partial^\gamma \tilde{\varphi}$ is in the space $L^\infty(\widetilde{T})$ for each $|\gamma| \leq m$. We therefore have $\tilde{\varphi} \in W^{m,\infty}(\widetilde{T})$. By passing to the limit $p \to \infty$ in (5.3.7) and because $\lim_{p\to\infty} \|\cdot\|_{L^p(\widetilde{T})} = \|\cdot\|_{L^\infty(\widetilde{T})}$, we have

$$|\tilde{\varphi}|_{W^{m,\infty}(\widetilde{T})} \leq c \max_{|\beta|=m} \left(h^{-\beta} \|\partial^\beta \hat{\varphi}\|_{L^\infty(\widehat{T})} \right). \qquad (5.3.8)$$

From the standard estimate in [30, Lemma 1.101], we have

$$|\varphi^s|_{W^{m,\infty}(T^s)} \leq c \|\widetilde{\mathcal{A}}^{-1}\|_2^m |\tilde{\varphi}|_{W^{m,\infty}(\widetilde{T})}. \qquad (5.3.9)$$

Inequality (5.3.1b) follows from (5.3.8) and (5.3.9).

Proof of (5.3.3)

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \mathbb{N}_0^d$ and $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}_0^d$ be multi-indies with $|\varepsilon| = |\gamma|$ and $|\delta| = |\beta|$. Let $p \in [1, \infty)$. Because the space $\mathcal{C}^\ell(\widehat{T})$ is dense in the space $W^{\ell,p}(\widehat{T})$, we show (5.3.3) for $\hat{\varphi} \in \mathcal{C}^\ell(\widehat{T})$ with $\tilde{\varphi} = \hat{\varphi} \circ \widehat{\Phi}^{-1}$ and $\varphi^s = \tilde{\varphi} \circ \widetilde{\Phi}^{-1}$. Through a simple calculation, we have

$$\begin{aligned}
|\partial^{\beta+\gamma} \hat{\varphi}| &= \left| \frac{\partial^\ell \hat{\varphi}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d} \partial \hat{x}_1^{\gamma_1} \cdots \partial \hat{x}_d^{\gamma_d}} \right| \\
&\leq c h^\beta \|\widetilde{\mathcal{A}}\|_{\max}^{|\beta|} \underbrace{\sum_{i_1^{(1)}=1}^d \cdots \sum_{i_{\beta_1}^{(1)}=1}^d}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(d)}=1}^d \cdots \sum_{i_{\beta_d}^{(d)}=1}^d}_{\beta_d \text{times}} \underbrace{\sum_{j_1^{(1)}=1}^d \cdots \sum_{j_{\gamma_1}^{(1)}=1}^d}_{\gamma_1 \text{times}} \cdots \underbrace{\sum_{j_1^{(d)}=1}^d \cdots \sum_{j_{\gamma_d}^{(d)}=1}^d}_{\gamma_d \text{times}} \\
&\quad \underbrace{\mathscr{H}_{j_1^{(1)}} \cdots \mathscr{H}_{j_{\varepsilon_1}^{(1)}}}_{\gamma_1 \text{times}} \cdots \underbrace{\mathscr{H}_{j_1^{(d)}} \cdots \mathscr{H}_{j_{\varepsilon_d}^{(d)}}}_{\gamma_d \text{times}}
\end{aligned}$$

$$\left|\underbrace{\frac{\partial^{\beta_1}}{\partial x^s_{i^{(1)}_1}\cdots\partial x^s_{i^{(1)}_{\beta_1}}}}_{\beta_1\text{times}}\cdots\underbrace{\frac{\partial^{\beta_d}}{\partial x^s_{i^{(d)}_1}\cdots\partial x^s_{i^{(d)}_{\beta_d}}}}_{\beta_d\text{times}}\underbrace{\frac{\partial^{\gamma_1}}{\partial x^s_{j^{(1)}_1}\cdots\partial x^s_{j^{(1)}_{\gamma_1}}}}_{\gamma_1\text{times}}\cdots\underbrace{\frac{\partial^{\gamma_d}}{\partial x^s_{j^{(d)}_1}\cdots\partial x^s_{j^{(d)}_{\gamma_d}}}}_{\gamma_d\text{times}}\varphi^s\right|$$

$$\leq ch^{\beta}\|\widetilde{\mathcal{A}}\|^{|\beta|}_{\max}\sum_{|\delta|=|\beta|}\sum_{|\varepsilon|=|\gamma|}\mathscr{H}^{\varepsilon}|\partial^{\delta}\partial^{\varepsilon}\varphi^s|.$$

We then have, using (1.3.1),

$$\begin{aligned}\int_{\widehat{T}}|\partial^{\beta}\partial^{\gamma}\hat{\varphi}|^p d\hat{x} &\leq c\|\widetilde{\mathcal{A}}\|_2^{mp}h^{\beta p}\sum_{|\delta|=|\beta|}\sum_{|\epsilon|=|\gamma|}\mathscr{H}^{\varepsilon p}\int_{\widehat{T}}|\partial^{\delta}\partial^{\varepsilon}\varphi^s|^p d\hat{x}\\ &= c|\det(\mathcal{A}^s)|^{-1}\|\widetilde{\mathcal{A}}\|_2^{mp}h^{\beta p}\sum_{|\delta|=|\beta|}\sum_{|\epsilon|=|\gamma|}\mathscr{H}^{\varepsilon p}\int_{T^s}|\partial^{\delta}\partial^{\varepsilon}\varphi^s|^p dx.\end{aligned}$$

Therefore, using (1.6.1), we have

$$\|\partial^{\beta}\partial^{\gamma}\hat{\varphi}\|_{L^p(\widehat{T})} \leq c|\det(\mathcal{A}^s)|^{-\frac{1}{p}}\|\widetilde{\mathcal{A}}\|_2^m\alpha^{\beta}\sum_{|\epsilon|=|\gamma|}\mathscr{H}^{\varepsilon}|\partial^{\epsilon}\varphi^s|_{W^{m,p}(T^s)},$$

which concludes (5.3.3).

We consider the case that $p=\infty$. A function $\varphi\in W^{\ell,\infty}(T)$ belongs to the space $W^{\ell,p}(T)$ for any $p\in[1,\infty)$. It therefore holds that $\hat{\varphi}\in W^{\ell,p}(\widehat{T})$ for any $p\in[1,\infty)$ and thus

$$\begin{aligned}\|\partial^{\beta}\partial^{\gamma}\hat{\varphi}\|_{L^p(\widehat{T})} &\leq c|\det(\mathcal{A}^s)|^{-\frac{1}{p}}\|\widetilde{\mathcal{A}}\|_2^m h^{\beta}\sum_{|\epsilon|=|\gamma|}\mathscr{H}^{\varepsilon}|\partial^{\epsilon}\varphi^s|_{W^{m,p}(T^s)}\\ &\leq c\|\widetilde{\mathcal{A}}\|_2^m h^{\beta}\sum_{|\epsilon|=|\gamma|}\mathscr{H}^{\varepsilon}|\partial^{\epsilon}\varphi^s|_{W^{m,\infty}(T^s)}<\infty. \qquad (5.3.10)\end{aligned}$$

This implies that the function $\partial^{\beta}\partial^{\gamma}\hat{\varphi}$ is in the space $L^{\infty}(\widehat{T})$. Inequality (5.3.3) for $p=\infty$ is obtained by passing to the limit $p\to\infty$ in (5.3.10) on the basis that $\lim_{p\to\infty}\|\cdot\|_{L^p(\widehat{T})}=\|\cdot\|_{L^{\infty}(\widehat{T})}$.

Proof of (5.3.2)

We follow the proof of (5.3.3). Let $p\in[1,\infty)$. Because the space $\mathcal{C}^{\ell}(\widehat{T})$ is dense in the space $W^{\ell,p}(\widehat{T})$, we show (5.3.2) for $\hat{\varphi}\in\mathcal{C}^{\ell}(\widehat{T})$ with $\tilde{\varphi}=\hat{\varphi}\circ\widehat{\Phi}^{-1}$, $\varphi^s=\tilde{\varphi}\circ\widetilde{\Phi}^{-1}$ and $\varphi=\varphi^s\circ\Phi_{T^s}^{-1}$, it holds that, for $1\leq i,k\leq d$,

$$\left|\partial^{\beta+\gamma}\hat{\varphi}\right| \leq ch^{\beta}\|\widetilde{\mathcal{A}}\|^{|\beta|}_{\max}\|\mathcal{A}_T\|^{|\beta|}_{\max}\sum_{|\delta|=|\beta|}\sum_{|\varepsilon|=|\gamma|}h^{\varepsilon}\left|\partial^{\delta}\partial_r^{\varepsilon}\varphi_0\right|.$$

Using (3.6.1c) and (1.3.1), we obtain (5.3.2) for $p\in[1,\infty]$ by an argument analogous to the proof of (5.3.3). □

Remark 5.3.5. In inequality (5.3.3), it is possible to obtain the estimates in T by specifically determining the matrix $\mathcal{A}_T$.

Let $\ell = 2$, $m = 1$ and $p = q = 2$. Recall that

$$\Phi_{T^s} : T^s \ni x^s \mapsto x = \mathcal{A}_T x^s + b_T \in T.$$

For $\varphi^s \in \mathcal{C}^2(T^s)$ with $\varphi = \varphi^s \circ \Phi_{T_0}^{-1}$ and $1 \le i, j \le d$, we have

$$\left| \frac{\partial^2 \varphi^s}{\partial x_i^s \partial x_j^s}(x) \right| = \left| \sum_{i_1^{(1)}, j_1^{(1)} = 1}^{2} [\mathcal{A}_T]_{i_1^{(1)} i} [\mathcal{A}_T]_{j_1^{(1)} j} \frac{\partial^2 \varphi}{\partial x_{i_1^{(1)}} \partial x_{j_1^{(1)}}}(x) \right|.$$

Let $d = 2$. We define the matrix $\mathcal{A}_T$ as

$$\mathcal{A}_T := \begin{pmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{pmatrix}.$$

Because $\|\mathcal{A}_T\|_{\max} = 1$, we have

$$\left| \frac{\partial^2 \varphi^s}{\partial x_i^s \partial x_j^s}(x) \right| \le \left| \frac{\partial^2 \varphi}{\partial x_{i+1} \partial x_{j+1}}(x) \right|,$$

where the indices i, $i+1$ and j, $j+1$ have to be understood mod 2. Because $|\det(\mathcal{A}_T)| = 1$, it holds that

$$\left\| \frac{\partial^2 \varphi^s}{\partial x_i^s \partial x_j^s} \right\|_{L^2(T^s)} \le \left\| \frac{\partial^2 \varphi}{\partial x_{i_{i+1}} \partial x_{j+1}} \right\|_{L^2(T)}.$$

We then have

$$\sum_{j=1}^{2} \mathscr{H}_j \left| \frac{\partial \varphi^s}{\partial x_j^s} \right|_{H^1(T^s)} \le \sum_{j=1}^{2} \mathscr{H}_j \left| \frac{\partial \varphi}{\partial x_{j+1}} \right|_{H^1(T)},$$

where the indices j, $j+1$ have to be understood mod 2.

We define the matrix $\mathcal{A}_T$ as

$$\mathcal{A}_T := \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix}.$$

We then have

$$\left| \frac{\partial^2 \varphi^s}{\partial x_i^s \partial x_j^s}(x) \right| \le \frac{1}{\sqrt{2}} \sum_{i_1^{(1)}, j_1^{(1)} = 1}^{2} \left| \frac{\partial^2 \varphi}{\partial x_{i_1^{(1)}} \partial x_{j_1^{(1)}}}(x) \right|,$$

which leads to

$$\left\| \frac{\partial^2 \varphi^s}{\partial x_i^s \partial x_j^s} \right\|_{L^2(T^s)}^2 \leq c \sum_{i_1^{(1)}, j_1^{(1)}=1}^{2} \left\| \frac{\partial^2 \varphi}{\partial x_{i_1^{(1)}} \partial x_{j_1^{(1)}}} \right\|_{L^2(T)}^2 \leq c |\varphi|_{H^2(T)}^2.$$

We then have, using (1.6.1),

$$\sum_{j=1}^{2} \mathscr{H}_j \left| \frac{\partial \varphi^s}{\partial x_j^s} \right|_{H^1(T^s)} \leq \sum_{j=1}^{2} \mathscr{H}_j |\varphi|_{H^2(T)} \leq c h_T |\varphi|_{H^2(T)}.$$

In this case, anisotropic interpolation error estimates cannot be obtained.

Lemma 5.3.6. *Let Φ_{T^s} be the affine mapping defined in Definition 3.4.1. Let $s \geq 0$ and $1 \leq p \leq \infty$. There exists positive constants c_1 and c_2 such that, for all $T \in \mathbb{T}_h$ and $\varphi \in W^{s,p}(T)$,*

$$c_1 |\varphi|_{W^{s,p}(T)} \leq |\varphi^s|_{W^{s,p}(T^s)} \leq c_2 |\varphi|_{W^{s,p}(T)}, \tag{5.3.11}$$

with $\varphi^s = \varphi \circ \Phi_{T^s}$.

Proof. The following inequalities are found in [30, Lemma 1.101]. There exists a positive constant c such that, for all $T \in \mathbb{T}_h$ and $\varphi \in W^{s,p}(T)$,

$$|\varphi^s|_{W^{s,p}(T^s)} \leq c \|\mathcal{A}_T\|_2^s |\det(\mathcal{A}_T)|^{-\frac{1}{p}} |\varphi|_{W^{s,p}(T)}, \tag{5.3.12}$$

$$|\varphi|_{W^{s,p}(T)} \leq c \|\mathcal{A}_T^{-1}\|_2^s |\det(\mathcal{A}_T)|^{\frac{1}{p}} |\varphi^s|_{W^{s,p}(T^s)}, \tag{5.3.13}$$

with $\varphi^s = \varphi \circ \Phi_{T^s}$.

Because the length of all edges of a simplex and measure of the simplex are not changed by a rotation and mirror imaging matrix and $\mathcal{A}_T, \mathcal{A}_T^{-1} \in O(d)$,

$$|\det(\mathcal{A}_T)| = \frac{|T|}{|T^s|} = 1, \quad \|\mathcal{A}_T\|_2 = 1, \quad \|\mathcal{A}_T^{-1}\|_2 = 1. \tag{5.3.14}$$

From (5.3.12), (5.3.13), and (5.3.14), we obtain the desired inequality (5.3.11). □

5.4 Classical interpolation error estimates

The following theorem is another representation of the standard interpolation error estimates, e.g., see [30, Theorem 1.103].

Theorem 5.4.1. *Let $1 \leq p \leq \infty$ and assume that there exists a nonnegative integer k such that*

$$\mathcal{P}^k \subset \widehat{P} \subset W^{k+1,p}(\widehat{T}) \subset V(\widehat{T}).$$

Let ℓ $(0 \leq \ell \leq k)$ be such that $W^{\ell+1,p}(\widehat{T}) \subset V(\widehat{T})$ with continuous embedding. Furthermore, assume that $\ell, m \in \mathbb{N} \cup \{0\}$ and $p, q \in [1, \infty]$ such that $0 \leq m \leq \ell + 1$ and

$$W^{\ell+1,p}(\widehat{T}) \hookrightarrow W^{m,q}(\widehat{T}). \tag{5.4.1}$$

It holds that, for any $m \in \{0, \ldots, \ell+1\}$ and any $\varphi \in W^{\ell+1,p}(T)$,

$$|\varphi^s - I_T \varphi|_{W^{m,q}(T)} \leq C_*^I |T|^{\frac{1}{q} - \frac{1}{p}} \left(\frac{h_{\max}}{h_{\min}} \right)^m \left(\frac{H_T}{h_T} \right)^m h_T^{\ell+1-m} |\varphi|_{W^{\ell+1,p}(T)}, \tag{5.4.2}$$

where C_*^I is a positive constant independent of h_T and H_T, and the parameters $h_{\max}$ and $h_{\min}$ are defined by

$$h_{\max} := \max\{h_1, \ldots, h_d\}, \quad h_{\min} := \min\{h_1, \ldots, h_d\}. \tag{5.4.3}$$

Proof. Let $\hat{\varphi} \in W^{\ell+1,p}(\widehat{T})$. Because $0 \leq \ell \leq k$, $\mathcal{P}^\ell \subset \mathcal{P}^k \subset \widehat{P}$. Therefore, for any $\hat{\eta} \in \mathcal{P}^\ell$, we have $I_{\widehat{T}} \hat{\eta} = \hat{\eta}$. Using (5.1.3) and (5.4.1), we obtain

$$\begin{aligned} |\tilde{\varphi} - I_{\widehat{T}} \hat{\varphi}|_{W^{m,q}(\widehat{T})} &\leq |\tilde{\varphi} - \hat{\eta}|_{W^{m,q}(\widehat{T})} + |I_{\widehat{T}}(\hat{\eta} - \hat{\varphi})|_{W^{m,q}(\widehat{T})} \\ &\leq c \|\tilde{\varphi} - \hat{\eta}\|_{W^{\ell+1,p}(\widehat{T})}, \end{aligned}$$

where we used the stability of the interpolation operator $I_{\widehat{T}}$, that is,

$$|I_{\widehat{T}}(\hat{\eta} - \hat{\varphi})|_{W^{m,q}(\widehat{T})} \leq \sum_{i=1}^{n_0} |\hat{\chi}_i(\hat{\eta} - \hat{\varphi})| |\hat{\theta}_i|_{W^{m,q}(\widehat{T})} \leq c \|\hat{\eta} - \tilde{\varphi}\|_{W^{\ell+1,p}(\widehat{T})}.$$

Using the classic Bramble–Hilbert–type lemma (e.g., [22, Lemma 4.3.8]), we obtain

$$|\tilde{\varphi} - I_{\widehat{T}} \hat{\varphi}|_{W^{m,q}(\widehat{T})} \leq c \inf_{\hat{\eta} \in \mathcal{P}^\ell} \|\hat{\eta} - \tilde{\varphi}\|_{W^{\ell+1,p}(\widehat{T})} \leq c |\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})}. \tag{5.4.4}$$

The inequalities (5.3.11), (5.3.1), (1.6.1), and (5.4.4) yield

$$\begin{aligned} &|\varphi - I_T \varphi|_{W^{m,q}(T)} \leq c |\varphi^s - I_{T^s} \varphi^s|_{W^{m,q}(T^s)} \\ &\leq c |\det(\mathcal{A}^s)|^{\frac{1}{q}} \|\widetilde{\mathcal{A}}^{-1}\|_2^m \left(\sum_{|\beta|=m} (h^{-\beta})^q \|\partial^\beta (\hat{\varphi} - I_{\widehat{T}} \hat{\varphi})\|_{L^q(\widehat{T})}^q \right)^{1/q} \\ &\leq c |\det(\mathcal{A}^s)|^{\frac{1}{q}} \|\widetilde{\mathcal{A}}^{-1}\|_2^m \max\{h_1^{-1}, \ldots, h_d^{-1}\}^{|\beta|} |\tilde{\varphi} - I_{\widehat{T}} \hat{\varphi}|_{W^{m,q}(\widehat{T})} \\ &\leq c |\det(\mathcal{A}^s)|^{\frac{1}{q}} \|\widetilde{\mathcal{A}}^{-1}\|_2^m h_{\min}^{-|\beta|} |\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})}. \end{aligned} \tag{5.4.5}$$

Using the inequalities (1.6.1), (5.3.11) and (5.3.2), we have

$$\begin{aligned}
|\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})} &\leq \sum_{|\gamma|=\ell+1-m} \sum_{|\beta|=m} \|\partial^\beta \partial^\gamma \hat{\varphi}\|_{L^p(\widehat{T})} \\
&\leq c|\det(\mathcal{A}^s)|^{-\frac{1}{p}} \|\widetilde{\mathcal{A}}\|_2^m \sum_{|\gamma|=\ell+1-m} \sum_{|\beta|=m} h^\beta \sum_{|\epsilon|=|\gamma|} h^\varepsilon |\partial_r^\epsilon \varphi|_{W^{m,p}(T)} \\
&\leq c|\det(\mathcal{A}^s)|^{-\frac{1}{p}} \|\widetilde{\mathcal{A}}\|_2^m \max\{h_1, \dots, h_d\}^{|\beta|} h_T^{\ell+1-m} |\varphi|_{W^{\ell+1,p}(T)} \\
&\leq c|\det((\mathcal{A}^s)|^{-\frac{1}{p}} \|\widetilde{\mathcal{A}}\|_2^m h_{\max}^{|\beta|} h_T^{\ell+1-m} |\varphi|_{W^{\ell+1,p}(T)}. \qquad (5.4.6)
\end{aligned}$$

From (5.4.5) and (5.4.6) together with (3.6.1) and (3.6.2), we have the desired estimate (5.4.2). □

Remark 5.4.2. We introduced the estimate (1.6.9) that is a variant of the Bramble–Hilbert lemma. However, because we prove estimate (5.4.4) with $p = q$ using the reference element, it is sufficient to use the standard estimate (e.g., [29, 22]) to achieve our goal.

Example 5.4.3. As the examples in [30, Example 1.106], we get local interpolation error estimates for a Lagrange finite element of degree k, a more general finite element, and the Crouzeix–Raviart finite element with $k = 1$.

(I) For a Lagrange finite element of degree k, we set $V(\widehat{T}) := \mathcal{C}^0(\widehat{T})$. The condition on ℓ in Theorem 5.4.1 is $\frac{d}{p} - 1 < \ell \leq k$ because $W^{\ell+1,p}(\widehat{T}) \subset \mathcal{C}^0(\widehat{T})$ if $\ell + 1 > \frac{d}{p}$ according to the Sobolev imbedding theorem.

(II) For a general finite element with $V(\widehat{T}) := \mathcal{C}^t(\widehat{T})$ and $t \in \mathbb{N}$. The condition on ℓ in Theorem 5.4.1 is $\frac{d}{p} - 1 + t < \ell \leq k$. When $t = 1$, there is a Hermite finite element.

(III) For the Crouzeix–Raviart finite element with $k = 1$, we set $V(\widehat{T}) := W^{1,1}(\widehat{T})$. The condition on ℓ in Theorem 5.4.1 is $0 \leq \ell \leq 1$.

Example 5.4.4. We introduce typical examples of the quantity $h_{\max}/h_{\min}$ in two dimensions. Let $0 < t \ll 1$, $t \in \mathbb{R}$ and $\varepsilon, \delta \in \mathbb{R}$. All examples degenerate in the y-axis direction.

(I) Let $T^s \subset \mathbb{R}^2$ be the simplex with vertices $P_1 := (0,0)^T$, $P_2 := (t,0)^T$ and $P_3 := (0, t^\varepsilon)^T$ with $1 < \varepsilon < 2$. We then have $h_1 = t$ and $h_2 = t^\varepsilon$, that is,

$$\frac{h_{\max}}{h_{\min}} = \frac{t}{t^\varepsilon} = t^{1-\varepsilon}, \quad \frac{H_{T^s}}{h_{T^s}} = \frac{t^{1+\varepsilon}}{\frac{1}{2} t^{1+\varepsilon}} = 2.$$

When $m = \ell = 1$ in (5.4.2) of Theorem 5.4.1, it holds that

$$|\varphi^s - I_{T^s}\varphi^s|_{W^{1,p}(T^s)} \leq c h_{T^s}^{2-\varepsilon} |\varphi^s|_{W^{2,p}(T^s)} \quad \forall \varphi^s \in W^{2,p}(T^s).$$

Remark that $|\overline{P_2P_3}| = \sqrt{t^2 + t^{2\varepsilon}} = h_{T^s}$. If $\varepsilon \geq 2$, the interpolation error estimates do not converge.

(II) Let $T^s \subset \mathbb{R}^2$ be the simplex with vertices $P_1 := (0,0)^T$, $P_2 := (t,0)^T$ and $P_3 := (t^\delta, t^\varepsilon)^T$ with $1 < \delta < \varepsilon < 2$. We then have $h_1 = \sqrt{(t - t^\delta)^2 + t^{2\varepsilon}}$ and $h_2 = \sqrt{t^{2\delta} + t^{2\varepsilon}}$, that is,

$$\frac{h_{\max}}{h_{\min}} = \frac{\sqrt{(t-t^\delta)^2 + t^{2\varepsilon}}}{\sqrt{t^{2\delta} + t^{2\varepsilon}}} \leq c t^{1-\delta}, \quad \frac{H_{T^s}}{h_{T^s}} = \frac{\sqrt{(t-t^\delta)^2 + t^{2\varepsilon}}\sqrt{t^{2\delta} + t^{2\varepsilon}}}{\frac{1}{2} t^{1+\varepsilon}}.$$

When $m = \ell = 1$ in (5.4.2) of Theorem 5.4.1, it holds that

$$|\varphi^s - I_{T^s}\varphi^s|_{W^{1,p}(T^s)} \leq c h_{T^s}^{2-\varepsilon} |\varphi^s|_{W^{2,p}(T^s)} \quad \forall \varphi^s \in W^{2,p}(T^s).$$

Remark that $|\overline{P_1P_2}| = t = h_{T^s}$. If $\varepsilon \geq 2$, the interpolation error estimates do not converge.

(III) Let $T^s \subset \mathbb{R}^2$ be the simplex with vertices $P_1 := (0,0)^T$, $P_2 := (2t,0)^T$ and $P_3 := (t, t^\varepsilon)^T$ with $1 < \varepsilon < 2$. We then have $h_1 = \sqrt{t^2 + t^{2\varepsilon}}$ and $h_2 = \sqrt{t^2 + t^{2\varepsilon}}$, that is,

$$\frac{h_{\max}}{h_{\min}} = 1, \quad \frac{H_{T^s}}{h_{T^s}} = \frac{t^2 + t^{2\varepsilon}}{t^{1+\varepsilon}}.$$

When $m = \ell = 1$ in (5.4.2) of Theorem 5.4.1, it holds that

$$|\varphi^s - I_{T^s}\varphi^s|_{W^{1,p}(T^s)} \leq c h_{T^s}^{2-\varepsilon} |\varphi^s|_{W^{2,p}(T^s)} \quad \forall \varphi^s \in W^{2,p}(T^s).$$

Remark that $|\overline{P_1P_2}| = 2t = h_{T^s}$. If $\varepsilon \geq 2$, the interpolation error estimates do not converge.

Example 5.4.5 (Lagrange finite element in $\mathbb{R}^3$)**.** Let $T^s \subset \mathbb{R}^3$ be the simplex with vertices $P_1 := (0,0,0)^T$, $P_2 := (t,0,0)^T$, $P_3 := (t/2, t^\varepsilon, 0)^T$, and $P_4 := (0,0,t)^T$ $(1 < \varepsilon < 2)$, and $0 < t \ll 1$, $t \in \mathbb{R}$. We then have $h_1 = t$, $h_2 = \sqrt{t^2/4 + t^{2\varepsilon}}$, and $h_3 := t$, that is,

$$\frac{h_{\max}}{h_{\min}} = \frac{t}{\sqrt{\frac{t^2}{4} + t^{2\varepsilon}}} \leq c.$$

Let

$$\varphi^s(x,y,z) := x^2 + \frac{1}{4}y^2 + z^2.$$

Let $I_{T^s}^L : \mathcal{C}^0(T^s) \to \mathcal{P}^1$ be the local Lagrange interpolation operator. We set

$$I_{T^s}^L \varphi^s(x,y,z) := ax + by + cz + d,$$

where $a,b,c,d \in \mathbb{R}$. For any nodes P of T^s, since $I_{T^s}^L \varphi^s(P) = \varphi^s(P)$, we have

$$I_{T^s}^L \varphi^s(x,y,z) = tx - \frac{1}{4}(t^{2-\varepsilon} - t^{\varepsilon})y + tz.$$

It thus holds that

$$(\varphi^s - I_{T^s}^L \varphi^s)(x,y,z) = x^2 + \frac{1}{4}y^2 + z^2 - tx + \frac{1}{4}(t^{2-\varepsilon} - t^{\varepsilon})y - tz,$$

Therefore, we have

$$\frac{|\varphi^s - I_{T^s}^L \varphi^s|_{W^{1,\infty}(T^s)}}{|\varphi^s|_{W^{2,\infty}(T^s)}} = \frac{\frac{1}{4}(t^{2-\varepsilon} + t^{\varepsilon})}{2} =: J_{T^s}.$$

By simple calculation, we have

$$\frac{J_{T^s}}{H_{T^s}} = \frac{t^4 + t^{2+2\varepsilon}}{48\sqrt{2}t^3\sqrt{(\frac{t}{2})^2 + t^{2\varepsilon}}} \geq \frac{t^4 + t^{2+2\varepsilon}}{24\sqrt{10}t^4} \geq \frac{t^4}{24\sqrt{10}t^4} = \frac{1}{24\sqrt{10}}.$$

We here used

$$H_{T^s} = \frac{6\sqrt{2}t^3\sqrt{(\frac{t}{2})^2 + t^{2\varepsilon}}}{t^{2+\varepsilon}} \approx h_{T^s}^{2-\varepsilon}.$$

We conclude that

$$|\varphi^s - I_{T^s}^L \varphi^s|_{W^{1,\infty}(T^s)} \geq \frac{1}{24\sqrt{10}} H_{T^s} |\varphi^s|_{W^{2,\infty}(T^s)}.$$

From (5.4.2), we have

$$|\varphi^s - I_{T^s}^L \varphi^s|_{W^{1,\infty}(T^s)} \leq c C_*^I H_{T^s} |\varphi^s|_{W^{2,\infty}(T^s)} \quad \forall \varphi^s \in W^{2,\infty}(T^s).$$

This example indicates that the convergence order on the simplex cannot be improved.

Example 5.4.6 ($\mathcal{P}^1$ + bubble finite element in $\mathbb{R}^2$)**.** We give a numerical example which is not optimal in the usual sense. Let $T^s \subset \mathbb{R}^2$ be the triangle with vertices $P_1 := (0,0)^T$, $P_2 := (t,0)^T$, $P_3 := (0,t^\varepsilon)^T$ (Case I, Example 5.4.4), where $t := \frac{1}{N}$, $N \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$, $1 < \varepsilon \leq 2$. Let P_4 be the barycentre of T^s.

Using the barycentric coordinates $\lambda_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, \ldots, 3$, we define the local basis functions as

$$\theta_4(x) := 27\lambda_1(x)\lambda_2(x)\lambda_3(x),$$
$$\theta_i(x) := \lambda_i(x) - \frac{1}{3}\theta_4(x), \quad i = 1, 2, 3.$$

The interpolation operator I_T^b defined by

$$I_{T^s}^b : H^2(T) \ni \varphi^s \mapsto I_{T^s}^b \varphi^s := \sum_{i=1}^4 \varphi^s(x_i)\theta_i \in \operatorname{span}\{\theta_1, \theta_2, \theta_3, \theta_4\}.$$

From Theorem 5.4.1, we have

$$|\varphi^s - I_{T^s}^b \varphi^s|_{H^1(T^s)} \leq c h_{T^s}^{2-\varepsilon} |\varphi^s|_{H^2(T^s)} \quad \forall \varphi^s \in H^2(T^s).$$

Let φ^s be a function such that

$$\varphi^s(x,y) := 2x^2 - xy + 3y^2.$$

We compute the convergence order with respect to the H^1 norm defined by

$$Err_t^b(H^1) := \frac{|\varphi^s - I_{T^s}^b \varphi^s|_{H^1(T^s)}}{|\varphi^s|_{H^2(T^s)}},$$

for the cases: $\varepsilon = 1.5$ (Table 5.3) and $\varepsilon = 2.0$ (Table 5.4). The convergence indicator r is defined by

$$r = \frac{1}{\log(2)} \log\left(\frac{Err_t^b(H^1)}{Err_{t/2}^b(H^1)} \right).$$

Remark 5.4.7. As stated above, we are not able to prove Theorem 2 under the original assumptions. If we are concerned with anisotropic elements, it would be desirable to remove the quantity $h_{\max}/h_{\min}$ from estimate (5.4.2). To this end, we employ the approach described in [4], and consider the case of a finite element with $V(\widehat{T}) := \mathcal{C}(\widehat{T})$ and $\widehat{P} := \mathcal{P}^k(\widehat{T})$.

Table 5.1: Error of the local interpolation operator ($\varepsilon = 1.5$)

N	t	$Err_t^b(H^1)$	r
128	7.8125e-03	2.9951e-02	
256	3.9062e-03	2.1101e-02	5.0529e-01
512	1.9531e-03	1.4874e-02	5.0452e-01
1024	9.7656e-04	1.0491e-02	5.0364e-01

Table 5.2: Error of the local interpolation operator ($\varepsilon = 2.0$)

N	t	$Err_t^b(H^1)$	r
128	7.8125e-03	3.3397e-01	
256	3.9062e-03	3.3366e-01	1.3398e-03
512	1.9531e-03	3.3350e-01	6.9198e-04
1024	9.7656e-04	3.3341e-01	3.8939e-04

5.5 Anisotropic Interpolation Analysis on the Reference Element

We introduce estimates on the reference element due to [10, 4] in order to obtain anisotropic interpolation error estimates.

For the reference element $\widehat{T}$ defined in Sections 3.1.1 and 3.1.2, let the triple $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be the reference finite element with associated normed vector space $V(\widehat{T})$.

The original research [10, Lemma3] and [4, Lemma 2.2] gives error estimates for the reference finite element $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$.

Theorem 5.5.1. *Let $I_{\widehat{T}} : \mathcal{C}(\widehat{T}) \to \mathcal{P}^k(\widehat{T})$ be a linear operator. Fix $m, \ell \in \mathbb{N}$ and $p, q \in [1, \infty]$ such that $0 \leq m \leq \ell \leq k+1$ and*

$$W^{\ell-m,p}(\widehat{T}) \hookrightarrow L^q(\widehat{T}). \tag{5.5.1}$$

Let β be a multi-index with $|\beta| = m$. We set $j := \dim(\partial_{\hat{x}}^{\beta} \mathcal{P}^k)$. Assume that there exist linear functionals $\mathscr{F}_i$, $i = 1, \ldots, j$, such that

$$\mathscr{F}_i \in W^{\ell-m,p}(\widehat{T})', \quad \forall i = 1, \ldots, j, \tag{5.5.2a}$$

$$\mathscr{F}_i(\partial^{\beta}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})) = 0 \quad \forall i = 1, \ldots, j, \quad \forall \hat{\varphi} \in \mathcal{C}(\widehat{T}) : \ \partial^{\beta}\hat{\varphi} \in W^{\ell-m,p}(\widehat{T}), \tag{5.5.2b}$$

$$\hat{\eta} \in \mathcal{P}^k, \quad \mathscr{F}_i(\partial^{\beta}\hat{\eta}) = 0 \quad \forall i = 1, \ldots, j \quad \Rightarrow \quad \partial^{\beta}\hat{\eta} = 0. \tag{5.5.2c}$$

It holds that for all $\hat{\varphi} \in \mathcal{C}(\widehat{T})$ *with* $\partial^\beta \hat{\varphi} \in W^{\ell-m,p}(\widehat{T})$,

$$\|\partial^\beta_{\hat{x}}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} \leq C^F |\partial^\beta_{\hat{x}}\hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})}. \tag{5.5.3}$$

Proof. We follow [4, Lemma 2.2].

For all $\hat{\eta} \in \mathcal{P}^{\ell-1}$, we have

$$\|\partial^\beta_{\hat{x}}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} \leq \|\partial^\beta_{\hat{x}}(\hat{\varphi} - \hat{\eta})\|_{L^q(\widehat{T})} + \|\partial^\beta_{\hat{x}}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})}. \tag{5.5.4}$$

Note that $\hat{\eta} - I_{\widehat{T}}\hat{\varphi} \in \mathcal{P}^k$, because $\ell \leq k + 1$. That is, $\partial^\beta_{\hat{x}}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi}) \in \partial^\beta_{\hat{x}}\mathcal{P}^k$. Because the polynomial spaces are finite-dimensional all norms are equivalent, that is, by the fact $\sum_{i=1}^{j} |\mathscr{F}_i(\hat{\eta})|$ is a norm on $\partial^\beta_{\hat{x}}\mathcal{P}^k$, together with (5.5.2a), (5.5.2b) and (5.5.2c), we have for any $\hat{\eta} \in \mathcal{P}^{\ell-1}$,

$$\begin{aligned}\|\partial^\beta_{\hat{x}}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} &\leq c \sum_{i=1}^{j} |\mathscr{F}_i(\partial^\beta_{\hat{x}}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi}))| = c \sum_{i=1}^{j} |\mathscr{F}_i(\partial^\beta_{\hat{x}}(\hat{\eta} - \hat{\varphi}))| \\ &\leq c \|\partial^\beta_{\hat{x}}(\hat{\eta} - \hat{\varphi})\|_{W^{\ell-m,p}(\widehat{T})}.\end{aligned}$$

Using (5.5.4) and (5.5.1), it holds that for any $\hat{\eta} \in \mathcal{P}^{\ell-1}$,

$$\begin{aligned}\|\partial^\beta_{\hat{x}}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} &\leq \|\partial^\beta_{\hat{x}}(\hat{\varphi} - \hat{\eta})\|_{L^q(\widehat{T})} + \|\partial^\beta_{\hat{x}}(\hat{\eta} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} \\ &\leq c \|\partial^\beta_{\hat{x}}(\hat{\eta} - \hat{\varphi})\|_{W^{\ell-m,p}(\widehat{T})}.\end{aligned}$$

By Lemma 1.6.7, we have

$$\begin{aligned}\|\partial^\beta_{\hat{x}}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} &\leq c \inf_{\hat{\eta} \in \mathcal{P}^{\ell-1}} \|\partial^\beta_{\hat{x}}(\hat{\eta} - \hat{\varphi})\|_{W^{\ell-m,p}(\widehat{T})} \\ &\leq c |\partial^\beta_{\hat{x}}\hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})}.\end{aligned}$$

□

Remark 5.5.2. Note that it is not required $I_{\widehat{T}}\hat{\eta} = \hat{\eta}$ for any $\hat{\eta} \in \mathcal{P}^{\ell-1}$.

Remark 5.5.3. In this thesis, we use the result of [10, 4], that is, Theorem 5.5.1.

Later, in [23], the authors showed another interpretation of anisotropic interpolation theory as follows.

We set $\widehat{P} := \mathcal{P}^k(\widehat{T})$ with $k \in \mathbb{N}_0$. Let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$ be a multi-index. Then, $\partial^\beta_{\hat{x}}\widehat{P}$ is a vector space on $\widehat{T}$. We set

$$j := \dim(\partial^\beta_{\hat{x}}\widehat{P}).$$

Let $\{\widehat{\Theta}_k\}_{k=1}^{j}$ be a set of basis functions of $\partial_{\hat{x}}^{\beta}\widehat{P}$, that is, there exists a set $\{d_{i,k}\}_{k=1}^{j} \subset \mathbb{R}$ such that

$$\partial_{\hat{x}}^{\beta}\hat{\theta}_i = \sum_{k=1}^{j} d_{i,k}\widehat{\Theta}_k, \quad 1 \leq i \leq n_0. \tag{5.5.5}$$

From (5.1.1) and (5.5.5), we have, for any $\hat{\varphi} \in V(\widehat{T})$,

$$\partial_{\hat{x}}^{\beta}(I_{\widehat{T}}\hat{\varphi}) = \sum_{i=1}^{n_0} \hat{\chi}_i(\hat{\varphi})\partial_{\hat{x}}^{\beta}\hat{\theta}_i = \sum_{k=1}^{j}\left(\sum_{i=1}^{n_0} d_{i,k}\hat{\chi}_i(\hat{\varphi})\right)\widehat{\Theta}_k. \tag{5.5.6}$$

We set

$$\Lambda_k(\hat{\varphi}) := \sum_{i=1}^{n_0} d_{i,k}\hat{\chi}_i(\hat{\varphi}). \tag{5.5.7}$$

From (5.1.2) and (5.5.7), we have

$$\Lambda_k(\hat{\varphi}) = \sum_{i=1}^{n_0} d_{i,k}\hat{\chi}_i(\hat{\varphi}) = \sum_{i=1}^{n_0} d_{i,k}\hat{\chi}_i(I_{\widehat{T}}\hat{\varphi}) = \Lambda_k(I_{\widehat{T}}\hat{\varphi}). \tag{5.5.8}$$

Theorem 5.5.4. *Let $\ell \in \mathbb{N}_0$ with $0 \leq \ell \leq k$. Let β be a multi-index such that $\mathcal{P}^{\ell}(\widehat{T}) \subset \partial_{\hat{x}}^{\beta}\widehat{P}$. Fix $m \in \mathbb{N}_0$ and $p, q \in [1, \infty]$ such that $0 \leq m \leq \ell + 1 \leq k + 1$ and*

$$W^{\ell+1,p}(\widehat{T}) \hookrightarrow W^{m,q}(\widehat{T}). \tag{5.5.9}$$

Let $I_{\widehat{T}} : V(\widehat{T}) := W^{\ell+1+|\beta|,p}(\widehat{T}) \to \widehat{P}$ be the interpolation operator on $\widehat{T}$ defined in (5.1.1) such that

$$I_{\widehat{T}} \in \mathcal{L}(W^{\ell+1+|\beta|,p}(\widehat{T}); W^{m+|\beta|,q}(\widehat{T})).$$

Assume that there exists an interpolation operator $J_{\widehat{T}} : W^{\ell+1,p}(\widehat{T}) \to \partial_{\hat{x}}^{\beta}\widehat{P}$ such that

$$J_{\widehat{T}} \in \mathcal{L}(W^{\ell+1,p}(\widehat{T}); W^{m,q}(\widehat{T})), \tag{5.5.10}$$

and

$$\partial_{\hat{x}}^{\beta} I_{\widehat{T}}\hat{\varphi} = J_{\widehat{T}}\partial_{\hat{x}}^{\beta}\hat{\varphi} \quad \forall \hat{\varphi} \in W^{\ell+1+|\beta|,p}(\widehat{T}). \tag{5.5.11}$$

It then holds that

$$|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})|_{W^{m,q}(\widehat{T})} \leq C(\widehat{T}, I_{\widehat{T}})|\partial_{\hat{x}}^{\beta}\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})} \quad \forall \hat{\varphi} \in W^{\ell+1+|\beta|,p}(\widehat{T}). \tag{5.5.12}$$

Proof. We follow [23]. From (5.5.11), we obviously have

$$|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})|_{W^{m,q}(\widehat{T})} = |\partial_{\hat{x}}^{\beta}\hat{\varphi} - J_{\widehat{T}}\partial_{\hat{x}}^{\beta}\hat{\varphi}|_{W^{m,q}(\widehat{T})}. \tag{5.5.13}$$

Because $\mathcal{P}^{\ell}(\widehat{T}) \subset \partial_{\hat{x}}^{\beta}\widehat{P}$, for any $\hat{p} \in \mathcal{P}^{\ell}(\widehat{T})$, there exists $\hat{q} \in \mathcal{P}^{\ell+|\beta|}(\widehat{T}) \subset \widehat{P}$ such that $\hat{p} = \partial_{\hat{x}}^{\beta}\hat{q}$. We then have, from (5.5.11) and (5.1.3),

$$J_{\widehat{T}}\hat{p} = J_{\widehat{T}}\partial_{\hat{x}}^{\beta}\hat{q} = \partial_{\hat{x}}^{\beta} I_{\widehat{T}}\hat{q} = \partial_{\hat{x}}^{\beta}\hat{q} = \hat{p}.$$

Therefore, we obtain, from (5.5.10) and (5.5.14),

$$\begin{aligned}
|\partial_{\hat{x}}^{\beta}\hat{\varphi} - J_{\widehat{T}}\partial_{\hat{x}}^{\beta}\hat{\varphi}|_{W^{m,q}(\widehat{T})} &\leq |\partial_{\hat{x}}^{\beta}\hat{\varphi} - \hat{p}|_{W^{m,q}(\widehat{T})} + |J_{\widehat{T}}(\hat{p} - \partial_{\hat{x}}^{\beta}\hat{\varphi})|_{W^{m,q}(\widehat{T})} \\
&= |\partial_{\hat{x}}^{\beta}(\hat{\varphi} - \hat{q})|_{W^{m,q}(\widehat{T})} + |J_{\widehat{T}}\{\partial_{\hat{x}}^{\beta}(\hat{q} - \hat{\varphi})\}|_{W^{m,q}(\widehat{T})} \\
&\leq c|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - \hat{q})|_{W^{\ell+1,p}(\widehat{T})}.
\end{aligned}$$

From the Bramble–Hilbert–type lemma (e.g., Lemma 1.6.10), we thus have

$$\begin{aligned}
|\partial_{\hat{x}}^{\beta}\hat{\varphi} - J_{\widehat{T}}\partial_{\hat{x}}^{\beta}\hat{\varphi}|_{W^{m,q}(\widehat{T})} &\leq c \inf_{\hat{p} \in \mathcal{P}^{\ell}(\widehat{T})} |\partial_{\hat{x}}^{\beta}\hat{\varphi} - \hat{p}|_{W^{\ell+1,p}(\widehat{T})} \\
&\leq c|\partial_{\hat{x}}^{\beta}\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})},
\end{aligned}$$

which is the target inequality. □

Theorem 5.5.5. *Let $\ell \in \mathbb{N}_0$ with $0 \leq \ell \leq k$. Let β be a multi-index such that $\mathcal{P}^{\ell}(\widehat{T}) \subset \partial_{\hat{x}}^{\beta}\widehat{P}$. Fix $m \in \mathbb{N}_0$ and $p, q \in [1, \infty]$ such that $0 \leq m \leq \ell + 1 \leq k + 1$ and*

$$W^{\ell+1,p}(\widehat{T}) \hookrightarrow W^{m,q}(\widehat{T}). \tag{5.5.14}$$

Let $I_{\widehat{T}} : V(\widehat{T}) := W^{\ell+1+|\beta|,p}(\widehat{T}) \to \widehat{P}$ be the interpolation operator on $\widehat{T}$ defined in (5.1.1) *such that*

$$I_{\widehat{T}} \in \mathcal{L}(W^{\ell+1+|\beta|,p}(\widehat{T}); W^{m+|\beta|,q}(\widehat{T})).$$

If $\Lambda_k(\hat{\varphi})$ of (5.5.7) *can be expressed by*

$$\Lambda_k(\hat{\varphi}) := \mathscr{J}_k(\partial_{\hat{x}}^{\beta}\hat{\varphi}), \quad k = 1, \ldots, j, \tag{5.5.15}$$

where

$$\mathscr{J}_k \in W^{\ell+1,p}(\widehat{T})', \quad k = 1, \ldots, j, \tag{5.5.16}$$

then (5.5.12) *holds.*

Proof. We follow [23]. We define the interpolation operator $J_{\widehat{T}} : W^{\ell+1,p}(\widehat{T}) \to \partial_{\hat{x}}^{\beta} \widehat{P}$ by

$$J_{\widehat{T}} \hat{\eta} := \sum_{k=1}^{j} \mathscr{I}_k(\hat{\eta}) \widehat{\Theta}_k \quad \forall \hat{\eta} \in W^{\ell+1,p}(\widehat{T}).$$

We then have, from (5.5.16),

$$\| J_{\widehat{T}} \hat{\eta} \|_{W^{m,q}(\widehat{T})} \leq \sum_{k=1}^{j} | \mathscr{I}_k(\hat{\eta}) | \| \widehat{\Theta}_k \|_{W^{m,q}(\widehat{T})} \leq c \| \hat{\eta} \|_{W^{\ell+1,p}(\widehat{T})},$$

that is, $J_{\widehat{T}} \in \mathcal{L}(W^{\ell+1,p}(\widehat{T}); W^{m,q}(\widehat{T}))$.

Furthermore, for any $\hat{\varphi} \in W^{\ell+1+|\beta|,p}(\widehat{T})$, we have, from (5.5.15) and (5.5.6),

$$J_{\widehat{T}} \partial_{\hat{x}}^{\beta} \hat{\varphi} = \sum_{k=1}^{j} \mathscr{I}_k(\partial_{\hat{x}}^{\beta} \hat{\varphi}) \widehat{\Theta}_k = \sum_{k=1}^{j} \Lambda_k(\hat{\varphi}) \widehat{\Theta}_k = \partial_{\hat{x}}^{\beta} (I_{\widehat{T}} \hat{\varphi}).$$

Therefore, the estimate (5.5.12) follows from Theorem 5.5.4. □

Remark 5.5.6 ([23])**.** From (5.5.15) and (5.5.8), we have

$$\mathscr{I}_k(\partial_{\hat{x}}^{\beta} I_{\widehat{T}} \hat{\varphi}) = \Lambda_k(I_{\widehat{T}} \hat{\varphi}) = \Lambda_k(\hat{\varphi}) = \mathscr{I}_k(\partial_{\hat{x}}^{\beta} \hat{\varphi}), \quad k = 1, \ldots, j.$$

For any $\hat{q} \in \partial_{\hat{x}}^{\beta} \widehat{P}$, there exists $\hat{p} \in \widehat{P}$ such that $\hat{q} = \partial_{\hat{x}}^{\beta} \hat{p}$. We also have, from (5.1.3), (5.5.6) and (5.5.15),

$$\hat{q} = \partial_{\hat{x}}^{\beta} \hat{p} = \partial_{\hat{x}}^{\beta} I_{\widehat{T}} \hat{p} = \sum_{k=1}^{j} \Lambda_k(\hat{p}) \widehat{\Theta}_k = \sum_{k=1}^{j} \mathscr{I}_k(\partial_{\hat{x}}^{\beta} \hat{p}) \widehat{\Theta}_k = \sum_{k=1}^{j} \mathscr{I}_k(\hat{q}) \widehat{\Theta}_k.$$

Because $\{\widehat{\Theta}_k\}_{k=1}^{j}$ is a basis for $\partial_{\hat{x}}^{\beta} \widehat{P}$, we obtain

$$\forall \hat{q} \in \partial_{\hat{x}}^{\beta} \widehat{P}, \quad \mathscr{I}_k(\hat{q}) = 0, \quad 1 \leq k \leq j \quad \Leftrightarrow \quad \hat{q} = 0.$$

5.6 Local Interpolation Error Estimates

Theorem 5.6.1 (Local interpolation)**.** *Let* $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ *be a finite element with the normed vector space* $V(\widehat{T}) := \mathcal{C}(\widehat{T})$ *and* $\widehat{P} := \mathcal{P}^k(\widehat{T})$ *with* $k \geq 1$*. Let* $I_{\widehat{T}} : V(\widehat{T}) \to \widehat{P}$ *be a linear operator. Fix* $\ell \in \mathbb{N}$*,* $m \in \mathbb{N}_0$*, and* $p, q \in [1, \infty]$ *such that* $0 \leq m \leq \ell \leq k+1$*,* $\ell - m \geq 1$*, and the embeddings* (1.6.2) *and*

(1.6.3) *with* $s := \ell - m$ *hold. Let* β *be a multi-index with* $|\beta| = m$. *We set* $j := \dim(\partial^\beta \mathcal{P}^k)$. *Assume that there exist linear functionals* $\mathscr{F}_i$, $i = 1, \ldots, j$, *satisfying the conditions* (5.5.2). *It then holds that, for all* $\hat{\varphi} \in W^{\ell,p}(\widehat{T}) \cap \mathcal{C}(\widehat{T})$ *with* $\varphi := \hat{\varphi} \circ \Phi^{-1}$,

$$|\varphi - I_T\varphi|_{W^{m,q}(T)} \leq C_1^I |T|^{\frac{1}{q}-\frac{1}{p}} \left(\frac{H_T}{h_T}\right)^m \sum_{|\varepsilon|=\ell-m} h^\varepsilon |\partial_r^\varepsilon \varphi|_{W^{m,p}(T)}, \tag{5.6.1}$$

where C_1^I *is a positive constant independent of* h_T *and* H_T. *In particular, if Condition 3.3.1 is imposed, it holds that, for all* $\hat{\varphi} \in W^{\ell,p}(\widehat{T}) \cap \mathcal{C}(\widehat{T})$ *with* $\varphi := \hat{\varphi} \circ \Phi^{-1}$,

$$|\varphi - I_T\varphi|_{W^{m,q}(T)} \leq C_2^I |T|^{\frac{1}{q}-\frac{1}{p}} \left(\frac{H_T}{h_T}\right)^m \sum_{|\varepsilon|=\ell-m} \mathscr{H}^\varepsilon |\partial^\varepsilon (\varphi \circ \Phi_{T^s})|_{W^{m,p}(\Phi_{T^s}^{-1}(T))}, \tag{5.6.2}$$

where C_2^I *is a positive constant independent of* h_{T^s} *and* H_{T^s}.

Proof. The introduction of the functionals $\mathscr{F}_i$ follows from [10, 4], also see Theorem 5.5.1. Actually, under the same assumptions as in Theorem 5.6.1, we have

$$\|\partial_{\hat{x}}^\beta (\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} \leq C^B |\partial_{\hat{x}}^\beta \hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})}, \tag{5.6.3}$$

where $|\beta| = m$, $\hat{\varphi} \in \mathcal{C}(\widehat{T})$, and $\partial_{\hat{x}}^\beta \hat{\varphi} \in W^{\ell-m,p}(\widehat{T})$.

The inequalities in (5.3.11), (1.6.1), (5.3.1), and (5.6.3) yield

$$\begin{aligned}
|\varphi - I_T\varphi|_{W^{m,q}(T)} &\leq c|\varphi^s - I_{T^s}\varphi^s|_{W^{m,q}(T^s)} \\
&\leq c|\det(\mathcal{A}^s)|^{\frac{1}{q}} \|\widetilde{\mathcal{A}}^{-1}\|_2^m \left(\sum_{|\beta|=m} (h^{-\beta})^q \|\partial_{\hat{x}}^\beta (\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})}^q \right)^{1/q} \\
&\leq c|\det(\mathcal{A}^s)|^{\frac{1}{q}} \|\widetilde{\mathcal{A}}^{-1}\|_2^m \sum_{|\beta|=m} (h^{-\beta}) \|\partial_{\hat{x}}^\beta (\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} \\
&\leq c|\det(\mathcal{A}^s)|^{\frac{1}{q}} \|\widetilde{\mathcal{A}}^{-1}\|_2^m \sum_{|\beta|=m} (h^{-\beta}) |\partial_{\hat{x}}^\beta \hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})}.
\end{aligned} \tag{5.6.4}$$

Inequalities (1.6.1) and (5.3.2) yield

$$
\begin{aligned}
&\sum_{|\beta|=m} (h^{-\beta}) |\partial_{\hat{x}}^{\beta} \hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})} \\
&\quad \leq \sum_{|\gamma|=\ell-m} \sum_{|\beta|=m} (h^{-\beta}) \|\partial_{\hat{x}}^{\beta} \partial_{\hat{x}}^{\gamma} \hat{\varphi}\|_{L^p(\widehat{T})} \\
&\quad \leq c |\det(\mathcal{A}^s)|^{-\frac{1}{p}} \|\widetilde{\mathcal{A}}\|_2^m \sum_{|\gamma|=\ell-m} \sum_{|\beta|=m} (h^{-\beta}) h^{\beta} \sum_{|\epsilon|=|\gamma|} h^{\varepsilon} |\partial_r^{\epsilon} \varphi|_{W^{m,p}(T)} \\
&\quad \leq c |\det(\mathcal{A}^s)|^{-\frac{1}{p}} \|\widetilde{\mathcal{A}}\|_2^m \sum_{|\epsilon|=\ell-m} h^{\varepsilon} |\partial_r^{\varepsilon} \varphi|_{W^{m,p}(T)}. \qquad (5.6.5)
\end{aligned}
$$

From (3.6.1), (3.6.2), (5.6.4), and (5.6.5), we have

$$
|\varphi - I_T \varphi|_{W^{m,q}(T)} \leq c |T|^{\frac{1}{q}-\frac{1}{p}} \left(\frac{H_T}{h_T} \right)^m \sum_{|\varepsilon|=\ell-m} h^{\varepsilon} |\partial_r^{\varepsilon} \varphi|_{W^{m,p}(T)},
$$

which is the inequality (5.6.1).

If Condition 3.3.1 is imposed, inequality (5.3.3) yields

$$
\begin{aligned}
&\sum_{|\beta|=m} (h^{-\beta}) |\partial_{\hat{x}}^{\beta} \hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})} \\
&\quad \leq \sum_{|\gamma|=\ell-m} \sum_{|\beta|=m} (h^{-\beta}) \|\partial_{\hat{x}}^{\beta} \partial_{\hat{x}}^{\gamma} \hat{\varphi}\|_{L^p(\widehat{T})} \\
&\quad \leq c |\det(\mathcal{A}^s)|^{-\frac{1}{p}} \|\widetilde{\mathcal{A}}\|_2^m \sum_{|\gamma|=\ell-m} \sum_{|\beta|=m} (h^{-\beta}) h^{\beta} \sum_{|\epsilon|=|\gamma|} \mathscr{H}^{\varepsilon} |\partial^{\epsilon} \varphi^s|_{W^{m,p}(T^s)} \\
&\quad \leq c |\det(\mathcal{A}^s)|^{-\frac{1}{p}} \|\widetilde{\mathcal{A}}\|_2^m \sum_{|\epsilon|=\ell-m} \mathscr{H}^{\varepsilon} |\partial^{\epsilon} \varphi^s|_{W^{m,p}(T^s)}. \qquad (5.6.6)
\end{aligned}
$$

From (3.6.1), (3.6.2), (5.6.4), and (5.6.6), we have

$$
|\varphi - I_T \varphi|_{W^{m,q}(T)} \leq c |T|^{\frac{1}{q}-\frac{1}{p}} \left(\frac{H_T}{h_T} \right)^m \sum_{|\varepsilon|=\ell-m} \mathscr{H}^{\varepsilon} |\partial^{\epsilon} \varphi^s|_{W^{m,p}(T^s)},
$$

which is the inequality (5.6.2) using $T^s = \Phi_{T^s}^{-1}(T)$ and $\varphi^s = \varphi \circ \Phi_{T^s}$. □

5.7 Global Interpolation Error Estimates

A global interpolation operator I_h is constructed as follows ([30, Section 1.4.2]). Its domain is defined by

$$
D(I_h) := \{\varphi \in L^1(\Omega);\ \varphi|_T \in V(T),\ \forall T \in \mathbb{T}_h\}.
$$

For $T \in \mathbb{T}_h$ and $\varphi \in D(I_h)$, the quantities $\chi_i(\varphi|_T)$ are meaningful on all the mesh elements and $1 \le i \le n_0$. The global interpolation $I_h\varphi$ can be specified elementwise using the local interpolation operators, that is,

$$(I_h\varphi)|_T := I_T(\varphi|_T) = \sum_{i=1}^{n_0} \chi_i(\varphi|_T)\theta_i \quad \forall T \in \mathbb{T}_h, \quad \forall \varphi \in D(I_h).$$

The global interpolation operator $I_h : D(I_h) \to V_h$ is defined as

$$I_h : D(I_h) \ni \varphi \mapsto I_h\varphi := \sum_{T\in\mathbb{T}_h} \sum_{i=1}^{n_0} \chi_i(\varphi|_T)\theta_i \in V_h,$$

where V_h is defined as

$$V_h := \{\varphi_h \in L^1(\Omega)^n;\ \varphi_h|_T \in P,\ \forall T \in \mathbb{T}_h\}.$$

Theorem 5.7.1. *Suppose that the assumptions of Theorem 5.6.1 are satisfied. We impose Condition 4.3.1. Let I_h be the corresponding global interpolation operator. It then holds that, for any $\varphi \in W^{\ell,p}(\Omega)$;*

(I) *if Condition 3.3.1 is not imposed,*

$$|\varphi - I_h\varphi|_{W^{m,q}(\Omega)} \le c \sum_{T\in\mathbb{T}_h} |T|^{\frac{1}{q}-\frac{1}{p}} \sum_{|\varepsilon|=\ell-m} h^\varepsilon |\partial_r^\varepsilon \varphi|_{W^{m,p}(T)}. \tag{5.7.1}$$

(II) *if Condition 3.3.1 is imposed,*

$$|\varphi - I_h\varphi|_{W^{m,q}(\Omega)} \le c \sum_{T\in\mathbb{T}_h} |T|^{\frac{1}{q}-\frac{1}{p}} \sum_{|\varepsilon|=\ell-m} \mathscr{H}^\varepsilon |\partial^\varepsilon (\varphi \circ \Phi_{T^s})|_{W^{m,p}(\Phi_{T^s}^{-1}(T))}. \tag{5.7.2}$$

Proof. If Condition 3.3.1 is not imposed, using (5.6.1),

$$\begin{aligned}
|\varphi - I_h\varphi|^q_{W^{m,q}(\Omega)} &= \sum_{T\in\mathbb{T}} |\varphi - I_T\varphi|^q_{W^{m,q}(T)} \\
&\le c \sum_{T\in\mathbb{T}} |T|^{q\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\frac{H_T}{h_T}\right)^{qm} \left(\sum_{|\varepsilon|=\ell-m} h^\varepsilon |\partial_r^\varepsilon \varphi|_{W^{m,p}(T)}\right)^q,
\end{aligned}$$

which leads to the desired result together with (1.6.1) and Condition 4.3.1.

If Condition 3.3.1 is imposed, using (5.6.2),

$$|\varphi - I_h\varphi|^q_{W^{m,q}(\Omega)} \leq c \sum_{T\in\mathbb{T}} |\varphi^s - I_{T^s}\varphi^s|^q_{W^{m,q}(T^s)}$$

$$\leq c \sum_{T\in\mathbb{T}} |T|^{q\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\frac{H_T}{h_T}\right)^{qm} \left(\sum_{|\varepsilon|=\ell-m} \mathscr{H}^\varepsilon |\partial^\varepsilon \varphi^s|_{W^{m,p}(T^s)} \right)^q,$$

which leads to the desired result together with (1.6.1) and Condition 4.3.1.

□

5.8 Examples satisfying Conditions (5.5.2) in Theorem 5.5.1

5.8.1 Lagrange Finite Element

5.8.1.1 Finite Element Generation on Standard Element

Let $\widehat{T} \subset \mathbb{R}^d$ be the reference element defined in Sections 3.1.1 and 3.1.2. Let α be a multi-index. We define the set of nodes as

$$\mathscr{P} := \{\widehat{P}_i\}_{i=1}^{N^{(2,k)}} := \left\{ \left(\frac{i_1}{k}, \frac{i_2}{k}\right)^T \in \mathbb{R}^2 \right\}_{0\leq i_1+i_2\leq k} = \left\{ \frac{1}{k}\alpha \in \mathbb{R}^2 \right\}_{|\alpha|\leq k}, \quad \text{if } d=2,$$

$$\mathscr{P} := \{\widehat{P}_i\}_{i=1}^{N^{(3,k)}} := \widehat{T} \cap \left\{ \left(\frac{i_1}{k}, \frac{i_2}{k}, \frac{i_3}{k}\right)^T \in \mathbb{R}^3 \right\}_{0\leq i_1,i_2,i_3\leq k}, \quad \text{if } d=3.$$

The Lagrange finite element on the reference element is defined by the triple $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ as follows.

(I) $\widehat{P} := \mathcal{P}^k(\widehat{T})$;

(II) $\widehat{\Sigma}$ is a set $\{\hat{\chi}_i\}_{1\leq i\leq N^{(d,k)}}$ of $N^{(d,k)}$ linear forms $\{\hat{\chi}_i\}_{1\leq i\leq N^{(d,k)}}$ with its components such that, for any $\hat{p} \in \widehat{P}$,

$$\hat{\chi}_i(\hat{p}) := \hat{p}(\widehat{P}_i) \quad \forall i \in \{1 : N^{(d,k)}\}. \tag{5.8.1}$$

The nodal basis functions associated with the degrees of freedom by (5.8.1) are defined as

$$\hat{\theta}_i(\widehat{P}_j) = \delta_{ij} \quad \forall i,j \in \{1 : N^{(d,k)}\}. \tag{5.8.2}$$

It then holds that $\hat{\chi}_i(\hat{\theta}_j) = \delta_{ij}$ for any $i, j \in \{1 : d+1\}$. Setting $V(\widehat{T}) := \mathcal{C}(\widehat{T})$ or $V(\widehat{T}) := W^{s,p}(\widehat{T})$ with $p \in [1, \infty]$ and $ps > d$ ($s \geq d$ if $p = 1$), the local operator $I_{\widehat{T}}^L$ is defined as

$$I_{\widehat{T}}^L : V(\widehat{T}) \ni \hat{\varphi} \mapsto I_{\widehat{T}}^L \hat{\varphi} := \sum_{i=1}^{N^{(d,k)}} \hat{\varphi}(\widehat{P}_i)\hat{\theta}_i \in \widehat{P}. \tag{5.8.3}$$

By analogous argument in Section 5.1, we assume that the Lagrange finite elements $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$, $\{T^s, P^s, \Sigma^s\}$ and $\{T, P, \Sigma\}$ are constructed. The local shape functions are $\tilde{\theta}_i = \psi_{\widetilde{T}}^{-1}(\hat{\theta}_i)$, $\theta_i^s = \psi_{\widetilde{T}}^{-1}(\tilde{\theta}_i)$ and $\theta_i = \psi_{T^s}^{-1}(\theta_i^s)$ for any $i \in \{1 : N^{(d,k)}\}$, and the associated local interpolation operators are respectively defined as

$$I_{\widetilde{T}}^L : V(\widetilde{T}) \ni \tilde{\varphi} \mapsto I_{\widetilde{T}}^L \tilde{\varphi} := \sum_{i=1}^{N^{(d,k)}} \tilde{\varphi}(\widetilde{P}_i)\tilde{\theta}_i \in \widetilde{P}, \tag{5.8.4}$$

$$I_{T^s}^L : V(T^s) \ni \varphi^s \mapsto I_{T^s}^L \varphi^s := \sum_{i=1}^{N^{(d,k)}} \varphi^s(P_i^s)\theta_i^s \in P^s, \tag{5.8.5}$$

$$I_T^L : V(T) \ni \varphi \mapsto I_T^L \varphi := \sum_{i=1}^{N^{(d,k)}} \varphi(P_i)\theta_i \in P, \tag{5.8.6}$$

where $\widetilde{P}_i = \widehat{\Phi}(\widehat{P}_i)$, $P_i^s = \widetilde{\Phi}(\widetilde{P}_i)$ and $P_i = \Phi_{T^s}(P_i^s)$ for $i \in \{1 : N^{(d,k)}\}$.

5.8.1.2 Local Error Estimates

Lemma 5.8.1 ($d = 2$)**.** *Let β be a multi-index with $m := |\beta|$ and $\hat{\varphi} \in \mathcal{C}(\widehat{T})$ a function such that $\partial_{\hat{x}}^{\beta} \hat{\varphi} \in W^{\ell-m,p}(\widehat{T})$, where $\ell, m \in \mathbb{N}_0$, $p \in [1, \infty]$ are such that $0 \leq m \leq \ell \leq k+1$ and*

$$p = \infty \quad \textit{if } m = 0 \textit{ and } \ell = 0, \tag{5.8.7a}$$

$$p > 2 \quad \textit{if } m = 0 \textit{ and } \ell = 1, \tag{5.8.7b}$$

$$m < \ell \quad \textit{if } \beta_1 = 0 \textit{ or } \beta_2 = 0, \textit{ and } m > 0. \tag{5.8.7c}$$

Fix $q \in [1, \infty]$ such that $W^{\ell-m,p}(\widehat{T}) \hookrightarrow L^q(\widehat{T})$. Let $I_{\widehat{T}} := I_{\widehat{T}}^L$. It then holds that

$$\|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}}^L \hat{\varphi})\|_{L^q(\widehat{T})} \leq c|\partial_{\hat{x}}^{\beta} \hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})}. \tag{5.8.8}$$

Proof. The proof is found in [4, Lemma 2.4]. □

Lemma 5.8.2 ($d = 3$)**.** *Let β be a multi-index with $m := |\beta|$ and $\hat{\varphi} \in \mathcal{C}(\widehat{T})$ a function such that $\partial_{\hat{x}}^{\beta}\hat{\varphi} \in W^{\ell-m,p}(\widehat{T})$, where $\ell, m \in \mathbb{N}_0$, $p \in [1,\infty]$ are such that $0 \le m \le \ell \le k+1$ and*

$$p = \infty \quad \text{if } m = 0 \text{ and } \ell = 0, \tag{5.8.9a}$$

$$p > \frac{3}{\ell} \quad \text{if } m = 0 \text{ and } \ell = 1, 2, \tag{5.8.9b}$$

$$m < \ell \quad \text{if } \beta_1 = 0,\ \beta_2 = 0,\ \text{or } \beta_3 = 0, \tag{5.8.9c}$$

$$p > 2 \quad \text{if } \beta \in \{(\ell-1,0,0); (0,\ell-1,0); (0,0,\ell-1)\}. \tag{5.8.9d}$$

Fix $q \in [1,\infty]$ such that $W^{\ell-m,p}(\widehat{T}) \hookrightarrow L^q(\widehat{T})$. Let $I_{\widehat{T}} := I_{\widehat{T}}^L$. It then holds that

$$\|\partial_{\hat{x}}^{\beta}(\hat{\varphi} - I_{\widehat{T}}\hat{\varphi})\|_{L^q(\widehat{T})} \le c|\partial_{\hat{x}}^{\beta}\hat{\varphi}|_{W^{\ell-m,p}(\widehat{T})}. \tag{5.8.10}$$

Proof. The proof is found in [4, Lemma 2.6]. □

Corollary 5.8.3. *Let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be the Lagrange finite element with $V(\widehat{T}) := \mathcal{C}(\widehat{T})$ and $\widehat{P} := \mathcal{P}^k(\widehat{T})$ with $k \ge 1$. Let $I_{\widehat{T}} := I_{\widehat{T}}^L$. Let $m \in \mathbb{N}_0$, $\ell \in \mathbb{N}$, and $p \in \mathbb{R}$ be such that $0 \le m \le \ell \le k+1$ and*

$$d = 2: \begin{cases} p \in (2,\infty] & \text{if } m = 0,\ \ell = 1, \\ p \in [1,\infty] & \text{if } m = 0,\ \ell \ge 2 \text{ or } m \ge 1,\ \ell - m \ge 1, \end{cases}$$

$$d = 3: \begin{cases} p \in \left(\frac{3}{\ell},\infty\right] & \text{if } m = 0,\ \ell = 1, 2, \\ p \in (2,\infty] & \text{if } m \ge 1,\ \ell - m = 1, \\ p \in [1,\infty] & \text{if } m = 0,\ \ell \ge 3 \text{ or } m \ge 1,\ \ell - m \ge 2. \end{cases}$$

Setting $q \in [1,\infty]$ such that $W^{\ell-m,p}(\widehat{T}) \hookrightarrow L^q(\widehat{T})$. Then, for all $\hat{\varphi} \in W^{\ell,p}(\widehat{T})$ with $\varphi := \hat{\varphi} \circ \Phi^{-1}$, we have

$$|\varphi - I_T^L\varphi|_{W^{m,q}(T)} \le c|T|^{\frac{1}{q}-\frac{1}{p}} \left(\frac{H_T}{h_T}\right)^m \sum_{|\varepsilon|=\ell-m} h^{\varepsilon}|\partial_r^{\varepsilon}\varphi|_{W^{m,p}(T)}. \tag{5.8.11}$$

In particular, if Condition 3.3.1 is imposed, it holds that,for all $\hat{\varphi} \in W^{\ell,p}(\widehat{T})$ with $\varphi := \hat{\varphi} \circ \Phi^{-1}$,

$$|\varphi - I_T^L\varphi|_{W^{m,q}(T)} \le C_2^I|T|^{\frac{1}{q}-\frac{1}{p}} \left(\frac{H_T}{h_T}\right)^m \sum_{|\varepsilon|=\ell-m} \mathscr{H}^{\varepsilon}|\partial^{\varepsilon}(\varphi \circ \Phi_{T^s})|_{W^{m,p}(\Phi_{T^s}^{-1}(T))}. \tag{5.8.12}$$

Furthermore, for any $\hat{\varphi} \in \mathcal{C}(\widehat{T})$ with $\varphi := \hat{\varphi} \circ \Phi^{-1}$, it holds that

$$\|\varphi_0 - I_T\varphi\|_{L^\infty(T)} \le c\|\varphi\|_{L^\infty(T)}.$$

5.8.2 Nodal Crouzeix–Raviart Finite Element

5.8.2.1 Finite Element Generation on Standard Element

Let $\widehat{T} \subset \mathbb{R}^d$ be the reference element defined in Sections 3.1.1 and 3.1.2. Let $\widehat{F}_i$ be the face of $\widehat{T}$ opposite to $\widehat{P}_i$ and let $\hat{x}_{\widehat{F}_i}$ the barycentre of the face $\widehat{F}_i$. The (nodal) Crouzeix–Raviart finite element on the reference element is defined by the triple $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ as follows.

(I) $\widehat{P} := \mathcal{P}^1(\widehat{T})$;

(II) $\widehat{\Sigma}$ is a set $\{\hat{\chi}_i\}_{1 \le i \le N^{(d,1)}}$ of $N^{(d,1)}$ linear forms $\{\hat{\chi}_i\}_{1 \le i \le N^{(d,1)}}$ with its components such that, for any $\hat{p} \in \widehat{P}$,

$$\hat{\chi}_i(\hat{p}) := \hat{p}(\hat{x}_{\widehat{F}_i}) \quad \forall i \in \{1 : d+1\}. \tag{5.8.13}$$

Using the barycentric coordinates $\{\hat{\lambda}_i\}_{i=1}^{d+1} : \mathbb{R}^d \to \mathbb{R}$ on the reference element, the nodal basis functions associated with the degrees of freedom by (5.8.13) are defined as

$$\hat{\theta}_i(\hat{x}) := d\left(\frac{1}{d} - \hat{\lambda}_i(\hat{x})\right) \quad \forall i \in \{1 : d+1\}. \tag{5.8.14}$$

It then holds that $\hat{\chi}_i(\hat{\theta}_j) = \delta_{ij}$ for any $i, j \in \{1 : d+1\}$. Setting $V(\widehat{T}) := \mathcal{C}(\widehat{T})$ or $V(\widehat{T}) := W^{s,p}(\widehat{T})$ with $p \in [1, \infty]$ and $ps > d$ ($s \ge d$ if $p = 1$), the local operator $I_{\widehat{T}}^{nCR}$ is defined as

$$I_{\widehat{T}}^{nCR} : V(\widehat{T}) \ni \hat{\varphi} \mapsto I_{\widehat{T}}^{nCR} \hat{\varphi} := \sum_{i=1}^{d+1} \hat{\varphi}(\hat{x}_{\widehat{F}_i}) \hat{\theta}_i \in \widehat{P}. \tag{5.8.15}$$

By analogous argument in Section 5.1, we assume that the (nodal) Crouzeix–Raviart finite elements $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$, $\{T^s, P^s, \Sigma^s\}$ and $\{T, P, \Sigma\}$ are constructed. The local shape functions are $\tilde{\theta}_i = \psi_{\widehat{T}}^{-1}(\hat{\theta}_i)$, $\theta_i^s = \psi_{\widetilde{T}}^{-1}(\tilde{\theta}_i)$ and $\theta_i = \psi_{T^s}^{-1}(\theta_i^s)$ for any $i \in \{1 : d+1\}$, and the associated local interpolation operators are respectively defined as

$$I_{\widetilde{T}}^{nCR} : V(\widetilde{T}) \ni \tilde{\varphi} \mapsto I_{\widetilde{T}}^{nCR} \tilde{\varphi} := \sum_{i=1}^{d+1} \tilde{\varphi}(\tilde{x}_{\widetilde{F}_i}) \tilde{\theta}_i \in \widetilde{P}, \tag{5.8.16}$$

$$I_{T^s}^{nCR} : V(T^s) \ni \varphi^s \mapsto I_{T^s}^{nCR} \varphi^s := \sum_{i=1}^{d+1} \varphi^s(x_{F_i^s}) \theta_i^s \in P^s, \tag{5.8.17}$$

$$I_{T}^{nCR} : V(T) \ni \varphi \mapsto I_{T}^{nCR} \varphi := \sum_{i=1}^{d+1} \varphi(x_{F_i}) \theta_i \in P, \tag{5.8.18}$$

where $\{\widetilde{F}_i := \widehat{\Phi}(\widehat{F}_i)\}_{i \in \{1:d+1\}}$, $\{F_i^s := \widetilde{\Phi}(\widetilde{F}_i)\}_{i \in \{1:d+1\}}$ and $\{F_i := \Phi_{T^s}(F_i^s)\}_{i \in \{1:d+1\}}$.

5.8.2.2 Local Error Estimates

Corollary 5.8.4. *Let* $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ *be the Crouzeix–Raviart finite element with* $V(\widehat{T}) := \mathcal{C}(\widehat{T})$ *and* $\widehat{P} := \mathcal{P}^1(\widehat{T})$*. Set* $I_{\widehat{T}}; = I_{\widehat{T}}^{nCR}$*. Let* $m \in \mathbb{N}_0$*,* $\ell \in \mathbb{N}$*, and* $p \in \mathbb{R}$ *be such that*

$$d = 2 : \begin{cases} p \in (2, \infty] & \text{if } m = 0,\ \ell = 1, \\ p \in [1, \infty] & \text{if } m = 0,\ \ell = 2 \text{ or } m = 1,\ \ell = 2, \end{cases}$$

$$d = 3 : \begin{cases} p \in \left(\frac{3}{\ell}, \infty\right] & \text{if } m = 0,\ \ell = 1, 2, \\ p \in (2, \infty] & \text{if } m = 1,\ \ell = 2. \end{cases}$$

Setting $q \in [1, \infty]$ *such that* $W^{\ell - m, p}(\widehat{T}) \hookrightarrow L^q(\widehat{T})$*. Then, for all* $\hat{\varphi} \in W^{\ell,p}(\widehat{T})$ *with* $\varphi := \hat{\varphi} \circ \Phi^{-1}$*, we have*

$$|\varphi - I_T^{nCR} \varphi|_{W^{m,q}(T)} \leq c |T|^{\frac{1}{q} - \frac{1}{p}} \left(\frac{H_T}{h_T} \right)^m \sum_{|\varepsilon| = \ell - m} h^{\varepsilon} |\partial_r^{\varepsilon} \varphi|_{W^{m,p}(T)}, \qquad (5.8.19)$$

In particular, if Condition 3.3.1 is imposed, it holds that, for all $\hat{\varphi} \in W^{\ell,p}(\widehat{T})$ *with* $\varphi := \hat{\varphi} \circ \Phi^{-1}$*,*

$$|\varphi - I_T \varphi|_{W^{m,q}(T)} \leq c |T|^{\frac{1}{q} - \frac{1}{p}} \left(\frac{H_T}{h_T} \right)^m \sum_{|\varepsilon| = \ell - m} \mathscr{H}^{\varepsilon} |\partial^{\varepsilon} (\varphi \circ \Phi_{T^s})|_{W^{m,p}(\Phi_{T^s}^{-1}(T))}. \qquad (5.8.20)$$

Furthermore, for any $\hat{\varphi} \in \mathcal{C}(\widehat{T})$ *with* $\varphi := \hat{\varphi} \circ \Phi^{-1}$*, it holds that*

$$\|\varphi_0 - I_T \varphi\|_{L^\infty(T)} \leq c \|\varphi\|_{L^\infty(T)}.$$

Proof. For $k = 1$, we only introduce functionals $\mathscr{F}_i$ satisfying (5.5.2) in Theorem 5.6.1 (or Theorem 5.5.1) for each ℓ and m.

Let $m = 0$, that is, $\beta = (0, \cdots, 0) \in \mathbb{N}_0^d$. We then have $j = \dim \mathcal{P}^1 = d+1$. From the Sobolev embedding theorem (Theorem 1.6.1), we have $W^{\ell,p}(\widehat{T}) \subset \mathcal{C}^0(\widehat{T})$ with $1 < p \leq \infty$, $d < \ell p$ or $p = 1$, $d \leq \ell$. Under this condition, we use

$$\mathscr{F}_i(\hat{\varphi}) := \hat{\varphi}(\hat{x}_{\widehat{F}_i}), \quad \hat{\varphi} \in W^{\ell,p}(\widehat{T}), \quad i = 1, \ldots, d+1.$$

It then holds that

$$|\mathscr{F}_i(\hat{\varphi})| \leq \|\hat{\varphi}\|_{\mathcal{C}^0(\widehat{T})} \leq c \|\hat{\varphi}\|_{W^{\ell,p}(\widehat{T})},$$

which means $\mathscr{F}_i \in W^{\ell,p}(\widehat{T})'$ for $i = 1, \ldots, d+1$, that is, (5.5.2a) is satisfied. Furthermore, we have

$$\mathscr{F}_i(I_{\widehat{T}}^{nCR}\hat{\varphi}) = (I_{\widehat{T}}^{nCR}\hat{\varphi})(\hat{x}_{\widehat{F}_i}) = \hat{\varphi}(\hat{x}_{\widehat{F}_i}) = \mathscr{F}_i(\hat{\varphi}), \quad i = 1, \ldots, d+1,$$

which satisfies (5.5.2b). For all $\hat{\eta} \in \mathcal{P}^1$, if $\mathscr{F}_i(\hat{\eta}) = 0$ for $i = 1, \ldots, d+1$, it obviously holds $\hat{\eta} = 0$. This means that (5.5.2c) is satisfied.

Let $d = 2$ and $m = 1$ ($\ell = 2$). We set $\beta = (1,0)$. We then have $j = \dim(\partial^\beta \mathcal{P}^1) = 1$. We consider a functional

$$\mathscr{F}_1(\hat{\varphi}) := \int_0^{\frac{1}{2}} \hat{\varphi}(\hat{x}_1, 1/2) d\hat{x}_1, \quad \hat{\varphi} \in W^{2,p}(\widehat{T}), \quad 1 < p.$$

We set $\widehat{I} := \{\hat{x} \in \widehat{T};\ \hat{x}_2 = \frac{1}{2}\}$. The continuity is then shown by the trace theorem (e.g., see Corollary 1.6.3):

$$|\mathscr{F}_1(\hat{\varphi})| \leq \|\hat{\varphi}\|_{L^1(\widehat{I})} \leq c\|\hat{\varphi}\|_{W^{1,p}(\widehat{T})},$$

which means $\mathscr{F}_1 \in W^{2,p}(\widehat{T})'$, that is, (5.5.2a) is satisfied. Furthermore, it holds that

$$\begin{aligned}\mathscr{F}_1(\partial^{(1,0)}(\hat{\varphi} - I_{\widehat{T}}^{nCR}\hat{\varphi})) &= \int_0^{\frac{1}{2}} \frac{\partial}{\partial \hat{x}_1}(\hat{\varphi} - I_{\widehat{T}}^{nCR}\hat{\varphi})(\hat{x}_1, 1/2) d\hat{x}_1 \\ &= \left[\hat{\varphi} - I_{\widehat{T}}^{nCR}\hat{\varphi}\right]_{(0,1/2)}^{(1/2,1/2)} = 0,\end{aligned}$$

which satisfy (5.5.2b). Let $\hat{\eta} := a\hat{x}_1 + b\hat{x}_2 + c$. We then have

$$\mathscr{F}_1(\partial^{(1,0)}\hat{\eta}) = \frac{1}{2}a.$$

If $\mathscr{F}_1(\partial^{(1,0)}\hat{\eta}) = 0$, $a = 0$. This implies that $\partial^{(1,0)}\hat{\eta} = 0$. This means that (5.5.2c) is satisfied.

By analogous argument, the case $\beta = (0,1)$ holds.

Let $d = 3$ and $m = 1$ ($\ell = 2$). We consider Type (i) in Section 3.1.2 in detail. That is, the reference element is $\widehat{T} = \operatorname{conv}\{0, e_1, e_2, e_3\}$. Here, $e_1, \ldots, e_3 \in \mathbb{R}^3$ are the canonical basis. We set $\beta = (1,0,0)$. We then have $j = \dim(\partial^\beta \mathcal{P}^1) = 1$. We consider a functional

$$\mathscr{F}_1(\hat{\varphi}) := \int_0^{\frac{1}{3}} \hat{\varphi}(\hat{x}_1, 1/3, 1/3) d\hat{x}_1, \quad \hat{\varphi} \in W^{2,p}(\widehat{T}), \quad \frac{3}{2} < p.$$

We set $\widehat{I} := \{\hat{x} \in \widehat{T};\ \hat{x}_2 = \frac{1}{3},\ \ \hat{x}_3 = \frac{1}{3}\}$. The continuity is then shown by the trace theorem:

$$|\mathscr{F}_1(\hat{\varphi})| \leq \|\hat{\varphi}\|_{L^1(\widehat{I})} \leq c\|\hat{\varphi}\|_{W^{2,p}(\widehat{T})} \quad \text{if } p > 2,$$

which means $\mathscr{F}_1 \in W^{2,p}(\widehat{T})'$, that is, (5.5.2a) is satisfied. Furthermore, it holds that

$$\mathscr{F}_1(\partial^{(1,0,0)}(\hat{\varphi} - I_{\widehat{T}}^{nCR}\hat{\varphi})) = \left[\hat{\varphi} - I_{\widehat{T}}^{nCR}\hat{\varphi}\right]_{(0,1/3,1/3)}^{(1/3,1/3,1/3)} = 0,$$

which satisfy (5.5.2b). Let $\hat{\eta} := a\hat{x}_1 + b\hat{x}_2 + c\hat{x}_3 + d$. We then have

$$\mathscr{F}_1(\partial^{(1,0,0)}\hat{\eta}) = \frac{1}{3}a.$$

If $\mathscr{F}_1(\partial^{(1,0,0)}\hat{\eta}) = 0$, $a = 0$. This implies that $\partial^{(1,0,0)}\hat{\eta} = 0$. This means that (5.5.2c) is satisfied.

By analogous argument, it holds the cases $\beta = (0,1,0), (0,0,1)$.

We consider Type (ii) in Section 3.1.2. That is, the reference element is $\widehat{T} = \mathrm{conv}\{0, e_1, e_1 + e_2, e_3\}$. We set $\beta = (1,0,0)$. We then have $j = \dim(\partial^\beta \mathcal{P}^1) = 1$. We consider a functional

$$\mathscr{F}_1(\hat{\varphi}) := \int_{\frac{1}{3}}^{\frac{2}{3}} \hat{\varphi}(\hat{x}_1, 1/3, 1/3) d\hat{x}_1, \quad \hat{\varphi} \in W^{2,p}(\widehat{T}).$$

By similar argument with Type (i), we can deduce the result.

When $m = \ell = 0$, $p = \infty$ and $q \in [1, \infty]$, it holds that

$$\|\hat{\varphi} - I_{\widehat{T}}^{nCR}\hat{\varphi}\|_{L^q(\widehat{T})} \le c\|\hat{\varphi}\|_{L^\infty(\widehat{T})},$$

because we have

$$|(I_{\widehat{T}}^{nCR}\hat{\varphi})(\hat{x})| \le \sum_{i=1}^{d+1} |\hat{\varphi}(\hat{x}_{\widehat{F}_i})||\hat{\theta}_i(\hat{x})| \le (d+1)\left(\max_{1 \le i \le d+1} \|\hat{\theta}_i\|_{L^\infty(\widehat{T})}\right) \|\hat{\varphi}\|_{L^\infty(\widehat{T})}.$$

□

5.9 Example that does not satisfy conditions (5.5.2) in Theorem 5.5.1

5.9.1 Motivation

The following lemma ([10, Lemma 4], [4, Lemma 2.3]) gives a criterion for the existence of linear functionals satisfying conditions (5.5.2b) and (5.5.2c).

Lemma 5.9.1. *Let P be an arbitrary polynomial space and β be a multi-index. We set $j := \dim(\partial^\beta P)$. Assume that $I : \mathcal{C}^\mu(\widehat{T}) \to P$, $\mu \in \mathbb{N}$, is a linear operator with $I\hat{\eta} = \hat{\eta}$ $\forall \hat{\eta} \in P$. Then, there exist linear functionals $\mathscr{F}_i : \mathcal{C}^\infty(\widehat{T}) \to \mathbb{R}$, $i = 1, \ldots, j$, such that*

$$\mathscr{F}_i(\partial^\beta(\hat{\varphi} - I\hat{\varphi})) = 0 \quad \forall i = 1, \ldots, j, \quad \forall \hat{\varphi} \in \mathcal{C}^\infty(\widehat{T}), \tag{5.9.1}$$

$$\hat{\eta} \in P, \quad \mathscr{F}_i(\partial^\beta \hat{\eta}) = 0 \quad \forall i = 1, \ldots, j \quad \Rightarrow \quad \partial^\beta \hat{\eta} = 0 \tag{5.9.2}$$

if and only if the condition

$$\hat{\varphi} \in \mathcal{C}^\infty(\widehat{T}), \quad \partial^\beta \hat{\varphi} = 0 \quad \Rightarrow \quad \partial^\beta I\hat{\varphi} = 0 \tag{5.9.3}$$

holds.

Proof. The proof can be found in [10, Lemma 4]. □

If Condition (5.9.3) is violated, estimate (5.5.3) does not hold. This means that one cannot obtain estimate (5.6.1), which is sharper than (5.4.2). The following is a counterexample that does not satisfy (5.9.3).

Let $\widehat{T} \subset \mathbb{R}^2$ be the reference element with vertices $\widehat{P}_1 := (0,0)^T$, $\widehat{P}_2 := (1,0)^T$, $\widehat{P}_3 := (0,1)^T$. We set $\widehat{P}_4 := (1/3, 1/3)^T$. We define the barycentric coordinates $\lambda_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, \ldots, 3$, on the reference element as

$$\lambda_1 := 1 - \hat{x}_1 - \hat{x}_2, \quad \lambda_2 := \hat{x}_1, \quad \lambda_3 := \hat{x}_2, \quad (\hat{x}_1, \hat{x}_2)^T \in \widehat{T}.$$

5.9.2 $\mathcal{P}^1$ + bubble Finite Element

As mentioned in Example 5.4.6, we define the local basis functions as

$$\begin{aligned} \theta_4(x) &:= 27\lambda_1(x)\lambda_2(x)\lambda_3(x), \\ \theta_i(x) &:= \lambda_i(x) - \frac{1}{3}\theta_4(x), \quad i = 1, 2, 3. \end{aligned}$$

The interpolation operator I_T^b defined by

$$I^b : \mathcal{C}^\mu(\widehat{T}) \ni \hat{\varphi} \mapsto I^b \hat{\varphi} := \sum_{i=1}^{4} \hat{\varphi}(\widehat{P}_i)\theta_i \in \operatorname{span}\{\theta_1, \theta_2, \theta_3, \theta_4\}.$$

Let $\beta = (1,0)$. Setting $\hat{\varphi}(\hat{x}_1, \hat{x}_2) := \hat{x}_2^2$, we have $\frac{\partial \hat{\varphi}}{\partial \hat{x}_1} = 0$. By simple calculation, we obtain

$$\begin{aligned} \frac{\partial}{\partial \hat{x}_1} I^b \hat{\varphi} &= \hat{\varphi}(\widehat{P}_1)\frac{\partial \theta_1}{\partial \hat{x}_1} + \hat{\varphi}(\widehat{P}_2)\frac{\partial \theta_1}{\partial \hat{x}_1} + \hat{\varphi}(\widehat{P}_3)\frac{\partial \theta_3}{\partial \hat{x}_1} + \hat{\varphi}(\widehat{P}_4)\frac{\partial \theta_4}{\partial \hat{x}_1} \\ &= \frac{\partial \theta_3}{\partial \hat{x}_1} + \frac{1}{3^2}\frac{\partial \theta_4}{\partial \hat{x}_1} = -\frac{1}{3}\frac{\partial \theta_4}{\partial \hat{x}_1} + \frac{1}{3^2}\frac{\partial \theta_4}{\partial \hat{x}_1} \not\equiv 0. \end{aligned}$$

Therefore, the condition (5.9.3) is not satisfied. This implies that the error estimate (5.5.3) on the reference element does not hold for the $\mathcal{P}^1$ + bubble finite element.

5.9.3 $\mathcal{P}^3$ Hermite Finite Element

Following [25, Theorem 2.2.8], we define the Hermite interpolation operator $I^H : H^3(T) \to \mathcal{P}^3$ as

$$
\begin{aligned}
I^H \hat{\varphi} := & \sum_{i=1}^{3} \left(-2\lambda_i^3 + 3\lambda_i^2 - 7\lambda_i \sum_{1 \le j < k \le 3, j \ne i, k \ne i} \lambda_j \lambda_k \right) \hat{\varphi}(\widehat{P}_i) + 27\lambda_1\lambda_2\lambda_3 \hat{\varphi}(\widehat{P}_4) \\
& + \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \lambda_i \lambda_j (2\lambda_i + \lambda_j - 1)(\widehat{P}_j^{(1)} - \widehat{P}_i^{(1)}) \right) \frac{\partial \hat{\varphi}}{\partial \hat{x}_1}(\widehat{P}_i) \\
& + \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \lambda_i \lambda_j (2\lambda_i + \lambda_j - 1)(\widehat{P}_j^{(2)} - \widehat{P}_i^{(2)}) \right) \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{P}_i),
\end{aligned}
$$

where $\widehat{P}_i^{(k)}$, $1 \le k \le 2$, are the components of a point $\widehat{P}_i \in \mathbb{R}^2$.

Let $\beta = (1, 0)$. Setting $\hat{\varphi}(\hat{x}_1, \hat{x}_2) := \hat{x}_2^4$, we have $\frac{\partial \hat{\varphi}}{\partial \hat{x}_1} = 0$. Furthermore, by a simple calculation, i.e.,

$$
\begin{aligned}
\frac{\partial}{\partial \hat{x}_1}(\lambda_1\lambda_2\lambda_3) &= -\hat{x}_2^2 - 2\hat{x}_1\hat{x}_2 - \hat{x}_2, \\
\frac{\partial}{\partial \hat{x}_1}\left\{\lambda_3\lambda_1(2\lambda_3 + \lambda_1 - 1)\right\} &= -\hat{x}_2 + 2\hat{x}_1\hat{x}_2, \\
\frac{\partial}{\partial \hat{x}_1}\left\{\lambda_3\lambda_2(2\lambda_3 + \lambda_2 - 1)\right\} &= -\hat{x}_2 + 2\hat{x}_1\hat{x}_2 + 2\hat{x}_2^2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{\partial}{\partial \hat{x}_1} I^H \hat{\varphi} = & \frac{\partial}{\partial \hat{x}_1} \left(-2\lambda_3^3 + 3\lambda_3^2 - 7\lambda_3 \sum_{1 \le j < k \le 3, j \ne 3, k \ne 3} \lambda_j \lambda_k \right) \hat{\varphi}(\widehat{P}_3) \\
& + 27 \frac{\partial}{\partial \hat{x}_1}(\lambda_1\lambda_2\lambda_3) \hat{\varphi}(\widehat{P}_4) \\
& + \frac{\partial}{\partial \hat{x}_1} \left(\sum_{j=1}^{3} \lambda_3 \lambda_j (2\lambda_3 + \lambda_j - 1)(\widehat{P}_j^{(2)} - \widehat{P}_3^{(2)}) \right) \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{P}_3) \\
= & -7 \frac{\partial}{\partial \hat{x}_1}(\lambda_1\lambda_2\lambda_3) \hat{\varphi}(\widehat{P}_3) + 27 \frac{\partial}{\partial \hat{x}_1}(\lambda_1\lambda_2\lambda_3) \hat{\varphi}(\widehat{P}_4) \\
& + \frac{\partial}{\partial \hat{x}_1} \left\{ \lambda_3\lambda_1(2\lambda_3 + \lambda_1 - 1)(\widehat{P}_1^{(2)} - \widehat{P}_3^{(2)}) \right\} \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{P}_3)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\partial}{\partial \hat{x}_1} \left\{ \lambda_3 \lambda_2 (2\lambda_3 + \lambda_2 - 1)(\widehat{P}_2^{(2)} - \widehat{P}_3^{(2)}) \right\} \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{P}_3) \\
&= -7(-\hat{x}_2^2 - 2\hat{x}_1\hat{x}_2 - \hat{x}_2) + \frac{1}{3}(-\hat{x}_2^2 - 2\hat{x}_1\hat{x}_2 - \hat{x}_2) \\
&\quad + 8(\hat{x}_2 - 2\hat{x}_1\hat{x}_2 - \hat{x}_2^2) \not\equiv 0.
\end{aligned}
$$

Here, we have used

$$
\begin{aligned}
&\hat{\varphi}(\widehat{P}_i) = 0, \quad \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{P}_i) = 0, \quad i = 1, 2, \\
&\hat{\varphi}(\widehat{P}_3) = 1, \quad \hat{\varphi}(\widehat{P}_4) = \frac{1}{3^4}, \quad \frac{\partial \hat{\varphi}}{\partial \hat{x}_2}(\widehat{P}_3) = 4, \\
&\widehat{P}_1^{(2)} - \widehat{P}_3^{(2)} = -1, \quad \widehat{P}_2^{(2)} - \widehat{P}_3^{(2)} = -1.
\end{aligned}
$$

Therefore, Condition (5.9.3) is not satisfied. This implies that error estimate (5.5.3) on the reference element does not hold for Hermitian finite elements.

5.10 Concluding remarks

As concluding remarks, we present some topics related to the above sections.

5.10.1 One dimensional Lagrange interpolation

Let $\Omega := (0,1) \subset \mathbb{R}$. For $N \in \mathbb{N}$, let $\mathbb{T}_h = \{0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1\}$ be a mesh of $\overline{\Omega}$ such as

$$
\overline{\Omega} := \bigcup_{i=1}^{N} I_i, \quad \operatorname{int} I_i \cap \operatorname{int} I_j = \emptyset \quad \text{for } i \neq j,
$$

where $I_i := [x_i, x_{i+1}]$ for $0 \le i \le N$. We denote $h_i := x_{i+1} - x_i$ for $0 \le i \le N$. For $\widehat{T} := [0,1] \subset \mathbb{R}$ and $\widehat{P} := \mathcal{P}^k$ with $k \in \mathbb{N}$, let $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ be the reference Lagrange finite element, e.g., see [30]. The corresponding interpolation operator is defined as

$$
I_{\widehat{T}}^k : \mathcal{C}(\widehat{T}) \ni \hat{v} \mapsto I_{\widehat{T}}^k(\hat{v}) := \sum_{m=0}^{k} \hat{v}(\hat{\xi}_m) \widehat{\mathcal{L}}_m^k,
$$

where $\hat{\xi}_m := \frac{m}{k}$ and $\{\widehat{\mathcal{L}}_0^k, \ldots, \widehat{\mathcal{L}}_k^k\}$ is the Lagrange polynomials associated with the nodes $\{\hat{\xi}_0, \ldots, \hat{\xi}_k\}$. For $i \in \{0, \ldots, N\}$, we consider the affine transformations

$$
\Phi_i : \widehat{T} \ni t \mapsto x = x_i + t h_i \in I_i.
$$

For $\hat{v} \in \mathcal{C}(\widehat{T})$, we set $\hat{v} = v \circ \Phi_i$.

Theorem 5.10.1. *Let* $1 \le p \le \infty$ *and assume that there exists a nonnegative integer* k *such that*

$$\mathcal{P}^k = \widehat{P} \subset W^{k+1,p}(\widehat{T}) \subset \mathcal{C}(\widehat{T}).$$

Let ℓ $(0 \le \ell \le k)$ *be such that* $W^{\ell+1,p}(\widehat{T}) \subset \mathcal{C}(\widehat{T})$ *with continuous embedding. Furthermore, assume that* $\ell, m \in \mathbb{N} \cup \{0\}$ *and* $p, q \in [1, \infty]$ *such that* $0 \le m \le \ell + 1$ *and*

$$W^{\ell+1,p}(\widehat{T}) \hookrightarrow W^{m,q}(\widehat{T}).$$

It then holds that, for any $v \in W^{\ell+1,p}(I_i)$ *with* $\hat{v} = v \circ \Phi_i$,

$$|v - I_{I_i}^k v|_{W^{m,q}(I_i)} \le c h_i^{\frac{1}{q} - \frac{1}{p} + \ell + 1 - m} |v|_{W^{\ell+1,p}(I_i)}. \tag{5.10.1}$$

Proof. We only show the outline of the proof. Scaling argument yields

$$|v - I_{I_i}^k v|_{W^{m,q}(I_i)} = h_i^{-m+\frac{1}{q}} |\hat{v} - I_{\hat{T}} \hat{v}|_{W^{m,q}(\widehat{T})},$$
$$|\hat{v}|_{W^{\ell+1,p}(\widehat{T})} = h_i^{\ell+1-\frac{1}{p}} |v|_{W^{\ell+1,p}(I_i)}.$$

Using the Sobolev embedding theorem and the Bramble–Hilbert–type lemma, we have

$$|\hat{v} - I_{\hat{T}} \hat{v}|_{W^{m,q}(\widehat{T})} \le c |\hat{v}|_{W^{\ell+1,p}(\widehat{T})}.$$

Therefore, we obtain the estimate (5.10.1). □

Remark 5.10.2. The assumptions of Theorem 5.10.1 are standard; that is, there is no need to show the existences of functionals such as Theorem 5.6.1. Furthermore, the quantity $h_{\max}/h_{\min}$ that deteriorates the convergent order does not appear in (5.10.1).

Remark 5.10.3. If we set $x_j := \frac{j}{N+1}$, $j = 0, 1, \dots, N, N+1$, the mesh $\mathbb{T}_h$ is said to be the uniform mesh. If we set $x_j := g\left(\frac{j}{N+1}\right)$, $j = 1, \dots, N, N+1$ with a grading function g, the mesh $\mathbb{T}_h$ is said to be the graded mesh with respect to $x = 0$, see [14]. In particular, when one sets $g(y) := y^{\varepsilon}$ $(\varepsilon > 0)$, the mesh is called the radical mesh.

Remark 5.10.4 (Optimal order)**.** If $p = q$, it is possible to have the optimal error estimates even if the scale is different for each element. In one dimensional case, when $q > p$, the convergence order of the interpolation operator may deteriorate.

5.10.2 Effect of the quantity $|T|^{\frac{1}{q}-\frac{1}{p}}$ in the interpolation error estimates for $d = 2, 3$

We consider the effect of the factor $|T|^{\frac{1}{q}-\frac{1}{p}}$.

5.10.2.1 Case that $q > p$

When $q > p$, the factor may affect the convergence order. In particular, the interpolation error estimate may diverge on anisotropic mesh partitions.

Let $T \subset \mathbb{R}^2$ be the triangle with vertices $P_1 := (0,0)^T$, $P_2 := (t,0)^T$, $P_3 := (0,t^\varepsilon)^T$ for $0 < t \ll 1$, $\varepsilon \geq 1$, $t \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$. Then,

$$\frac{h_{\max}}{h_{\min}} = t^{1-\varepsilon}, \quad |T| = \frac{1}{2} t^{1+\varepsilon}.$$

Let $k = 1$, $\ell = 2$, $m = 1$, $q = 2$, and $p \in (1,2)$. Then, $W^{1,p}(T) \hookrightarrow L^2(T)$ and Theorem B lead to

$$|\varphi - I_T \varphi|_{H^1(T)} \leq c t^{-(1+\varepsilon)\frac{2-p}{2p}} \left(t \left| \frac{\partial \varphi}{\partial x_1} \right|_{W^{1,p}(T)} + t^\varepsilon \left| \frac{\partial \varphi}{\partial x_2} \right|_{W^{1,p}(T)} \right).$$

When $\varepsilon = 1$ (the case of the isotropic element), we get

$$|\varphi - I_T \varphi|_{H^1(T)} \leq c h_T^{\frac{2(p-1)}{p}} |\varphi|_{W^{2,p}(T)}, \quad \frac{2(p-1)}{p} > 0.$$

However, when $\varepsilon > 1$ (the case of the anisotropic element), the estimate may diverge as $t \to 0$. Therefore, if $q > p$, the convergence order of the interpolation operator may deteriorate.

We next set $m = 0$, $\ell = 2$, $q = \infty$, and $p = 2$. Let

$$\varphi(x, y) := x^2 + y^2.$$

Let $I_T^L : \mathcal{C}^0(T) \to \mathcal{P}^1$ be the local Lagrange interpolation operator. For any nodes P of T, because $I_T^L \varphi(P) = \varphi(P)$, we have

$$I_T^L \varphi(x, y) = tx + t^\varepsilon y.$$

It thus holds that

$$(\varphi - I_T^L \varphi)(x, y) = \left(x - \frac{t}{2} \right)^2 + \left(y - \frac{t^\varepsilon}{2} \right)^2 - \frac{1}{4}(t^2 + t^{2\varepsilon}).$$

We therefore have, because $H^2(T) \hookrightarrow L^\infty(T)$,

$$\|\varphi - I_T^L \varphi\|_{L^\infty(T)} = \frac{1}{4}(t^2 + t^{2\varepsilon}), \quad \sum_{|\gamma|=2} \mathscr{H}^\gamma \|\partial^\gamma \varphi\|_{L^2(T)} = 2|T|^{\frac{1}{2}}(t^2 + t^{2\varepsilon}),$$

and thus,

$$\frac{\|\varphi - I_T^L \varphi\|_{L^\infty(T)}}{|T|^{-\frac{1}{2}} \sum_{|\gamma|=2} \mathscr{H}^\gamma \|\partial^\gamma \varphi\|_{L^2(T)}} = \frac{1}{8}.$$

This example implies that the convergence order is not optimal, but the estimate converges on anisotropic meshes.

5.10.2.2 Case that $q < p$

We consider Theorem B. Let $I_T^L : \mathcal{C}^0(T) \to \mathcal{P}^k$ ($k \in \mathbb{N}$) be the local Lagrange interpolation operator. Let $\varphi \in W^{\ell,\infty}(T)$ be such that $\ell \in \mathbb{N}$, $2 \leq \ell \leq k+1$. It then holds that, for any $m \in \{0, \dots, \ell - 1\}$ and $q \in [1, \infty]$,

$$|\varphi - I_T^L \varphi|_{W^{m,q}(T)} \leq c|T|^{\frac{1}{q}} \left(\frac{H_T}{h_T} \right)^m \sum_{|\gamma|=\ell-m} \mathscr{H}^\gamma |\partial^\gamma (\varphi \circ \Phi_{T^s})|_{W^{m,\infty}(\Phi_{T^s}^{-1}(T))}. \tag{5.10.2}$$

The convergence order is therefore improved by $|T|^{\frac{1}{q}}$.

We do numerical tests to confirm this. Let $k = 1$ and

$$\varphi(x, y, z) := x^2 + \frac{1}{4}y^2 + z^2.$$

Let $t := \frac{1}{N}$, $N \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$, $1 < \varepsilon$. We compute the convergence order with respect to the H^1 norm defined by

$$Err_t^\varepsilon(H^1) := |\varphi - I_T^L \varphi|_{H^1(T)}.$$

The convergence indicator r is defined by

$$r = \frac{1}{\log(2)} \log \left(\frac{Err_t^\varepsilon(H^1)}{Err_{t/2}^\varepsilon(H^1)} \right).$$

(I) Let $T \subset \mathbb{R}^3$ be the simplex with vertices $P_1 := (0,0,0)^T$, $P_2 := (t,0,0)^T$, $P_3 := (0, t^\varepsilon, 0)^T$, and $P_4 := (0,0,t^\delta)^T$ ($1 < \delta \leq \varepsilon$), and $0 < t \ll 1$, $t \in \mathbb{R}$. We then have $h_1 = \sqrt{t^2 + t^{2\varepsilon}}$, $h_2 = t^\varepsilon$ and $h_3 := \sqrt{t^{2\varepsilon} + t^{2\delta}}$; i.e.,

$$\frac{h_{\max}}{h_{\min}} \leq ct^{1-\varepsilon}, \quad \frac{H_T}{h_T} \leq c.$$

From (5.10.2) with $m = 1$, $\ell = 2$, and $q = 2$, because $|T| \approx t^{1+\varepsilon+\delta}$, we have the estimate

$$|\varphi - I_T^L \varphi|_{H^1(T)} \le c h_T^{\frac{3+\varepsilon+\delta}{2}}.$$

Computational results are for the case that $\varepsilon = 3.0$ and $\delta = 2.0$ (Table 5.3).

Table 5.3: Error of the local interpolation operator ($\varepsilon = 3.0, \delta = 2.0$)

N	t	$Err_t^{3.0}(H^1)$	r
64	1.5625e-02	2.4336e-08	
128	7.8125e-03	1.5209e-09	4.00
256	3.9062e-03	9.5053e-11	4.00

(II) Let $T \subset \mathbb{R}^3$ be the simplex with vertices $P_1 := (0,0,0)^T$, $P_2 := (t,0,0)^T$, $P_3 := (t/2, t^\varepsilon, 0)^T$, and $P_4 := (0,0,t)^T$ $(1 < \varepsilon \le 6)$ and $0 < t \ll 1$, $t \in \mathbb{R}$. We then have $h_1 = t$, $h_2 = \sqrt{t^2/4 + t^{2\varepsilon}}$ and $h_3 := t$; i.e.,

$$\frac{h_{\max}}{h_{\min}} = \frac{t}{\sqrt{t^2/4 + t^{2\varepsilon}}} \le c, \quad \frac{H_T}{h_T} \le c t^{1-\varepsilon}.$$

From (5.10.2) with $m = 1$, $\ell = 2$, and $q = 2$, because $|T| \approx t^{2+\varepsilon}$, we have the estimate

$$|\varphi - I_T^L \varphi|_{H^1(T)} \le c h_T^{3-\frac{\varepsilon}{2}}.$$

Computational results are for the cases that $\varepsilon = 3.0, 6.0$ (Table 5.4).

Table 5.4: Error of the local interpolation operator ($\varepsilon = 3.0, 6.0$)

N	t	$Err_t^{3.0}(H^1)$	r	$Err_t^{6.0}(H^1)$	r
64	1.5625e-02	1.9934e-04		1.0206e-01	
128	7.8125e-03	7.0477e-05	1.50	1.0206e-01	0
256	3.9062e-03	2.4917e-05	1.50	1.0206e-01	0

5.10.3 What happens if violating the maximum-angle condition?

This subsection introduces two negative points by violating the maximum-angle condition. One is that it is practically disadvantageous. As an example, let $T^s \subset \mathbb{R}^2$ be the triangle with vertices $P_1 := (0,0)^T$, $P_2 := (t,0)^T$, $P_3 := (t/2, t^{\varepsilon})^T$ for $0 < t \ll 1$, $\varepsilon \geq 1$, $t \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$. From Theorem 5.6.1 with $k = 1$, $\ell = 2$, $m = 1$, $p = q = 2$, we have

$$|\varphi^s - I_{T^s}\varphi^s|_{H^1(T^s)} \leq ct^{2-\varepsilon} \left| \frac{\partial \varphi^s}{\partial x_1} \right|_{H^1(T^s)} + t \left| \frac{\partial \varphi^s}{\partial x_2} \right|_{H^1(T^s)}.$$

Even if one wants to reduce the step size in a specific direction (y-axis direction), the interpolation error may diverge as $t \to 0$ when $\varepsilon > 2$. This loses the benefits of using anisotropic meshes.

Another is that violating the condition makes it challenging to show mathematical validity in the finite element method. One of the answers can be found in [11], also see Section 11.8. That is, the maximum-angle condition is sufficient to do numerical calculations safely.

Chapter 6

L^2-orthogonal projection

In this chapter, we consider error estimates of the L^2-orthogonal projection, e.g., for standard argument, see [30, Section 1.4.3] and [31, Section 11.5.3].

6.1 Finite Element Generation on Standard Element

Let $k \in \mathbb{N}_0$. Let $\widehat{T} \subset \mathbb{R}^d$ be the reference element defined in Sections 3.1.1 and 3.1.2. The finite element of the L^2-orthogonal projection on the reference element is defined by the triple $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ as follows.

(I) $\widehat{P} := \mathcal{P}^k(\widehat{T})$;

(II) For any $\hat{p} \in \widehat{P}$, we define a linear form $\hat{\chi}_i$, $i \in \{1 : N^{(d,k)}\}$ as

$$\hat{\chi}_i(\hat{p}) := \frac{1}{|\widehat{T}|} \int_{\widehat{T}} \hat{p} \hat{\kappa}_i d\hat{x}, \tag{6.1.1}$$

where $\hat{\kappa}_i$ is a smooth function on $\widehat{T}$. We set $\widehat{\Sigma} := \{\hat{\chi}_i\}_{i=1}^{N^{(d,k)}}$. In particular, if $k = 0$, we set $\hat{\kappa}_1 = 1$, and a linear form $\hat{\chi}_1$ is defined as

$$\hat{\chi}_1(\hat{p}) := \frac{1}{|\widehat{T}|} \int_{\widehat{T}} \hat{p} d\hat{x}.$$

The nodal basis functions $\{\hat{\theta}_i\}_{i=1}^{N^{(d,k)}}$ associated with the degree of freedom by (6.1.1) is defined as

$$\hat{\chi}_i(\hat{\theta}_j) = \delta_{ij} \quad i, j \in \{1 : N^{(d,k)}\}. \tag{6.1.2}$$

If $k = 0$, we choose $\hat{\theta}_1 = 1$. Setting $V(\widehat{T}) := L^1(\widehat{T})$, the local operator $\Pi^k_{\widehat{T}}$ is defined as

$$\Pi^k_{\widehat{T}} : V(\widehat{T}) \ni \hat{\varphi} \mapsto \Pi^k_{\widehat{T}}\hat{\varphi} := \sum_{i=1}^{N^{(d,k)}} \hat{\chi}_i(\hat{\varphi})\hat{\theta}_i = \sum_{i=1}^{N^{(d,k)}} \left(\frac{1}{|\widehat{T}|} \int_{\widehat{T}} \hat{\varphi}\hat{\kappa}_i d\hat{x} \right) \hat{\theta}_i \in \widehat{P}, \tag{6.1.3}$$

By analogous argument in Section 5.1, we assume that finite elements $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$, $\{T^s, P^s, \Sigma^s\}$ and $\{T, P, \Sigma\}$ are constructed. For $i \in \{1 : N^{(d,k)}\}$, the local shape functions are $\tilde{\theta}_i = \psi^{-1}_{\widehat{T}}(\hat{\theta}_i)$, $\theta^s_i = \psi^{-1}_{\widetilde{T}}(\tilde{\theta}_i)$ and $\theta_i = \psi^{-1}_{T^s}(\theta^s_i)$, and the associated local interpolation operators are respectively defined by

$$\Pi^k_{\widetilde{T}} : V(\widetilde{T}) \ni \tilde{\varphi} \mapsto \Pi^k_{\widetilde{T}}\tilde{\varphi} := \sum_{i=1}^{N^{(d,k)}} \tilde{\chi}_i(\tilde{\varphi})\tilde{\theta}_i = \sum_{i=1}^{N^{(d,k)}} \left(\frac{1}{|\widetilde{T}|} \int_{\widetilde{T}} \tilde{\varphi}\tilde{\kappa}_i d\tilde{x} \right) \tilde{\theta}_i \in \widetilde{P}, \tag{6.1.4}$$

$$\Pi^k_{T^s} : V(T^s) \ni \varphi^s \mapsto \Pi^k_{T^s}\varphi^s := \sum_{i=1}^{N^{(d,k)}} \chi^s_i(\varphi^s)\theta^s_i = \sum_{i=1}^{N^{(d,k)}} \left(\frac{1}{|T^s|} \int_{T^s} \varphi^s\kappa^s_i dx \right) \theta^s_i \in P^s, \tag{6.1.5}$$

$$\Pi^k_{T} : V(T) \ni \varphi \mapsto \Pi^k_{T}\varphi := \sum_{i=1}^{N^{(d,k)}} \chi_i(\varphi)\theta_i = \sum_{i=1}^{N^{(d,k)}} \left(\frac{1}{|T|} \int_{T} \varphi\kappa_i dx \right) \theta_i \in P, \tag{6.1.6}$$

where $\tilde{\kappa}_i = \hat{\kappa}_i \circ \widehat{\Phi}^{-1}$, $\kappa^s_i = \tilde{\kappa}_i \circ \widetilde{\Phi}^{-1}$ and $\kappa_i = \kappa^s_i \circ \Phi^{-1}_{T^s}$ for $i \in \{1 : N^{(d,k)}\}$.

6.2 Local Error Estimates of the Projection

Lemma 6.2.1. *$\widehat{P}$ is invariant under $\Pi^k_{\widehat{T}}$, that is, $\Pi^k_{\widehat{T}}\hat{p} = \hat{p}$ for any $\hat{p} \in \widehat{P}$.*

Proof. Let $\hat{p} := \sum_{j=1}^{N^{(d,k)}} \alpha_j\hat{\theta}_j$. We then have

$$\Pi^k_{\widehat{T}}\hat{p} = \sum_{i,j=1}^{N^{(d,k)}} \alpha_j\hat{\chi}_i(\hat{\theta}_j)\hat{\theta}_i = \hat{p}.$$

□

Lemma 6.2.2. *Let $q \in [1, \infty]$. It holds that*

$$\|\Pi^k_{\widehat{T}}\hat{\varphi}\|_{L^q(\widehat{T})} \leq c\|\hat{\varphi}\|_{L^q(\widehat{T})} \quad \forall \hat{\varphi} \in L^1(\widehat{T}). \tag{6.2.1}$$

Proof. Using the definition of the projection and Hölder's inequality yields

$$\begin{aligned}\|\Pi_{\widehat{T}}^k \hat{\varphi}\|_{L^q(\widehat{T})} &\le \frac{1}{|\widehat{T}|} \left(\sum_{i=1}^{N^{(d,k)}} \|\hat{\kappa}_i\|_{L^\infty(\widehat{T})} \|\hat{\theta}_i\|_{L^q(\widehat{T})} \right) \|\hat{\varphi}\|_{L^1(\widehat{T})} \\ &\le |\widehat{T}|^{2-\frac{1}{p}} \left(\sum_{i=1}^{N^{(d,k)}} \|\hat{\kappa}_i\|_{L^\infty(\widehat{T})} \|\hat{\theta}_i\|_{L^q(\widehat{T})} \right) \|\hat{\varphi}\|_{L^p(\widehat{T})}.\end{aligned}$$

Because $|\widehat{T}| = \frac{1}{d!}$ and

$$\left(\sum_{i=1}^{N^{(d,k)}} \|\hat{\kappa}_i\|_{L^\infty(\widehat{T})} \|\hat{\theta}_i\|_{L^q(\widehat{T})} \right) \le c,$$

we conclude (6.2.1). □

The following theorem gives an anisotropic error estimate of the projection Π_T^k.

Theorem 6.2.3. *For $k \in \mathbb{N}_0$, let $\ell \in \mathbb{N}_0$ be such that $0 \le \ell \le k$. Let $p, q \in [1, \infty]$ be such that $W^{1,p}(T) \hookrightarrow L^q(T)$. It then holds that, for any $\hat{\varphi} \in W^{\ell+1,p}(\widehat{T})$ with $\varphi := \hat{\varphi} \circ \Phi^{-1}$,*

$$\|\Pi_T^k \varphi - \varphi\|_{L^q(T)} \le c|T|^{\frac{1}{q}-\frac{1}{p}} \sum_{|\epsilon|=\ell+1} h^\varepsilon \|\partial_r^\varepsilon \varphi\|_{L^p(T)}. \tag{6.2.2}$$

In particular, if Condition 3.3.1 is imposed, it holds that, for any $\hat{\varphi} \in W^{\ell+1,p}(\widehat{T})$ with $\varphi := \hat{\varphi} \circ \Phi^{-1}$,

$$\|\Pi_T^k \varphi - \varphi\|_{L^q(T)} \le c|T|^{\frac{1}{q}-\frac{1}{p}} \sum_{|\epsilon|=\ell+1} \mathscr{H}^\varepsilon \|\partial^\varepsilon (\varphi \circ \Phi_{T^s})\|_{L^p(\Phi_{T^s}^{-1}(T))}. \tag{6.2.3}$$

Proof. Using the scaling argument, we have

$$\|\Pi_T^k \varphi - \varphi\|_{L^q(T)} \le c\|\Pi_{T^s}^k \varphi^s - \varphi^s\|_{L^q(T^s)} = c|\det(\mathcal{A}^s)|^{\frac{1}{q}} \|\Pi_{\widehat{T}}^k \hat{\varphi} - \hat{\varphi}\|_{L^q(\widehat{T})}. \tag{6.2.4}$$

For any $\hat{\eta} \in \mathcal{P}^\ell \subset \widehat{P}$ with $0 \le \ell \le k$, from the triangle inequality and Lemma 6.2.1, that is, $\Pi_{\widehat{T}}^k \hat{\eta} = \hat{\eta}$, we have

$$\|\Pi_{\widehat{T}}^k \hat{\varphi} - \hat{\varphi}\|_{L^q(\widehat{T})} \le \|\Pi_{\widehat{T}}^k (\hat{\varphi} - \hat{\eta})\|_{L^q(\widehat{T})} + \|\hat{\eta} - \hat{\varphi}\|_{L^q(T)}. \tag{6.2.5}$$

Using (6.2.1) for the first term on the right-hand side of (6.2.5), we have

$$\|\Pi_{\widehat{T}}^k(\hat{\varphi} - \hat{\eta})\|_{L^q(\widehat{T})} \leq c\|\hat{\varphi} - \hat{\eta}\|_{L^q(\widehat{T})}. \tag{6.2.6}$$

Using the Sobolev embedding theorem for the second term on the right-hand side of (6.2.5) and (6.2.6), we obtain

$$\|\hat{\varphi} - \hat{\eta}\|_{L^q(\widehat{T})} \leq c\|\hat{\varphi} - \hat{\eta}\|_{W^{1,p}(\widehat{T})}. \tag{6.2.7}$$

Combining (6.2.4), (6.2.5), (6.2.6), and (6.2.7), we have

$$\|\Pi_{T^s}^k \varphi^s - \varphi^s\|_{L^q(T^s)} \leq C(\widehat{T})|\det(\mathcal{A}^s)|^{\frac{1}{q}} \inf_{\hat{\eta}\in\mathcal{P}^\ell} \|\hat{\varphi} - \hat{\eta}\|_{W^{1,p}(\widehat{T})}. \tag{6.2.8}$$

From the Blamble–Hilbert-type lemma (e.g., see Subsection 1.6.4), there exists a constant $\hat{\eta}_\beta \in \mathcal{P}^\ell$ such that, for any $\hat{\varphi} \in W^{\ell+1,p}(\widehat{T})$,

$$|\hat{\varphi} - \hat{\eta}_\beta|_{W^{t,p}(\widehat{T})} \leq C^{BH}(\widehat{T})|\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})}, \quad t = 0, 1. \tag{6.2.9}$$

If Condition 3.3.1 is not imposed, using (5.3.2) ($m = 0$) and (6.2.9), we then have

$$\begin{aligned} \|\hat{\varphi} - \hat{\eta}_\beta\|_{W^{1,p}(\widehat{T})} &\leq c|\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})} \\ &\leq c|\det(\mathcal{A}^s)|^{-\frac{1}{p}} \sum_{|\epsilon|=\ell+1} h^\varepsilon \|\partial_r^\varepsilon \varphi\|_{L^p(T)}. \end{aligned} \tag{6.2.10}$$

If Condition 3.3.1 is imposed, using (5.3.3) ($m = 0$) and (6.2.9), we then have

$$\begin{aligned} \|\hat{\varphi} - \hat{\eta}_\beta\|_{W^{1,p}(\widehat{T})} &\leq c|\hat{\varphi}|_{W^{\ell+1,p}(\widehat{T})} \\ &\leq c|\det(\mathcal{A}^s)|^{-\frac{1}{p}} \sum_{|\epsilon|=\ell+1} \mathscr{H}^\varepsilon \|\partial^\varepsilon \varphi^s\|_{L^p(T^s)}. \end{aligned} \tag{6.2.11}$$

Therefore, combining (6.2.8), (6.2.10), and (6.2.11) with (3.6.2), we have (6.2.2) and (6.2.3) using $T^s = \Phi_{T^s}^{-1}(T)$ and $\varphi^s = \varphi \circ \Phi_{T^s}$. □

We have the following stability estimate of the projection $\Pi_{T^s}^k$.

Lemma 6.2.4. *Let $k \in \mathbb{N}_0$ and let $p, q \in [1, \infty]$ be such that $L^p(T) \hookrightarrow L^q(T)$. It then holds that, for any $\varphi \in L^p(T)$,*

$$\|\Pi_T^k \varphi\|_{L^q(T)} \leq c|T|^{\frac{1}{q}-\frac{1}{p}} \|\varphi\|_{L^p(T)}. \tag{6.2.12}$$

Proof. Using the scaling argument, we have

$$\|\Pi_T^k \varphi\|_{L^q(T)} \le c\|\Pi_{T^s}^k \varphi^s\|_{L^q(T^s)} = c|\det(\mathcal{A}^s)|^{\frac{1}{q}} \|\Pi_{\widehat{T}}^k \hat{\varphi}\|_{L^q(\widehat{T})}. \tag{6.2.13}$$

The stability estimate (6.2.1) on the reference element and the Sobolev embedding theorem yield

$$\|\Pi_T^k \varphi\|_{L^q(T)} \le c|\det(\mathcal{A}^s)|^{\frac{1}{q}} \|\hat{\varphi}\|_{L^q(\widehat{T})} \le c|\det(\mathcal{A}^s)|^{\frac{1}{q}} \|\hat{\varphi}\|_{L^p(\widehat{T})}. \tag{6.2.14}$$

Using the scaling argument, we have

$$\|\hat{\varphi}\|_{L^p(\widehat{T})} = |\det((\mathcal{A}^s)^{-1})|^{\frac{1}{p}} \|\varphi^s\|_{L^p(T^s)} \le c|\det((\mathcal{A}^s)^{-1})|^{\frac{1}{p}} \|\varphi\|_{L^p(T)}. \tag{6.2.15}$$

Therefore, combining (6.2.14), and (6.2.15) with (3.6.2), we have (6.2.12). □

6.3 Global Error Estimates of the Projection

We define the standard piecewise constant space as

$$M_h^k := \left\{ \varphi_h \in L^1(\Omega);\ \varphi_h|_T \in \mathcal{P}^k(T)\ \forall T \in \mathbb{T}_h \right\}.$$

We also define the global interpolation Π_h^k to the space M_h^k as

$$(\Pi_h^k \varphi)|_T := \Pi_T^k(\varphi|_T) = \sum_{i=1}^{N(d,k)} \left(\frac{1}{|T|} \int_T \varphi|_T \kappa_i dx \right) \theta_i,$$

for any $T \in \mathbb{T}_h$ and any $\varphi \in L^1(\Omega)$, that is

$$\Pi_h^k : L^1(\Omega) \ni \varphi \mapsto \Pi_h^k \varphi := \sum_{T \in \mathbb{T}_h} \sum_{i=1}^{N(d,k)} \left(\frac{1}{|T|} \int_T \varphi|_T \kappa_i dx \right) \theta_i \in M_h^k.$$

Theorem 6.3.1. *Suppose that the assumptions of Theorem 6.2.3 are satisfied. Let Π_h^k be the corresponding global interpolation operator. It then holds that, for any $\varphi \in W^{\ell+1,p}(\Omega)$;*

(I) *if Condition 3.3.1 is not imposed,*

$$\|\varphi - \Pi_h^k \varphi\|_{L^q(\Omega)} \le c \sum_{T \in \mathbb{T}_h} |T|^{\frac{1}{q} - \frac{1}{p}} \sum_{|\varepsilon| = \ell+1} h^{\varepsilon} \|\partial_r^{\varepsilon} \varphi\|_{L^p(T)}. \tag{6.3.1}$$

(II) *if Condition 3.3.1 is imposed,*

$$\|\varphi - \Pi_h^k \varphi\|_{L^q(\Omega)} \le c \sum_{T \in \mathbb{T}_h} |T|^{\frac{1}{q} - \frac{1}{p}} \sum_{|\varepsilon| = \ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon} \varphi^s\|_{L^p(\Phi_{T^s}^{-1}(T))}. \tag{6.3.2}$$

Proof. This theorem is proved in the same argument as Theorem 5.7.1. □

6.4 Further Insight

In some case, the coefficient of (6.3.1) (or (6.3.2)) can be calculated explicitly. Furthermore, we do not use the concept of the standard elements in this section.

Theorem 6.4.1. *Let* $T \subset \mathbb{R}^d$ *be a simplex. Let* $\Pi_T^0 : L^2(T) \to \mathcal{P}^0(T)$ *be the local* L^2*-projection defined by*

$$\Pi_T^0 \varphi := \frac{1}{|T|} \int_T \varphi dx \quad \forall \varphi \in L^2(T).$$

It then holds that

$$\|\Pi_T^0 \varphi - \varphi\|_{L^2(T)} \leq \frac{h_T}{\pi} |\varphi|_{H^1(T)} \quad \forall \varphi \in H^1(T). \tag{6.4.1}$$

Proof. For any $\varphi \in H^1(T)$, we set $w := \Pi_T^0 \varphi - \varphi$. It then holds that

$$\int_T w dx = \int_T (\Pi_T^0 \varphi - \varphi) dx = \frac{1}{|T|} \int_T \varphi dx |T| - \int_T \varphi dx = 0.$$

Therefore, using the Poincaré inequality (1.6.12), we conclude (6.4.1). □

Chapter 7

Crouzeix–Raviart Interpolation

7.1 Finite Element Generation on Standard Element

Let $\widehat{T} \subset \mathbb{R}^d$ be the reference element defined in Sections 3.1.1 and 3.1.2. Let $\widehat{F}_i$ be the face of $\widehat{T}$ opposite to $\widehat{P}_i$. The Crouzeix–Raviart finite element on the reference element is defined by the triple $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ as follows.

(I) $\widehat{P} := \mathcal{P}^1(\widehat{T})$;

(II) $\widehat{\Sigma}$ is a set $\{\hat{\chi}_i\}_{1 \le i \le N^{(d,1)}}$ of $N^{(d,1)}$ linear forms $\{\hat{\chi}_i\}_{1 \le i \le N^{(d,1)}}$ with its components such that, for any $\hat{p} \in \widehat{P}$,

$$\hat{\chi}_i(\hat{p}) := \frac{1}{|\widehat{F}_i|} \int_{\widehat{F}_i} \hat{p} d\hat{s} \quad \forall i \in \{1 : d+1\}. \tag{7.1.1}$$

Using the barycentric coordinates $\{\hat{\lambda}_i\}_{i=1}^{d+1} : \mathbb{R}^d \to \mathbb{R}$ on the reference element, the nodal basis functions associated with the degrees of freedom by (7.1.1) are defined in (5.8.14):

$$\hat{\theta}_i(\hat{x}) := d \left(\frac{1}{d} - \hat{\lambda}_i(\hat{x}) \right) \quad \forall i \in \{1 : d+1\}. \tag{7.1.2}$$

It then holds that $\hat{\chi}_i(\hat{\theta}_j) = \delta_{ij}$ for any $i, j \in \{1 : d+1\}$. Setting $V(\widehat{T}) := W^{1,1}(\widehat{T})$, the local operator $I_{\widehat{T}}^{CR}$ is defined as

$$I_{\widehat{T}}^{CR} : V(\widehat{T}) \ni \hat{\varphi} \mapsto I_{\widehat{T}}^{CR} \hat{\varphi} := \sum_{i=1}^{d+1} \left(\frac{1}{|\widehat{F}_i|} \int_{\widehat{F}_i} \hat{\varphi} d\hat{s} \right) \hat{\theta}_i \in \widehat{P}. \tag{7.1.3}$$

By analogous argument in Section 5.1, we assume that the Crouzeix–Raviart finite elements $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$, $\{T^s, P^s, \Sigma^s\}$ and $\{T, P, \Sigma\}$ are constructed. The local shape functions are $\tilde{\theta}_i = \psi_{\widehat{T}}^{-1}(\hat{\theta}_i)$, $\theta_i^s = \psi_{\widetilde{T}}^{-1}(\tilde{\theta}_i)$ and $\theta_i = \psi_{T^s}^{-1}(\theta_i^s)$ for any $i \in \{1 : d+1\}$, and the associated local interpolation operators are respectively defined as

$$I_{\widetilde{T}}^{CR} : V(\widetilde{T}) \ni \tilde{\varphi} \mapsto I_{\widetilde{T}}^{CR} \tilde{\varphi} := \sum_{i=1}^{d+1} \left(\frac{1}{|\widetilde{F}_i|} \int_{\widetilde{F}_i} \tilde{\varphi} d\tilde{s} \right) \tilde{\theta}_i \in \widetilde{P}, \tag{7.1.4}$$

$$I_{T^s}^{CR} : V(T^s) \ni \varphi^s \mapsto I_{T^s}^{CR} \varphi^s := \sum_{i=1}^{d+1} \left(\frac{1}{|F_i^s|} \int_{F_i^s} \varphi^s ds \right) \theta_i^s \in P^s, \tag{7.1.5}$$

$$I_{T}^{CR} : V(T) \ni \varphi \mapsto I_{T}^{CR} \varphi := \sum_{i=1}^{d+1} \left(\frac{1}{|F_i|} \int_{F_i} \varphi ds \right) \theta_i \in P, \tag{7.1.6}$$

where $\{\widetilde{F}_i := \widehat{\Phi}(\widehat{F}_i)\}_{i \in \{1:d+1\}}$, $\{F_i^s := \widetilde{\Phi}(\widetilde{F}_i)\}_{i \in \{1:d+1\}}$ and $\{F_i := \Phi_{T^s}(F_i^s)\}_{i \in \{1:d+1\}}$.

7.2 Local Error Estimates

We present anisotropic Crouzeix–Raviart interpolation error estimates (also see [8]).

Theorem 7.2.1. *Let $p \in [1, \infty)$ and $q \in [1, \infty]$ be such that* (5.5.1) *with $\ell = 1$ and $m = 0$ holds. It then holds that*

$$|I_T^{CR} \varphi - \varphi|_{W^{1,q}(T)} \leq c |T|^{\frac{1}{q} - \frac{1}{p}} \sum_{i,j=1}^{d} h_i \left\| \frac{\partial^2 \varphi}{\partial r_i \partial x_j} \right\|_{L^p(T)} \quad \forall \varphi \in W^{2,p}(T), \tag{7.2.1}$$

$$\|I_T^{CR} \varphi - \varphi\|_{L^q(T)} \leq c |T|^{\frac{1}{q} - \frac{1}{p}} \sum_{i=1}^{d} h_i \left\| \frac{\partial \varphi}{\partial r_i} \right\|_{L^p(T)} \quad \forall \varphi \in W^{1,p}(T). \tag{7.2.2}$$

In particular, if Condition 3.3.1 is imposed, it holds that

$$|I_T^{CR} \varphi - \varphi|_{W^{1,q}(T)} \leq c |T|^{\frac{1}{q} - \frac{1}{p}} \sum_{i,j=1}^{d} \mathscr{H}_i \left\| \frac{\partial^2}{\partial x_i \partial x_j} (\varphi \circ \Phi_{T^s}) \right\|_{L^p(\Phi_{T^s}^{-1}(T))} \quad \forall \varphi \in W^{2,p}(T), \tag{7.2.3}$$

$$\|I_T^{CR} \varphi - \varphi\|_{L^q(T)} \leq c |T|^{\frac{1}{q} - \frac{1}{p}} \sum_{i=1}^{d} \mathscr{H}_i \left\| \frac{\partial}{\partial x_i} (\varphi \circ \Phi_{T^s}) \right\|_{L^p(\Phi_{T^s}^{-1}(T))} \quad \forall \varphi \in W^{1,p}(T). \tag{7.2.4}$$

Proof. For $\varphi \in W^{2,p}(T)$, Green's formula and the definition of the Crouzeix–Raviart interpolation imply that, because $I_T^{CR}\varphi \in \mathcal{P}^1$,

$$\begin{aligned}\frac{\partial}{\partial x_j}(I_T^{CR}\varphi) &= \frac{1}{|T|}\int_T \frac{\partial}{\partial x_j}(I_T^{CR}\varphi)dx = \frac{1}{|T|}\sum_{i=1}^{d+1} n_T^{(j)}\int_{F_i} I_T^{CR}\varphi ds \\ &= \frac{1}{|T|}\sum_{i=1}^{d+1} n_T^{(j)}\int_{F_i}\varphi ds = \frac{1}{|T|}\int_T \frac{\partial \varphi}{\partial x_j}dx = \Pi_T^0\left(\frac{\partial \varphi}{\partial x_j}\right)\end{aligned}$$

for $j = 1, \dots, d$, where $n_T^{(j)}$ denotes the j-th component of the outer unit normal vector n_T. Using (6.2.2), it then holds that

$$\begin{aligned}|I_T^{CR}\varphi - \varphi|^q_{W^{1,q}(T)} &= \sum_{j=1}^{d}\left\| \frac{\partial}{\partial x_j}(I_T^{CR}\varphi - \varphi)\right\|^q_{L^q(T)} \\ &= \sum_{j=1}^{d}\left\|\Pi_T^0\left(\frac{\partial \varphi}{\partial x_j}\right) - \left(\frac{\partial \varphi}{\partial x_j}\right)\right\|^q_{L^q(T)} \\ &\le c|T|^{\left(\frac{1}{q}-\frac{1}{p}\right)q}\sum_{j=1}^{d}\sum_{i=1}^{d} h_i^q\left\|\frac{\partial^2\varphi}{\partial r_i \partial x_j}\right\|^q_{L^p(T)},\end{aligned}$$

which leads to (7.2.1) using (1.6.1). Furthermore, if Condition 3.3.1 is imposed, using (6.2.3), the following holds:

$$\begin{aligned}|I_T^{CR}\varphi - \varphi|^q_{W^{1,q}(T)} &= \sum_{j=1}^{d}\left\| \frac{\partial}{\partial x_j}(I_T^{CR}\varphi - \varphi)\right\|^q_{L^q(T)} \\ &= \sum_{j=1}^{d}\left\|\Pi_T^0\left(\frac{\partial \varphi}{\partial x_j}\right) - \left(\frac{\partial \varphi}{\partial x_j}\right)\right\|^q_{L^q(T)} \\ &\le c|T|^{\left(\frac{1}{q}-\frac{1}{p}\right)q}\sum_{j=1}^{d}\sum_{i=1}^{d} \mathscr{H}_i^q\left\|\frac{\partial^2}{\partial x_i \partial x_j}(\varphi \circ \Phi_{T^s})\right\|^q_{L^p(\Phi_{T^s}^{-1}(T))},\end{aligned} \tag{7.2.5}$$

which leads to (7.2.3) using (1.6.1).

Let $\varphi \in W^{1,p}(T)$. Using the scaling argument, we have

$$\|I_T^{CR}\varphi - \varphi\|_{L^q(T)} \le c\|I_{T^s}^{CR}\varphi^s - \varphi^s\|_{L^q(T^s)} = |\det(\mathcal{A}^s)|^{\frac{1}{q}}\|I_{\widehat{T}}^{CR}\hat{\varphi} - \hat{\varphi}\|_{L^q(\widehat{T})}. \tag{7.2.6}$$

For any $\hat{\eta} \in \mathcal{P}^0$, from the triangle inequality and $I_{\widehat{T}}^{CR}\hat{\eta} = \hat{\eta}$, we have

$$\|I_{\widehat{T}}^{CR}\hat{\varphi} - \hat{\varphi}\|_{L^q(\widehat{T})} \le \|I_{\widehat{T}}^{CR}(\hat{\varphi} - \hat{\eta})\|_{L^q(\widehat{T})} + \|\hat{\eta} - \hat{\varphi}\|_{L^q(\widehat{T})}. \tag{7.2.7}$$

Using the definition of the Crouzeix–Raviart interpolation (7.1.3) and the trace theorem, we have

$$\|I_{\widehat{T}}^{CR}(\hat{\varphi} - \hat{\eta})\|_{L^q(\widehat{T})} \leq \sum_{i=1}^{d+1} \frac{1}{|\widehat{F}_i|} \int_{\widehat{F}_i} |\hat{\varphi} - \hat{\eta}| d\hat{s} \|\hat{\theta}_i\|_{L^q(\widehat{T})} \leq c \|\hat{\varphi} - \hat{\eta}\|_{W^{1,p}(\widehat{T})}. \tag{7.2.8}$$

Combining (6.2.7), (7.2.7), and (7.2.8) with (7.2.6), we have

$$\|I_T^{CR}\varphi - \varphi\|_{L^q(T)} \leq C(\widehat{T}) |\det(\mathcal{A}^s)|^{\frac{1}{q}} \inf_{\hat{\eta} \in \mathcal{P}^0} \|\hat{\varphi} - \hat{\eta}\|_{W^{1,p}(\widehat{T})}. \tag{7.2.9}$$

Therefore, combining (6.2.10), (6.2.11), and (7.2.9) with (3.6.2), we have (7.2.2) and (7.2.4). □

We have the following stability estimate of the projection I_T^{CR}.

Lemma 7.2.2. *It holds that, for any $\varphi \in W^{1,p}(T)$,*

$$|I_T^{CR}\varphi|_{W^{1,p}(T)} \leq c|\varphi|_{W^{1,p}(T)}. \tag{7.2.10}$$

Proof. Using the triangle inequality, we have

$$|I_T^{CR}\varphi|_{W^{1,p}(T)} \leq |I_T^{CR}\varphi - \varphi|_{W^{1,p}(T^s)} + |\varphi|_{W^{1,p}(T^s)}.$$

From (7.2.5) and (6.2.12), we have

$$\begin{aligned} |I_T^{CR}\varphi - \varphi|^p_{W^{1,p}(T)} &= \sum_{j=1}^{d} \left\| \Pi_T^0 \left(\frac{\partial \varphi}{\partial x_j} \right) - \left(\frac{\partial \varphi}{\partial x_j} \right) \right\|^p_{L^p(T)} \\ &\leq c \sum_{j=1}^{d} \left\| \frac{\partial \varphi}{\partial x_j} \right\|^p_{L^p(T)} = c|\varphi|^p_{W^{1,p}(T)}. \end{aligned}$$

Gathering the above inequalities leads to (7.2.10). □

7.3 Global Error Estimates

We define the Crouzeix–Raviart finite element space as

$$V_h^{CR} := \left\{ \varphi_h \in L^\infty(\Omega);\ \varphi_h|_T \in \mathcal{P}^1(T)\ \forall T \in \mathbb{T}_h,\ \int_F [\![\varphi_h]\!] ds = 0\ \forall F \in \mathcal{F}_h^i \right\}.$$

We also define the global interpolation $I_h^{CR} : W^{1,1}(\Omega) \to V_h^{CR}$ as follows (also see [30, pp. 44,45] and [32, pp. 177,178]).

$$(I_h^{CR}\varphi)|_T := I_T^{CR}(\varphi|_T) = \sum_{i=1}^{d+1} \left(\frac{1}{|F_i|} \int_{F_i} \varphi|_T ds \right) \theta_i\ \forall T \in \mathbb{T}_h,\ \forall \varphi \in W^{1,1}(\Omega),$$

Theorem 7.3.1. *Suppose that the assumptions of Theorem 7.2.1 are satisfied. Let I_h^{CR} be the corresponding global Crouzeix–Raviart interpolation operator. It then holds that*

(I) *if Condition 3.3.1 is not imposed,*

$$|\varphi - I_h^{CR}\varphi|_{W^{1,q}(\mathbb{T}_h)} \leq c \sum_{T \in \mathbb{T}_h} |T|^{\frac{1}{q}-\frac{1}{p}} \sum_{i,j=1}^{d} h_i \left\| \frac{\partial^2 \varphi}{\partial r_i \partial x_j} \right\|_{L^p(T)} \quad \forall \varphi \in W^{2,p}(\Omega), \tag{7.3.1}$$

$$\|\varphi - I_h^{CR}\varphi\|_{L^q(\Omega)} \leq c \sum_{T \in \mathbb{T}_h} |T|^{\frac{1}{q}-\frac{1}{p}} \sum_{i=1}^{d} h_i \left\| \frac{\partial \varphi}{\partial r_i} \right\|_{L^p(T)} \quad \forall \varphi \in W^{1,p}(\Omega). \tag{7.3.2}$$

(II) *if Condition 3.3.1 is imposed,*

$$\begin{aligned} &|\varphi - I_h^{CR}\varphi|_{W^{1,q}(\mathbb{T}_h)} \\ &\quad \leq c \sum_{T \in \mathbb{T}_h} |T|^{\frac{1}{q}-\frac{1}{p}} \sum_{i,j=1}^{d} \mathscr{H}_i \left\| \frac{\partial^2}{\partial x_i \partial x_j} (\varphi \circ \Phi_{T^s}) \right\|_{L^p(\Phi_{T^s}^{-1}(T))} \quad \forall \varphi \in W^{2,p}(\Omega), \end{aligned} \tag{7.3.3}$$

$$\begin{aligned} &\|\varphi - I_h^{CR}\varphi\|_{L^q(\Omega)} \\ &\quad \leq c \sum_{T \in \mathbb{T}_h} |T|^{\frac{1}{q}-\frac{1}{p}} \sum_{i=1}^{d} \mathscr{H}_i \left\| \frac{\partial}{\partial x_i} (\varphi \circ \Phi_{T^s}) \right\|_{L^p(\Phi_{T^s}^{-1}(T))} \quad \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{7.3.4}$$

Proof. This theorem is proved in the same argument as Theorem 5.7.1. □

Theorem 7.3.2. *Let I_h^{CR} be the corresponding global Crouzeix–Raviart interpolation operator. It then holds that*

$$|I_h^{CR}\varphi|_{W^{1,p}(\mathbb{T}_h)} \leq c|\varphi|_{W^{1,p}(\Omega)} \quad \forall \varphi \in W^{1,p}(\Omega). \tag{7.3.5}$$

Proof. Using (7.2.10) and (5.3.11)

$$|I_h^{CR}\varphi|^p_{W^{1,p}(\mathbb{T}_h)} = \sum_{T \in \mathbb{T}_h} |I_{T_0}^{CR}\varphi|^p_{W^{1,p}(T)} \leq c \sum_{T \in \mathbb{T}_h} |\varphi|^p_{W^{1,p}(T)} \leq c \sum_{T \in \mathbb{T}_h} |\varphi|^p_{W^{1,p}(T)},$$

which leads to (7.3.5). □

7.4 Further Insight

In some cases, the coefficient of (7.2.1) (or (7.2.3)) can be calculated explicitly. Furthermore, we do not use the concept of the standard elements in this section.

Theorem 7.4.1. *Let $T \subset \mathbb{R}^d$ be a simplex. Let $I_T^{CR} : H^1(T) \to \mathcal{P}^1(T)$ be the local Crouzeix–Raviart interpolation operator defined as*

$$I_T^{CR} : H^1(T) \ni \varphi \mapsto I_T^{CR}\varphi := \sum_{i=1}^{d+1} \left(\frac{1}{|F_i|} \int_{F_i} \varphi ds \right) \theta_i \in \mathcal{P}^1.$$

It then holds that

$$|I_T^{CR}\varphi - \varphi|_{H^1(T)} \leq \frac{h_T}{\pi} |\varphi|_{H^2(T)} \quad \forall \varphi \in H^2(T). \tag{7.4.1}$$

Proof. By the same argument as the proof of Theorem 7.2.1, we have, using (6.4.1).

$$\begin{aligned} |I_T^{CR}\varphi - \varphi|_{H^1(T)}^2 &= \sum_{j=1}^{d} \left\| \frac{\partial}{\partial x_j} (I_T^{CR}\varphi - \varphi) \right\|_{L^2(T)}^2 \\ &= \sum_{j=1}^{d} \left\| \Pi_T^0 \left(\frac{\partial \varphi}{\partial x_j} \right) - \left(\frac{\partial \varphi}{\partial x_j} \right) \right\|_{L^2(T)}^2 \\ &\leq \left(\frac{h_T}{\pi} \right)^2 \sum_{i,j=1}^{d} \left\| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\|_{L^2(T)}^2 \\ &= \left(\frac{h_T}{\pi} \right)^2 |\varphi|_{H^2(T)}^2, \end{aligned}$$

which conclude (7.4.1). □

Chapter 8

Morley Interpolation

8.1 Finite Element Generation on Standard Element

Any dimensional Morley finite element is introduced in [88].

Let $\widehat{T} \subset \mathbb{R}^d$, $d \in \{2,3\}$, be the reference element defined in Sections 3.1.1 and 3.1.2. Let $\widehat{F}_i$, $1 \le i \le d+1$, be the $(d-1)$-dimensional subsimplex of $\widehat{T}$ without $\widehat{P}_i$ and $\widehat{S}_{i,j}$, $1 \le i < j \le d+1$, the $(d-2)$-dimensional subsimplex of $\widehat{T}$ without $\widehat{P}_i$ and $\widehat{P}_j$. The d-dimensional Morley finite element on the reference element is defined by the triple $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ as follows.

(I) $\widehat{P} := \mathcal{P}^2(\widehat{T})$;

(II) $\widehat{\Sigma}$ is a set $\{\hat{\chi}_i\}_{1 \le i \le N^{(d,2)}}$ of $N^{(d,2)}$ linear forms $\{\hat{\chi}_{i,j}^{(1)}\}_{1 \le i < j \le d+1} \cup \{\hat{\chi}_i^{(2)}\}_{1 \le i \le d+1}$ with its components such that, for any $\hat{p} \in \widehat{P}$,

$$\hat{\chi}_{i,j}^{(1)}(\hat{p}) := \frac{1}{|\widehat{S}_{i,j}|} \int_{\widehat{S}_{i,j}} \hat{p} d\hat{s}, \quad 1 \le i < j \le d+1, \tag{8.1.1a}$$

$$\hat{\chi}_i^{(2)}(\hat{p}) := \frac{1}{|\widehat{F}_i|} \int_{\widehat{F}_i} \frac{\partial \hat{p}}{\partial \hat{\nu}} d\hat{s}, \quad 1 \le i \le d+1, \tag{8.1.1b}$$

where $\frac{\partial}{\partial \hat{\nu}} = \nu_{\widehat{T}} \cdot \nabla$, and $\nu_{\widehat{T}}$ is the unit outer normal to $\widehat{F}_i \subset \partial \widehat{T}$. For $d = 2$, $\hat{\chi}_{i,j}^{(1)}(\hat{p})$ is interpreted as

$$\hat{\chi}_{i,j}^{(1)}(\hat{p}) = \hat{p}(\widehat{P}_k), \quad k = 1, 2, 3, \quad k \neq i, j.$$

It is shown that for the Morley finite element, $\widehat{\Sigma}$ is unisolvent (see [88, Lemma 2]). The nodal basis functions associated with the degrees of freedom by

(8.1.1) are defined by

$$\hat{\theta}_{i,j}^{(1)} := 1 - (d-1)(\hat{\lambda}_i + \hat{\lambda}_j) + d(d-1)\hat{\lambda}_i\hat{\lambda}_j - (d-1)(\nabla\hat{\lambda}_i)^T\nabla\hat{\lambda}_j \sum_{k=i,j} \frac{\hat{\lambda}_k(d\hat{\lambda}_k - 2)}{2|\nabla\hat{\lambda}_k|_E^2}, \quad 1 \le i < j \le d+1, \tag{8.1.2a}$$

$$\hat{\theta}_i^{(2)} := \frac{\hat{\lambda}_i(d\hat{\lambda}_i - 2)}{2|\nabla\hat{\lambda}_i|_E}, \quad 1 \le i \le d+1, \tag{8.1.2b}$$

where $\hat{\lambda}_i : \mathbb{R}^d \to \mathbb{R}$, $i = 1, \ldots, d+1$ is the barycentric coordinates on the reference element. It then proven in [88, Theorem 1] that, for $1 \le i < j \le d+1$,

$$\hat{\chi}_{k,\ell}^{(1)}(\hat{\theta}_{i,j}^{(1)}) = \delta_{ik}\delta_{j\ell}, \quad 1 \le k < \ell \le d+1, \quad \hat{\chi}_k^{(2)}(\hat{\theta}_{i,j}^{(1)}) = 0, \quad 1 \le k \le d+1, \tag{8.1.3}$$

and, for $1 \le i \le d+1$,

$$\hat{\chi}_{k,\ell}^{(1)}(\hat{\theta}_i^{(2)}) = 0, \quad 1 \le k < \ell \le d+1, \quad \hat{\chi}_k^{(2)}(\hat{\theta}_i^{(2)}) = \delta_{ik}, \quad 1 \le k \le d+1. \tag{8.1.4}$$

The local interpolation operator $I_{\widehat{T}}^M$ is defined by

$$I_{\widehat{T}}^M : W^{2,1}(\widehat{T}) \ni \hat{\varphi} \mapsto I_{\widehat{T}}^M \hat{\varphi} \in \widehat{P}, \tag{8.1.5}$$

with

$$I_{\widehat{T}}^M \hat{\varphi} := \sum_{1 \le i < j \le d+1} \hat{\chi}_{i,j}^{(1)}(\hat{\varphi})\hat{\theta}_{i,j}^{(1)} + \sum_{1 \le i \le d+1} \hat{\chi}_i^{(2)}(\hat{\varphi})\hat{\theta}_i^{(2)}. \tag{8.1.6}$$

It then holds that $I_{\widehat{T}}^M \hat{p} = \hat{p}$ for any $\hat{p} \in \widehat{P}$ and, for any $\hat{\varphi} \in W^{2,1}(\widehat{T})$,

$$\hat{\chi}_{i,j}^{(1)}(I_{\widehat{K}}^M \hat{\varphi}) = \hat{\chi}_{i,j}^{(1)}(\hat{\varphi}), \quad 1 \le i < j \le d+1, \tag{8.1.7a}$$

$$\hat{\chi}_i^{(2)}(I_{\widehat{K}}^M \hat{\varphi}) = \hat{\chi}_i^{(2)}(\hat{\varphi}), \quad 1 \le i \le d+1. \tag{8.1.7b}$$

By analogous argument in Section 5.1, the Morley finite elements $\{\widetilde{T}, \widetilde{P}, \widetilde{\Sigma}\}$ and $\{T^s, P^s, \Sigma^s\}$ are constructed. The Morley finite element $\{T^s, P^s, \Sigma^s\}$ is thus defined as

$$\begin{cases} T^s = \Phi^s(\widehat{T}); \\ P^s = \{(\psi^s)^{-1}(\hat{p});\ \hat{p} \in \widehat{P}\}; \\ \Sigma^s = \{\{\chi_i^s\}_{1 \le i \le N^{(d,2)}};\ \chi_i^s = \hat{\chi}_i(\psi^s(p^s)), \forall p^s \in P^s,\ \hat{\chi}_i \in \widehat{\Sigma}\}. \end{cases}$$

The local shape functions are

$$\theta_{i,j}^{(s,1)} = (\psi^s)^{-1}(\hat{\theta}_{i,j}^{(1)}), \quad 1 \le i < j \le d+1, \quad \theta_i^{(s,2)} = (\psi^s)^{-1}(\hat{\theta}_i^{(2)}), \quad 1 \le i \le d+1.$$

The associated local Morley interpolation operator is defined by

$$I_{T^s}^M : W^{2,1}(T^s) \ni \varphi^s \mapsto I_{T^s}^M \varphi^s \in P^s, \tag{8.1.8}$$

with, for any $\varphi^s \in W^{2,1}(T^s)$,

$$\chi_{i,j}^{(s,1)}(I_{T^s}^M \varphi^s) = \chi_{i,j}^{(s,1)}(\varphi^s), \quad 1 \le i < j \le d+1, \tag{8.1.9a}$$
$$\chi_i^{(s,2)}(I_{T^s}^M \varphi^s) = \chi_i^{(s,2)}(\varphi^s), \quad 1 \le i \le d+1, \tag{8.1.9b}$$

where $\{F_i^s := \Phi^s(\widehat{F}_i)\}_{i \in \{1:d+1\}}$ and $\{S_{i,j}^s := \Phi^s(\widehat{S}_{i,j})\}_{1 \le i < j \le d+1}$.

Furthermore, the Morley finite element $\{T, P, \Sigma\}$ is thus defined as

$$\begin{cases} T = \Phi_{T^s}(T^s); \\ P = \{(\psi_{T^s})^{-1}(p^s);\ p^s \in P^s\}; \\ \Sigma = \{\{\chi_i\}_{1 \le i \le N^{(d,2)}};\ \chi_i = \chi_i^s(\psi_{T^s}(p)), \forall p \in P,\ \chi_i^s \in \Sigma^s\}. \end{cases}$$

The local shape functions are

$$\theta_{i,j}^{(1)} = \psi_{T^s}^{-1}(\theta_{i,j}^{(s,1)}), \quad 1 \le i < j \le d+1, \quad \theta_i^{(2)} = \psi_{T^s}^{-1}(\theta_i^{(s,2)}), \quad 1 \le i \le d+1.$$

The associated local Morley interpolation operator is defined by

$$I_T^M : W^{2,1}(T) \ni \varphi \mapsto I_T^M \varphi \in P, \tag{8.1.10}$$

with, for any $\varphi \in W^{2,1}(T)$,

$$\chi_{i,j}^{(1)}(I_T^M \varphi) = \chi_{i,j}^{(1)}(\varphi), \quad 1 \le i < j \le d+1, \tag{8.1.11a}$$
$$\chi_i^{(2)}(I_T^M \varphi) = \chi_i^{(2)}(\varphi), \quad 1 \le i \le d+1, \tag{8.1.11b}$$

where $\{F_i := \Phi_{T^s}(F_i^s)\}_{i \in \{1:d+1\}}$ and $\{S_{i,j} := \Phi_{T^s}(S_{i,j}^s)\}_{1 \le i < j \le d+1}$.

8.2 Local Error Estimates

This section gives the Morley interpolation error estimates in $\mathbb{R}^d$.

Lemma 8.2.1. *Let $T \subset \mathbb{R}^d$ be a simplex. Let F_k, $k = 1, \ldots, d+1$, be a face ($(d-1)$-dimensional subsimplex) of T. Denote by $\nu = (\nu_1, \ldots, \nu_d)^T$ the unit normal of F_k, by $S_1, \ldots, S_d$ all $(d-2)$-dimensional subsimplexes of F_k, and* *by* $n^{(\ell)}$ *the unit out normal of* S_ℓ*. For any* $v \in \mathcal{C}^1(T)$ *and any constant vector* $\xi \in \mathbb{R}^d$*, setting* $\tau := \xi - (\xi \cdot \nu)\nu$*, it holds that* $\tau \cdot \nu = 0$ *and*

$$\int_{F_k} (\xi \cdot \nabla) v = \sum_{\ell=1}^{d} \tau \cdot n^{(\ell)} \int_{S_\ell} v + (\xi \cdot \nu) \int_{F_k} \frac{\partial v}{\partial \nu} \tag{8.2.1}$$

Note that the $(d-2)$*-dimensional subsimplexes* $S_1, \dots, S_d$ *are viewed as the boundary of an* $(d-1)$*-simplex in* $(d-1)$*-dimensional space.*

Proof. We follow [88, Lemma 1].

Let $v \in \mathcal{C}^1(T)$ and $\xi \in \mathbb{R}^d$. We easily have

$$\tau \cdot \nu = \xi \cdot \nu - (\xi \cdot \nu)\nu \cdot \nu = 0.$$

Therefore, τ is the tangent vector of F_k. Furthermore, it follows that, using the Gauss–Green formula,

$$\begin{aligned} \int_{F_k} (\xi \cdot \nabla) v &= \int_{F_k} \{(\tau \cdot \nabla) v + (\xi \cdot \nu)(\nu \cdot \nabla) v\} \\ &= \sum_{\ell=1}^{d} \int_{S_\ell} (\tau \cdot n^{(\ell)}) v + \int_{F_k} (\xi \cdot \nu) \frac{\partial v}{\partial \nu}. \end{aligned}$$

Because the quantities $\tau \cdot n^{(\ell)}$ and $\xi \cdot \nu$ are constants on S_ℓ and F_k, respectively, we have the desired result (8.2.1). □

Corollary 8.2.2. *We keep the notations of Lemma 8.2.1. Let* $v \in \mathcal{C}^1(T)$ *be such that*

$$\int_{S_\ell} v = 0, \quad \int_{F_k} \frac{\partial v}{\partial \nu} = 0, \tag{8.2.2}$$

for any $\ell = 1, \dots, d$ *and* $k = 1, \dots, d+1$*. It then holds that*

$$\int_{F_k} \frac{\partial v}{\partial x_i} = 0, \quad i = 1, \dots, d. \tag{8.2.3}$$

Proof. Let $e_1, \dots, e_d \in \mathbb{R}^d$ be the canonical basis. Setting $\xi := e_i$ in (8.2.1), we have the desired result (8.2.3) under the assumptions (8.2.2). □

Theorem 8.2.3. *Let* $p \in [1, \infty)$ *and* $q \in [1, \infty]$ *be such that* (5.5.1) *with* $\ell = 1$ *and* $m = 0$ *holds. It then holds that, for any* $\varphi \in W^{3,p}(T) \cap \mathcal{C}^1(T)$*,*

$$|I_T^M \varphi - \varphi|_{W^{2,q}(T)} \leq c |T|^{\frac{1}{q} - \frac{1}{p}} \sum_{i,j,k=1}^{d} h_i \left\| \frac{\partial^3 \varphi}{\partial r_i \partial x_j \partial x_k} \right\|_{L^p(T)}. \tag{8.2.4}$$

In particular, if Condition 3.3.1 is imposed, it holds that, for any $\varphi \in W^{3,p}(T) \cap \mathcal{C}^1(T)$,

$$|I_T^M \varphi - \varphi|_{W^{2,q}(T)} \leq c|T|^{\frac{1}{q}-\frac{1}{p}} \sum_{i,j,k=1}^{d} \mathscr{H}_i \left\| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} (\varphi \circ \Phi_{T^s}) \right\|_{L^p(\Phi_{T^s}^{-1}(T))} . \tag{8.2.5}$$

Proof. Let $\varphi \in W^{3,p}(T) \cap \mathcal{C}^1(T)$. We set $v := I_T^M \varphi - \varphi$. From the definition of the Morley interpolation operator (8.1.8), we have

$$\int_{S_{i,j}} v ds = 0, \quad 1 \leq i < j \leq d+1, \quad \int_{F_i} \frac{\partial v}{\partial \nu} ds = 0, \quad 1 \leq i \leq d+1.$$

Therefore, Corollary 8.2.2 yields

$$\int_{F_i} \frac{\partial v}{\partial x_k} = 0, \quad i = 1, \ldots, d+1, \quad k = 1, \ldots, d. \tag{8.2.6}$$

From (8.2.6), it follows that, for $1 \leq j, k \leq d$,

$$\int_T \frac{\partial^2 v}{\partial x_j \partial x_k} dx = \sum_{i=1}^{d+1} \nu_{T,j} \int_{F_i} \frac{\partial v}{\partial x_k} ds = 0,$$

which leads to

$$\frac{\partial^2}{\partial x_j \partial x_k} (I_T^M \varphi) = \frac{1}{|T|} \int_T \frac{\partial^2 \varphi}{\partial x_j \partial x_k} dx = \Pi_T^0 \left(\frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right),$$

because $\frac{\partial^2}{\partial x_j \partial x_k} (I_T^M \varphi) \in \mathcal{P}^0(T)$ for $1 \leq j, k \leq d$.

Using (6.2.2), it holds that

$$\begin{aligned} |I_T^M \varphi - \varphi|_{W^{2,q}(T)}^q &= \sum_{j,k=1}^{d} \left\| \frac{\partial^2}{\partial x_j \partial x_k} (I_T^M \varphi - \varphi) \right\|_{L^q(T)}^q \\ &= \sum_{j,k=1}^{d} \left\| \Pi_T^0 \left(\frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right) - \left(\frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right) \right\|_{L^q(T)}^q \\ &\leq c|T|^{\left(\frac{1}{q}-\frac{1}{p}\right)q} \sum_{j,k=1}^{d} \sum_{i=1}^{d} h_i^q \left\| \frac{\partial^3 \varphi}{\partial r_i \partial x_j \partial x_k} \right\|_{L^p(T)}^q, \end{aligned}$$

which leads to (8.2.4) using the Jensen-type inequality (1.6.1).

Furthermore, if Condition 3.3.1 is imposed, using (6.2.3), the following holds:

$$
\begin{aligned}
|I_T^M \varphi - \varphi|^q_{W^{2,q}(T)} &= \sum_{j,k=1}^{d} \left\| \Pi_T^0 \left(\frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right) - \left(\frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right) \right\|^q_{L^q(T)} \\
&\leq c |T|^{\left(\frac{1}{q} - \frac{1}{p}\right) q} \sum_{j,k=1}^{d} \sum_{i=1}^{d} \mathscr{H}_i^q \left\| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} (\varphi \circ \Phi_{T^s}) \right\|^q_{L^p(\Phi_{T^s}^{-1}(T))},
\end{aligned}
$$

which leads to (8.2.5) using the Jensen-type inequality (1.6.1). □

8.3 Global Error Estimates

We define the Morley finite element space by

$$
\begin{aligned}
V_h^M := \Bigg\{ & \varphi_h \in L^2(\Omega);\ \varphi_h|_T \in \mathcal{P}^2(T)\ \forall T \in \mathbb{T}_h,\ \int_F \left[\left[\frac{\partial \varphi_h}{\partial \nu} \right]\right]_F ds = 0, \\
& \forall F \in \mathcal{F}_h^i,\ \text{the integral average of } \varphi_h \text{ over each } (d-2)\text{-dimensional} \\
& \text{subsimplex of } T \in \mathbb{T}_h \text{ is continuous} \Bigg\},
\end{aligned}
$$

$$
V_{0h}^M := \{ \varphi_h \in V_h^M;\ \text{degrees of freedom of } \varphi_h \text{ in (8.1.1) vanish on } \partial\Omega \}.
$$

We also define the global interpolation $I_h^M : W^{2,1}(\Omega) \to V_h^M$ (or $I_h^M : W_0^{2,1}(\Omega) \to V_{0h}^M$) as follows.

$$
(I_h^M \varphi)|_T := I_T^M (\varphi|_T) \quad \forall T \in \mathbb{T}_h,\ \forall \varphi \in W^{2,1}(\Omega).
$$

Theorem 8.3.1. *Suppose that the assumptions of Theorem 8.2.3 are satisfied. Let I_h^M be the corresponding global Morley interpolation operator. It then holds that, for any $\varphi \in W^{3,p}(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$,*

(I) *if Condition 3.3.1 is not imposed,*

$$
|I_T^M \varphi - \varphi|_{W^{2,q}(\mathbb{T}_h)} \leq c \sum_{T \in \mathbb{T}_h} |T|^{\frac{1}{q} - \frac{1}{p}} \sum_{i,j,k=1}^{d} h_i \left\| \frac{\partial^3 \varphi}{\partial r_i \partial x_j \partial x_k} \right\|_{L^p(T)}. \quad (8.3.1)
$$

(II) *if Condition 3.3.1 is imposed,*

$$
|I_T^M \varphi - \varphi|_{W^{2,q}(\mathbb{T}_h)} \leq c \sum_{T \in \mathbb{T}_h} |T|^{\frac{1}{q} - \frac{1}{p}} \sum_{i,j,k=1}^{d} \mathscr{H}_i \left\| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} (\varphi \circ \Phi_{T^s}) \right\|_{L^p(\Phi_{T^s}^{-1}(T))}. \quad (8.3.2)
$$

Proof. This theorem is proved in the same argument as Theorem 5.7.1. □

8.4 Further Insight

In some case, the coefficient of (8.2.4) (or (8.2.5)) can be calculated explicitly. Furthermore, we do not use the concept of the standard elements in this section.

Theorem 8.4.1. *Let $T \subset \mathbb{R}^d$ be a simplex. Let $I_T^M : H^2(T) \to \mathcal{P}^2(T)$ be the local Morley interpolation operator defined as*

$$I_T^M : H^2(T) \ni \varphi \mapsto I_T^M \varphi \in \mathcal{P}^2(T),$$

with

$$\begin{aligned} \chi_{i,j}^{(1)}(I_T^M \varphi) &= \chi_{i,j}^{(1)}(\varphi), \quad 1 \le i < j \le d+1, \\ \chi_i^{(2)}(I_T^M \varphi) &= \chi_i^{(2)}(\varphi), \quad 1 \le i \le d+1. \end{aligned}$$

for any $\varphi \in H^2(T)$. It then holds that

$$|I_T^M \varphi - \varphi|_{H^2(T)} \le \frac{h_T}{\pi} |\varphi|_{H^3(T)} \quad \forall \varphi \in H^3(T). \tag{8.4.2}$$

Proof. By the same argument as the proof of Theorem 8.2.3, we have, using (6.4.1).

$$\begin{aligned} |I_T^M \varphi - \varphi|^2_{H^2(T)} &= \sum_{j,k=1}^{d} \left\| \frac{\partial^2}{\partial x_j \partial x_k} (I_T^M \varphi - \varphi) \right\|^2_{L^2(T)} \\ &= \sum_{j,k=1}^{d} \left\| \Pi_T^0 \left(\frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right) - \left(\frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right) \right\|^2_{L^2(T)} \\ &\le \left(\frac{h_T}{\pi} \right)^2 \sum_{j,k=1}^{d} \left| \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right|^2_{H^1(T)} = \left(\frac{h_T}{\pi} \right)^2 |\varphi|^2_{H^3(T)}, \end{aligned}$$

which conclude (8.4.2). □

Chapter 9

Raviart–Thomas Interpolation

9.1 Finite Element Generation on Standard Element

Let $\widehat{T} \subset \mathbb{R}^d$ be the reference element defined in Sections 3.1.1 and 3.1.2. The Raviart–Thomas finite element on the reference element is defined by the triple $\{\widehat{T}, \widehat{P}, \widehat{\Sigma}\}$ as follows:

(I) $\widehat{P} := RT^k(\widehat{T})$;

(II) $\widehat{\Sigma}$ is a set $\{\hat{\chi}_i\}_{1 \leq i \leq N^{(RT)}}$ of $N^{(RT)}$ linear forms with its components such that, for any $\hat{p} \in \widehat{P}$,

$$\int_{\widehat{F}} \hat{p} \cdot \hat{n}_{\widehat{F}} \hat{q}_k d\hat{s}, \quad \forall \hat{q}_k \in \mathcal{P}^k(\widehat{F}), \quad \widehat{F} \subset \partial \widehat{T}, \tag{9.1.1}$$

$$\int_{\widehat{T}} \hat{p} \cdot \hat{q}_{k-1} d\hat{x}, \quad \forall \hat{q}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T})^d, \tag{9.1.2}$$

shere $\hat{n}_{\widehat{F}}$ denotes the outer unit normal vector of $\widehat{T}$ on the face $\widehat{F}$. Note that for $k = 0$, the local degrees of freedom of type (9.1.2) are violated.

For the simplicial Raviart–Thomas element in $\mathbb{R}^d$, it holds that

$$\dim RT^k(\widehat{T}) = \begin{cases} (k+1)(k+3) & \text{if } d = 2, \\ \frac{1}{2}(k+1)(k+2)(k+4) & \text{if } d = 3. \end{cases} \tag{9.1.3}$$

The Raviart–Thomas finite element with the local degrees of freedom with respect to (9.1.1) and (9.1.2) is unisolvent; for example, see [18, Proposition 2.3.4].

We set the domain of the local Raviart–Thomas interpolation to $V(\widehat{T}) := W^{s,p}(\widehat{T})^d$ with $sp > 1$, $p \in (1, \infty)$ or $s = 1$, $p = 1$; for example, see [31, p. 188]. The local Raviart–Thomas interpolation $I_{\widehat{T}}^{RT^k} : V(\widehat{T}) \to \widehat{P}$ is then defined as follows: For any $\hat{v} \in V(\widehat{T})$,

$$\int_{\widehat{F}} (I_{\widehat{T}}^{RT^k} \hat{v} - \hat{v}) \cdot \hat{n}_{\widehat{F}} \hat{p}_k d\hat{s} = 0 \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{F}), \quad \widehat{F} \subset \partial \widehat{T}, \tag{9.1.4}$$

and if $k \geq 1$,

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k} \hat{v} - \hat{v}) \cdot \hat{q}_{k-1} d\hat{x} = 0 \quad \forall \hat{q}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T})^d. \tag{9.1.5}$$

In particular, when $k = 0$, the degrees of freedom by (9.1.1) are describe as

$$\hat{\chi}_i(\hat{p}) := \int_{\widehat{F}_i} \hat{p} \cdot \hat{n}_{\widehat{F}_i} d\hat{s} \quad \forall \hat{p} \in RT^0(\widehat{T}), \quad \forall i \in \{1 : d+1\}. \tag{9.1.6}$$

The nodal basis functions associated with the degrees of freedom by (9.1.6) are defined as, for any $i \in \{1 : d+1\}$,

$$\hat{\theta}_i := \frac{1}{d|\widehat{T}|} (\hat{x} - \widehat{P}_i), \quad \hat{x} = (\hat{x}_1, \ldots, \hat{x}_d)^T. \tag{9.1.7}$$

Indeed, $\hat{\theta}_i \in RT^0(\widehat{T})$ and $\hat{\chi}_i(\hat{\theta}_j) = \delta_{ij}$ for any $i, j \in \{1 : d+1\}$. The local Raviart–Thomas interpolation $I_{\widehat{T}}^{RT^0} : V(\widehat{T}) \to RT^0(\widehat{T})$ is then described as

$$I_{\widehat{T}}^{RT^0} : V(\widehat{T}) \ni \hat{v} \mapsto I_{\widehat{T}}^{RT^0} \hat{v} := \sum_{i=1}^{d+1} \left(\int_{\widehat{F}_i} \hat{v} \cdot \hat{n}_{\widehat{F}_i} d\hat{s} \right) \hat{\theta}_i \in RT^0(\widehat{T}). \tag{9.1.8}$$

Let Φ^s, $\widetilde{\Phi}$, and $\widehat{\Phi}$ be the affine mappings defined in Definition 3.4.1. Let Ψ^s, $\widetilde{\Psi}$, and $\widehat{\Psi}$ be the Piola transformations defined in Definition 3.4.2. Let $T^s \in \mathfrak{T}^{(d)}$ satisfy Condition 3.2.1 or Condition 3.2.2. The triples $\{\widetilde{T}, RT^k(\widetilde{T}), \widetilde{\Sigma}\}$ and $\{T^s, RT^k(T^s), \Sigma^s\}$ are defined as

$$\begin{cases} \widetilde{T} = \widehat{\Phi}(\widehat{T}); \\ RT^k(\widetilde{T}) = \{\widehat{\Psi}(\hat{p}); \ \hat{p} \in RT^k(\widehat{T})\}; \\ \widetilde{\Sigma} = \{\{\tilde{\chi}_i\}_{1 \leq i \leq N^{(RT)}}; \ \tilde{\chi}_i = \hat{\chi}_i(\widehat{\Psi}^{-1}(\tilde{p})), \forall \tilde{p} \in RT^k(\widetilde{T}), \hat{\chi}_i \in \widehat{\Sigma}\}; \end{cases}$$

and

$$\begin{cases} T^s = \widetilde{\Phi}(\widetilde{T}); \\ RT^k(T^s) = \{\widetilde{\Psi}(\tilde{p}); \ \tilde{p} \in RT^k(\widetilde{T})\}; \\ \Sigma^s = \{\{\chi_i^s\}_{1 \leq i \leq N^{(RT)}}; \ \chi_i^s = \tilde{\chi}_i(\widetilde{\Psi}^{-1}(p^s)), \forall p^s \in RT^k(T^s), \tilde{\chi}_i \in \widetilde{\Sigma}\}. \end{cases}$$

The triples $\{\widetilde{T}, RT^k(\widetilde{T}), \widetilde{\Sigma}\}$ and $\{T^s, RT^k(T^s), \Sigma^s\}$ are then the Raviart–Thomas finite elements. Furthermore, let

$$I_{\widetilde{T}}^{RT^k} : V(\widetilde{T}) \to RT^k(\widetilde{T}) \tag{9.1.9}$$

and

$$I_{T^s}^{RT^k} : V(T^s) \to RT^k(T^s) \tag{9.1.10}$$

be the associated local Raviart–Thomas interpolation defined in (9.1.4) and (9.1.5), respectively.

For any $T \in \mathbb{T}_h$, let Φ_{T^s} be the affine mapping defined in (3.4.2). Let $\Psi_{T^s} : V(T^s) \to V(T)$ be the Piola transformation defined in (3.4.5).

For the Raviart–Thomas finite element $\{T^s, RT^k(T^s), \Sigma^s\}$ with $k \in \mathbb{N}_0$, we define $\{T, RT^k(T), \Sigma\}$ as

$$\begin{cases} T = \Phi_T(T^s); \\ RT^k(T) = \{\Psi_{T^s}(p^s);\ p^s \in RT^k(T^s)\}; \\ \Sigma = \{\{\chi_i\}_{1 \le i \le N^{(RT)}};\ \chi_i = \chi_i^s(\Psi_{T^s}^{-1}(p)), \forall p \in RT^k(T), \chi_i^s \in \Sigma^s\}. \end{cases}$$

The triple $\{T, RT^k(T), \Sigma\}$ is then the Raviart–Thomas finite element. Let

$$I_T^{RT^k} : V(T) \to RT^k(T) \tag{9.1.11}$$

be the associated local Raviart–Thomas interpolation defined in (9.1.4) and (9.1.5), where $V(T) := W^{s,p}(T)^d$ with $sp > 1$, $p \in (1, \infty)$ or $s = 1$, $p = 1$.

Proposition 9.1.1. *Let $p \in [1, \infty)$. For any $\hat{v} \in W^{1,p}(\widehat{T})^d$ with $v := \Psi(\hat{v})$, it holds that*

$$\Psi(I_{\widehat{T}}^{RT^k} \hat{v}) = I_T^{RT^k}(\Psi \hat{v}),$$

that is, the diagrams

$$\begin{array}{ccccccc} V(T) & \xrightarrow{\Psi_{T^s}^{-1}} & V(T^s) & \xrightarrow{\widetilde{\Psi}^{-1}} & V(\widetilde{T}) & \xrightarrow{\widehat{\Psi}^{-1}} & V(\widehat{T}) \\ \downarrow I_T^{RT^k} & & \downarrow I_{T^s}^{RT^k} & & \downarrow I_{\widetilde{T}}^{RT^k} & & \downarrow I_{\widehat{T}}^{RT^k} \\ P & \xrightarrow[\Psi_{T^s}^{-1}]{} & P^s & \xrightarrow[\widetilde{\Psi}^{-1}]{} & \widetilde{P} & \xrightarrow[\widehat{\Psi}^{-1}]{} & \widehat{P} \end{array}$$

commute.

Proof. We extend the proof of [17, Lemma 3.4]. Recall that $\Psi = \Psi_{T^s} \circ \widetilde{\Psi} \circ \widehat{\Psi}$. Let $v \in W^{1,p}(T)^d$ and $F_i \subset \partial T$, $i \in \{1 : d+1\}$ be a face of T. We then check that $\Psi^{-1} I_T^{RT^k}(\Psi \hat{v})$ satisfies the conditions defining $I_{\widehat{T}}^{RT^k} \hat{v}$,

$$\int_{\widehat{F}_i} (\Psi^{-1} I_T^{RT^k}(\Psi \hat{v}) \cdot \hat{n}_i) \hat{p}_k d\hat{s} = \int_{\widehat{F}_i} (\hat{v} \cdot \hat{n}_i) \hat{p}_k d\hat{s} \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{F}_i), \tag{9.1.12}$$

where $\widehat{F}_i = \Phi^{-1}(F_i)$, and if $k \geq 1$,

$$\int_{\widehat{T}} (\Psi^{-1} I_T^{RT^k}(\Psi \hat{v})) \cdot \hat{q}_{k-1} d\hat{x} = \int_{\widehat{T}} \hat{v} \cdot \hat{q}_{k-1} d\hat{x} \quad \forall \hat{q}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T})^d. \tag{9.1.13}$$

Given $\hat{p}_k \in \mathcal{P}^k(\widehat{F}_i)$, we have

$$\int_{\widehat{F}_i} (\hat{v} \cdot \hat{n}_i) \hat{p}_k d\hat{s} = \int_{F_i} (v_0 \cdot n_i) p_k ds. \tag{9.1.14}$$

Indeed, this follows from (3.4.11) by a density argument. However, we cannot apply (3.4.11). Because the function obtained by extending p_k by zero to the other faces of T is not in $W^{\frac{1}{p},p'}(\partial T)$ and therefore, it is not the restriction to the boundary of a function $\varphi \in W^{1,p'}(T)$.

We take a sequence of functions $q_j \in \mathcal{C}_0^\infty(F_i)$ such that $q_j \to p_k$ in $L^{p'}(F_i)$, and because the extension by zero to ∂T of q_j is in $W^{\frac{1}{p},p'}(\partial T)$, there exists $\varphi_j \in W^{1,p'}(T)$ such that the restriction of φ_j to F_i is equal to q_j. Hence, applying (3.4.11),

$$\int_{\widehat{F}_i} (\hat{v} \cdot \hat{n}_i) \hat{q}_j d\hat{s} = \int_{F_i} (v \cdot n_i) q_j ds.$$

Because $v \cdot n_i \in L^p(F_i)$, we can pass to the limit to obtain (9.1.14). Similarly, from the definition of the Raviart–Thomas finite element interpolation and (9.1.14),

$$\begin{aligned} \int_{\widehat{F}_i} (\Psi^{-1} I_T^{RT^k} c \cdot \hat{n}_i) \hat{p}_k d\hat{s} &= \int_{F_i} (I_T^{RT^k} v \cdot n_i) p_k ds \\ &= \int_{F_i} (v \cdot n_i) p_k ds = \int_{\widehat{F}_i} (\hat{v} \cdot \hat{n}_i) \hat{p}_k d\hat{s}, \end{aligned}$$

which is (9.1.12).

For any $\hat{q}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T})^d$, we have

$$\begin{aligned}
\int_{\widehat{T}} (\Psi^{-1} I_T^{RT^k}(\Psi \hat{v})) \cdot \hat{q}_{k-1} d\hat{x} &= \int_{\widehat{T}} (\Psi^{-1} I_T^{RT^k}(\Psi \hat{v})) \cdot (\Psi^{-1} q_{k-1}) d\hat{x} \\
&= \int_T (I_T^{RT} v) \cdot q_{k-1} |\det(\mathcal{A})|^{-1} dx \\
&= \int_T v \cdot q_{k-1} |\det(\mathcal{A})|^{-1} dx \\
&= \int_T (\Psi \hat{v}) \cdot (\Psi \hat{q}_{k-1}) |\det(\mathcal{A})|^{-1} dx \\
&= \int_{\widehat{T}} \hat{v} \cdot \hat{q}_{k-1} d\hat{x},
\end{aligned}$$

which is (9.1.13). □

Lemma 9.1.2. *Let $k \in \mathbb{N}_0$ and $p \in [1, \infty)$. Let $\Pi_T^k : L^1(T) \to \mathcal{P}^k(T)$ be the L^2-orthogonal projection defined as*

$$\int_T (\Pi_T^k \varphi - \varphi) p_k dx = 0 \quad \forall p_k \in \mathcal{P}^k(T).$$

Then, the following diagram commutes:

$$\begin{array}{ccc}
W^{1,p}(T)^d & \xrightarrow{\text{div}} & L^1(T) \\
\Big\downarrow {\scriptstyle I_T^{RT^k}} & & \Big\downarrow {\scriptstyle \Pi_T^k} \\
RT^k(T) & \xrightarrow[\text{div}]{} & \mathcal{P}^k(T)
\end{array}$$

In other words, it holds that

$$\operatorname{div}(I_T^{RT^k} v) = \Pi_T^k(\operatorname{div} v) \quad \forall v \in W^{1,p}(T)^d. \tag{9.1.15}$$

Proof. The proof is found in [31, Lemma 16.2]. □

9.2 Remarks on the Anisotropic Raviart–Thomas Interpolation

In the proof of Theorem 3 in [47], we first proved the following lemmata in [47, Lemmas 6 and 7].

Lemma 6. *Let ℓ be such that $0 \le \ell \le k$. It holds that, for any $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_d)^T \in L^2(\widehat{T})^d$ with $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_d)^T := \widehat{\Psi}\hat{v}$ and $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_d)^T \in H^{\ell+1}(\widehat{T})^d$ with $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_d)^T := \widehat{\Psi}\hat{w}$,*

$$\frac{\|\tilde{v}\|_{L^2(\widetilde{T})^d}}{|\tilde{w}|_{H^{\ell+1}(\widetilde{T})^d}} \le \max_{1\le i\le d}\{h_i^{\ell+1}\} \frac{\left(\sum_{i=1}^d h_i^2 \|\hat{v}_i\|^2_{L^2(\widehat{T})}\right)^{1/2}}{\left(\sum_{i=1}^d h_i^2 |\hat{w}_i|^2_{H^{\ell+1}(\widehat{T})}\right)^{1/2}}. \tag{9.2.1}$$

Proof. From the definition of the Piola transformation, for $i = 1, \ldots, d$,

$$\tilde{w}_i(\tilde{x}) = \frac{1}{\det(\widehat{\mathcal{A}}^{(d)})} \sum_{j=1}^d [\widehat{\mathcal{A}}^{(d)}]_{ij} \hat{w}_j(\hat{x}) = \frac{1}{\det(\widehat{\mathcal{A}}^{(d)})} h_i \hat{w}_i(\hat{x}).$$

Let β be a multi-index with $|\beta| = \ell + 1$. We then have

$$\partial^\beta_{\tilde{x}} \tilde{w}_i(\tilde{x}) = \frac{1}{\det(\widehat{\mathcal{A}}^{(d)})} h_i (\partial^\beta_{\hat{x}} \hat{w}_i) h_1^{-\beta_1} \cdots h_d^{-\beta_d} = \frac{1}{\det(\widehat{\mathcal{A}}^{(d)})} h_i (\partial^\beta_{\hat{x}} \hat{w}_i) h^{-\beta}.$$

We here used $\hat{x}_j = \mathcal{A}_j^{-1} \tilde{x}_j$.

For any $\tilde{v} \in L^2(\widetilde{T})^d$, from the definition of the Piola transformation, we have

$$\|\tilde{v}\|^2_{L^2(\widetilde{T})^d} = \frac{1}{|\det(\widehat{\mathcal{A}}^{(d)})|} \|\widehat{\mathcal{A}}^{(d)} \hat{v}\|^2_{L^2(\widehat{T})^d} = \frac{1}{|\det(\widehat{\mathcal{A}}^{(d)})|} \sum_{i=1}^d h_i^2 \|\hat{v}_i\|^2_{L^2(\widehat{T})}.$$

Meanwhile, we have, for any $\tilde{w} \in H^{\ell+1}(\widetilde{T})^d$,

$$\begin{aligned}
|\tilde{w}|^2_{H^{\ell+1}(\widetilde{T})^d} &= \sum_{i=1}^d |\tilde{w}_i|^2_{H^{\ell+1}(\widetilde{T})} = \sum_{i=1}^d \sum_{|\beta|=\ell+1} \|\partial^\beta_{\tilde{x}} \tilde{w}_i\|^2_{L^2(\widetilde{T})} \\
&= \frac{1}{|\det(\widehat{\mathcal{A}}^{(d)})|} \sum_{i=1}^d h_i^2 \sum_{|\beta|=\ell+1} (h_1^{-\beta_1} \cdots h_d^{-\beta_d})^2 \|\partial^\beta_{\hat{x}} \hat{w}_i\|^2_{L^2(\widehat{T})} \\
&\ge \frac{1}{|\det(\widehat{\mathcal{A}}^{(d)})|} \min_{1\le j\le d}\{h_j^{-2|\beta|}\} \sum_{i=1}^d h_i^2 \sum_{|\beta|=\ell+1} \|\partial^\beta_{\hat{x}} \hat{w}_i\|^2_{L^2(\widehat{T})}.
\end{aligned}$$

These inequalities conclude (9.2.1). □

Lemma 7. *Let ℓ be such that $0 \le \ell \le k$. For any $\tilde{v} \in L^2(\widetilde{T})^d$ with $v^s := \widetilde{\Psi}\tilde{v}$ and $\tilde{w} \in H^{\ell+1}(\widetilde{T})^d$ with $w^s := \widetilde{\Psi}\tilde{w}$, we have*

$$\frac{\|v^s\|_{L^2(T^s)^d}}{|w^s|_{H^{\ell+1}(T^s)^d}} \le C^{P,d} \frac{H_{T^s}}{h_{T^s}} \frac{\|\tilde{v}\|_{L^2(\widetilde{T})^d}}{|\tilde{w}|_{H^{\ell+1}(\widetilde{T})^d}}, \tag{9.2.2}$$

where $C^{P,2} := 2^{\frac{\ell+1}{2}} C^{vec}$*, and* $C^{P,3} := \frac{2^{\ell+2}}{3} C^{vec}$*, where* C^{vec} *is a constant independent of* T^s *and* $\widetilde{T}$*.*

Proof. Using the standard estimates in [30, Lemma 1.113], we easily get

$$\frac{\|v^s\|_{L^2(T^s)^d}}{|w^s|_{H^{\ell+1}(T^s)^d}} \leq C^{vec} \left(\|\widetilde{\mathcal{A}}\|_2 \|\widetilde{\mathcal{A}}^{-1}\|_2 \right) \|\widetilde{\mathcal{A}}\|_2^{\ell+1} \frac{\|\tilde{v}\|_{L^2(\widetilde{T})^d}}{|\tilde{w}|_{H^{\ell+1}(\widetilde{T})^d}}, \quad d = 2, 3. \quad (9.2.3)$$

Therefore, (9.2.2) follows from (9.2.3), and (3.6.1b). □

For any $v^s \in H^1(T^s)^d$, using (9.2.1) and (9.2.2) yield

$$\frac{\|I_{T^s}^{RT} v^s - v^s\|_{L^2(T^s)^d}}{|v^s|_{H^1(T^s)^d}} \leq C^{P,d} \frac{H_{T^s}}{h_{T^s}} h_{T^s} \frac{\left(\sum_{i=1}^d h_i^2 \|(I_{\widehat{T}}^{RT^k} \hat{v})_i - \hat{v}_i\|_{L^2(\widehat{T})}^2 \right)^{1/2}}{\left(\sum_{i=1}^d h_i^2 |\hat{v}_i|_{H^1(\widehat{T})}^2 \right)^{1/2}}.$$

If the component-wise stability of the Raviart–Thomas interpolation on the reference element $\widehat{T}$

$$\|(I_{\widehat{T}}^{RT^k} \hat{v})_i - \hat{v}_i\|_{L^2(\widehat{T})} \leq c |\hat{v}_i|_{H^1(\widehat{T})}, \quad i = 1, \ldots, d \quad (9.2.4)$$

holds, then the target estimate

$$\|I_{T^s}^{RT^k} v^s - v^s\|_{L^2(T^s)^d} \leq c \frac{H_{T^s}}{h_{T^s}} h_{T^s} |v^s|_{H^1(T^s)^d}$$

is obtained. However, the estimate (9.2.4) generally does not hold (see [1, Introduction]): that is, we cannot apply the Babuška and Aziz technique [13]. We provide a counterexample of [1, Introduction].

We consider the simplex $\widehat{T} \subset \mathbb{R}^2$ with vertices $\widehat{P}_1 := (0,0)^T$, $\widehat{P}_2 := (1,0)^T$, and $\widehat{P}_3 := (0,1)^T$. For $1 \leq i \leq 3$, let $\widehat{F}_i$ be the face of $\widehat{T}$ opposite to $\widehat{P}_i$. The Raviart–Thomas interpolation of $\hat{v}$ is defined as

$$I_{\widehat{T}}^{RT^0} \hat{v} = \sum_{i=1}^{3} \left(\int_{\widehat{F}_i} \hat{v} \cdot \hat{n}_i d\hat{s} \right) \hat{\theta}_i \in RT^0,$$

where

$$\hat{\theta}_i := \frac{1}{2|\widehat{T}|} (\hat{x} - \widehat{P}_i), \quad \hat{x} = (\hat{x}_1, \hat{x}_2)^T.$$

Setting $\hat{v} := (0, \hat{x}_2^2)^T$ yields

$$\begin{aligned} I_{\widehat{T}}^{RT^0} \hat{v} &= \frac{1}{\sqrt{2}} \left(\int_{\widehat{F}_1} \hat{x}_2^2 d\hat{s} \right) (\hat{x}_1, \hat{x}_2)^T - \left(\int_{\widehat{F}_3} \hat{x}_2^2 d\hat{s} \right) (\hat{x}_1, \hat{x}_2 - 1)^T \\ &= \frac{1}{3} (\hat{x}_1, \hat{x}_2)^T. \end{aligned}$$

This implies that $(I_{\widehat{T}}^{RT^0}\hat{v})_1 - \hat{v}_1 \not\equiv 0$ for any $\hat{x} \in \mathbb{R}^2$ and the following component-wise stability does not hold:

$$\|(I_{\widehat{T}}^{RT^0}\hat{v})_1\|_{L^2(\widehat{T})} \leq c|\hat{v}_1|_{H^1(\widehat{T})}.$$

In other words, $(I_{\widehat{T}}^{RT^0}\hat{v})_1$ depends on both $\hat{v}_1$ and $\hat{v}_2$. Meanwhile, setting $\hat{v} := (0, \hat{x}_1^2)^T$ yields $I_{\widehat{T}}^{RT^0}\hat{v} = \frac{1}{3}(0,1)^T$.

A key observation is that if $\hat{r} := (0, g(\hat{x}_1))^T$, then $(I_{\widehat{T}}^{RT^0}\hat{r})_1 = 0$. In the next section, we introduce component-wise stabilities of the Raviart–Thomas interpolation on the reference element by [1].

9.3 Component-wise stability of the Raviart–Thomas interpolation on the reference element

9.3.1 Two-dimensional case

Let $\widehat{T} \subset \mathbb{R}^2$ be the reference triangle with vertices $\widehat{A}_1 := (1,0)^T$, $\widehat{A}_2 := (0,1)^T$, and $\widehat{A}_3 := (0,0)^T$ with $\hat{N}_1 := (-1,0)^T$, $\hat{N}_2 := (0,-1)^T$, and $\hat{N}_3 := \frac{1}{\sqrt{2}}(1,1)^T$. For $1 \leq i \leq 3$, let $\widehat{E}_i$ be the edge of $\widehat{T}$ opposite to $\widehat{A}_i$.

We use the same notation for a function of some variable than for its extension to $\widehat{T}$ as a function independent of the other variable. For example, $f(\hat{x}_2)$ denotes a function define on $\widehat{E}_1$ as well as one is defined in $\widehat{T}$. Furthermore, the same notation is used to denote a polynomial $\hat{p}_k$ on a edge and a polynomial in two variables such that its restriction to that edge agrees with $\hat{p}_k$. For example, for $\hat{p}_k \in \mathcal{P}^k(\widehat{E}_3)$, we write $\hat{p}_k(1-\hat{x}_2, \hat{x}_2)$.

Lemma 9.3.1. *Let $\hat{f}_i \in L^p(\widehat{E}_i)$, $i \in \{1,2\}$. If*

$$\hat{u}(\hat{x}) = (\hat{f}_1(\hat{x}_2), 0)^T, \quad \hat{v}(\hat{x}) = (0, \hat{f}_2(\hat{x}_1))^T,$$

then there exist polynomials $\hat{q}_i \in \mathcal{P}^k(\widehat{E}_i)$, $i \in \{1,2\}$, such that

$$I_{\widehat{T}}^{RT^k}\hat{u} = (\hat{q}_1(\hat{x}_2), 0)^T, \quad I_{\widehat{T}}^{RT^k}\hat{v} = (0, \hat{q}_2(\hat{x}_1))^T.$$

Proof. The proof is provided in [1, Lemma 3.2] (also see Lemma 9.3.3) for the case $d = 3$. The estimate in the case $d = 2$ can be proved analogously. □

Lemma 9.3.2. *For $k \in \mathbb{N}_0$, there exists a constant c such that, for all $\hat{u} = (\hat{u}_1, \hat{u}_2)^T \in W^{1,p}(\widehat{T})^2$,*

$$\|(I_{\widehat{T}}^{RT^k}\hat{u})_i\|_{L^p(\widehat{T})} \leq c\left(\|\hat{u}_i\|_{W^{1,p}(\widehat{T})} + \|\widehat{\mathrm{div}}\hat{u}\|_{L^p(\widehat{T})}\right), \quad i = 1,2. \tag{9.3.1}$$

Proof. The proof is provided in [1, Lemma 3.3] (also see Lemma 9.3.4) for the case $d = 3$. The estimate in the case $d = 2$ can be proved analogously. $\square$

9.3.2 Three-dimensional case: Type i

Let $\widehat{T} \subset \mathbb{R}^3$ be the reference triangle with vertices $\widehat{A}_1 := (1,0,0)^T$, $\widehat{A}_2 := (0,1,0)^T$, $\widehat{A}_3 := (0,0,1)^T$, and $\widehat{A}_4 := (0,0,0)^T$ with $\widehat{N}_1 := (-1,0,0)^T$, $\widehat{N}_2 := (0,-1,0)^T$, $\widehat{N}_3 := (0,0,-1)^T$, and $\widehat{N}_4 := \frac{1}{\sqrt{3}}(1,1,1)^T$. For $1 \leq i \leq 4$, let $\widehat{E}_i$ be the edge of $\widehat{T}$ opposite to $\widehat{A}_i$.

As two-dimensional case, we use the same notation for a function of some variable than for its extension to $\widehat{T}$ as a function independent of the other variable. For example, $f(\hat{x}_2, \hat{x}_3)$ denotes a function define on $\widehat{E}_1$ as well as one is defined in $\widehat{T}$. Furthermore, the same notation is used to denote a polynomial $\hat{p}_k$ on a edge and a polynomial in two variables such that its restriction to that edge agrees with $\hat{p}_k$. For example, for $\hat{p}_k \in \mathcal{P}^k(\widehat{E}_4)$, we write $\hat{p}_k(1 - \hat{x}_2 - \hat{x}_3, \hat{x}_2, \hat{x}_3)$.

Lemma 9.3.3. *Let $k \in \mathbb{N}_0$. Let $\hat{f}_i \in L^p(\widehat{E}_i)$, $i \in \{1,2,3\}$. If*

$$\begin{aligned} &\hat{u}(\hat{x}) = (\hat{f}_1(\hat{x}_2, \hat{x}_3), 0, 0)^T, \quad \hat{v}(\hat{x}) = (0, \hat{f}_2(\hat{x}_1, \hat{x}_3), 0)^T, \\ &\hat{w}(\hat{x}) = (0, 0, \hat{f}_3(\hat{x}_1, \hat{x}_2))^T, \end{aligned}$$

then there exist polynomials $\hat{q}_i \in \mathcal{P}^k(\widehat{E}_i)$, $i \in \{1,2,3\}$, such that

$$\begin{aligned} &I_{\widehat{T}}^{RT^k} \hat{u} = (\hat{q}_1(\hat{x}_2, \hat{x}_3), 0, 0)^T, \quad I_{\widehat{T}}^{RT^k} \hat{v} = (0, \hat{q}_2(\hat{x}_1, \hat{x}_3), 0)^T, \\ &I_{\widehat{T}}^{RT^k} \hat{w} = (0, 0, \hat{q}_3(\hat{x}_1, \hat{x}_2))^T. \end{aligned}$$

Proof. We follow [1, Lemma 3.2]. Because $\widehat{\mathrm{div}} \hat{u} = 0$, from the definition of the Raviart–Thomas interpolation and the Green's formula, we have, for any $\hat{p}_k \in \mathcal{P}^k(\widehat{T})$,

$$\begin{aligned} 0 &= \int_{\widehat{T}} \hat{p}_k \widehat{\mathrm{div}} \hat{u} d\hat{x} \\ &= \sum_{i=1}^{4} \int_{\widehat{E}_i} (\hat{p}_k \widehat{N}_i) \cdot \hat{u} d\hat{s} - \int_{\widehat{T}} (\hat{u} \cdot \widehat{\nabla}) \hat{p}_k d\hat{x} \\ &= \sum_{i=1}^{4} \int_{\widehat{E}_i} (\hat{p}_k \widehat{N}_i) \cdot (I_{\widehat{T}}^{RT^k} \hat{u}) d\hat{s} - \int_{\widehat{T}} ((I_{\widehat{T}}^{RT^k} \hat{u}) \cdot \widehat{\nabla}) \hat{p}_k d\hat{x} \\ &= \int_{\widehat{T}} \hat{p}_k \widehat{\mathrm{div}} (I_{\widehat{T}}^{RT^k} \hat{u}) d\hat{x}, \end{aligned}$$

which leads to $\widehat{\mathrm{div}}(I_{\widehat{T}}^{RT^k}\hat{u}) = 0$. Therefore, form the property of the Raviart–Thomas interpolation, $I_{\widehat{T}}^{RT^k}\hat{u} \in \mathcal{P}^k(\widehat{T})^3$, e.g. see [17, Lemma 3.1].

Using (9.1.4) for $i = 2, 3$, and $\hat{u}_2 = \hat{u}_3 = 0$, we have

$$\int_{\widehat{E}_i} (I_{\widehat{T}}^{RT^k}\hat{u})_i \hat{p}_k d\hat{s} = 0 \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_i), \quad i = 2, 3.$$

Setting $\hat{p}_k := (I_{\widehat{T}}^{RT^k}\hat{u})_i$, we obtain that $(I_{\widehat{T}}^{RT^k}\hat{u})_i|_{\widehat{E}_i} = 0$ for $i = 2, 3$.

For $k = 0$, because $I_{\widehat{T}}^{RT^0}\hat{u} \in \mathcal{P}^0(\widehat{T})^3$ and $(I_{\widehat{T}}^{RT^0}\hat{u})_i|_{\widehat{E}_i} = 0$ for $i = 2, 3$, it holds that $(I_{\widehat{T}}^{RT^0}\hat{u})_i = 0$ in $\widehat{T}$ for $i = 2, 3$. This implies that the first result holds.

For $k \geq 1$, there exists a polynomial $\hat{r}_i \in \mathcal{P}^{k-1}(\widehat{T})$, $i = 2, 3$, such that $(I_{\widehat{T}}^{RT^k}\hat{u})_i = \hat{x}_i \hat{r}_i$. Using (9.1.5) for $i = 2, 3$, and $\hat{u}_2 = \hat{u}_3 = 0$, we have, for $i = 2, 3$,

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k}\hat{u})_i \hat{r}_i d\hat{x} = 0. \quad \text{as } \hat{q}_{k-1} := (0, \hat{r}_i, 0)^T \text{ in (9.1.5)},$$

which leads to

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k}\hat{u})_i^2 d\hat{x} = \int_{\widehat{T}} \hat{x}_i \hat{x}_i \hat{r}_i^2 d\hat{x} \leq \|\hat{x}_i\|_{L^\infty(\widehat{T})} \int_{\widehat{T}} \hat{x}_i \hat{r}_i^2 d\hat{x} = 0.$$

Note that $\hat{x}_i \geq 0$ in $\widehat{T}$ for $i = 2, 3$. We hence conclude that $(I_{\widehat{T}}^{RT^k}\hat{u})_i = 0$ in $\widehat{T}$ for $i = 2, 3$.

Because $\widehat{\mathrm{div}}(I_{\widehat{T}}^{RT}\hat{u}) = 0$, it follows that

$$\frac{\partial (I_{\widehat{T}}^{RT^k}\hat{u})_1}{\partial \hat{x}_1} = 0.$$

This means that $(I_{\widehat{T}}^{RT^k}\hat{u})_1$ is independent of $\hat{x}_1$.

The other two results are analogous. □

Lemma 9.3.4. *For $k \in \mathbb{N}_0$, there exists a constant c such that, for all $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)^T \in W^{1,p}(\widehat{T})^3$,*

$$\|(I_{\widehat{T}}^{RT^k}\hat{u})_i\|_{L^p(\widehat{T})} \leq c \left(\|\hat{u}_i\|_{W^{1,p}(\widehat{T})} + \|\widehat{\mathrm{div}}\hat{u}\|_{L^p(\widehat{T})} \right), \quad i = 1, 2, 3. \tag{9.3.2}$$

Proof. We follow [1, Lemma 3.3]. From Lemma 9.3.3, if

$$\hat{v} := (\hat{u}_1, \hat{u}_2 - \hat{u}_2(\hat{x}_1, 0, \hat{x}_3), \hat{u}_3 - \hat{u}_3(\hat{x}_1, \hat{x}_2, 0))^T,$$

it holds that

$$I_{\widehat{T}}^{RT^k}\hat{v} = I_{\widehat{T}}^{RT^k}\hat{u} - I_{\widehat{T}}^{RT^k}(0, \hat{u}_2(\hat{x}_1, 0, \hat{x}_3), 0)^T - I_{\widehat{T}}^{RT^k}(0, 0, \hat{u}_3(\hat{x}_1, \hat{x}_2, 0))^T,$$

and thus, $(I_{\widehat{T}}^{RT^k}\hat{v})_1 = (I_{\widehat{T}}^{RT^k}\hat{u})_1$.

Let $k = 0$. Because $\hat{v}_2|_{\widehat{E}_2} = 0$ and $\hat{v}_3|_{\widehat{E}_3} = 0$, $I_{\widehat{T}}^{RT^0}\hat{v}$ is determined by the equations

$$\int_{\widehat{E}_1} (I_{\widehat{T}}^{RT^0}\hat{v})_1 d\hat{s} = \int_{\widehat{E}_1} \hat{v}_1 d\hat{s}, \tag{9.3.3a}$$

$$\int_{\widehat{E}_2} (I_{\widehat{T}}^{RT^0}\hat{v})_2 d\hat{s} = 0, \tag{9.3.3b}$$

$$\int_{\widehat{E}_3} (I_{\widehat{T}}^{RT^0}\hat{v})_3 d\hat{s} = 0, \tag{9.3.3c}$$

$$\int_{\widehat{E}_4} \{(I_{\widehat{T}}^{RT^0}\hat{v})_1 + (I_{\widehat{T}}^{RT^0}\hat{v})_2 + (I_{\widehat{T}}^{RT^0}\hat{v})_3\} d\hat{s} = \int_{\widehat{E}_4} (\hat{v}_1 + \hat{v}_2 + \hat{v}_3) d\hat{s}. \tag{9.3.3d}$$

From the divergence formula and the definition of $\hat{v}$, we have

$$\begin{aligned} \int_{\widehat{T}} \widehat{\operatorname{div}}\hat{v} d\hat{x} &= \int_{\partial\widehat{T}} \hat{v}\cdot\hat{n} d\hat{s} = \frac{1}{\sqrt{3}}\int_{\widehat{E}_4} (\hat{v}_1 + \hat{v}_2 + \hat{v}_3) d\hat{s} + \int_{\partial\widehat{T}\setminus\widehat{E}_4} \hat{v}\cdot\hat{n} d\hat{s} \\ &= \frac{1}{\sqrt{3}}\int_{\widehat{E}_4} (\hat{v}_1 + \hat{v}_2 + \hat{v}_3) d\hat{s} + \int_{\widehat{E}_1} \hat{v}_1 d\hat{s}. \end{aligned} \tag{9.3.4}$$

Because $\hat{u}_1 = \hat{v}_1$, $\widehat{\operatorname{div}}\hat{u} = \widehat{\operatorname{div}}\hat{v}$, $(I_{\widehat{T}}^{RT^0}\hat{u})_1 = (I_{\widehat{T}}^{RT^0}\hat{v})_1$, (9.3.3), (9.3.4), the definition of the Raviart–Thomas interpolation, and the trace theorem, we have

$$\begin{aligned} \|(I_{\widehat{T}}^{RT^0}\hat{u})_1\|_{L^p(\widehat{T})} &= \|(I_{\widehat{T}}^{RT^0}\hat{v})_1\|_{L^p(\widehat{T})} \\ &\leq \sum_{i=1}^{4} \left| \int_{\widehat{E}_i} \hat{v}\cdot\widehat{N}_i d\hat{s} \right| \|(\hat{\theta}_i)_1\|_{L^p(\widehat{T})} \\ &\leq c \left| \int_{\widehat{E}_1} \hat{v}_1 d\hat{s} + \frac{1}{\sqrt{3}}\int_{\widehat{E}_4} (\hat{v}_1 + \hat{v}_2 + \hat{v}_3) d\hat{s} \right| \\ &\leq c \left(\|\hat{u}_1\|_{W^{1,p}(\widehat{T})} + \|\widehat{\operatorname{div}}\hat{u}\|_{L^p(\widehat{T})} \right), \end{aligned}$$

which is the desired result for $k = 0$.

Let $k \geq 1$. Let $\hat{q}_i \in \mathcal{P}^{k-1}(\widehat{T})$, $i = 1, 2$, be such that

$$\int_{\widehat{T}} (\hat{v}_{i+1} - \hat{x}_{i+1}\hat{q}_i)\hat{p}_{k-1} d\hat{x} = 0 \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}), \quad i = 1, 2. \tag{9.3.5}$$

Note that there exist $\hat{q}_i \in \mathcal{P}^{k-1}(\widehat{T})$, $i = 1, 2$. Indeed, we can prove uniqueness (and therefore existence) of solutions of the linear systems.

We set

$$\hat{w} := (\hat{v}_1, \hat{v}_2 - \hat{x}_2\hat{q}_1, \hat{v}_3 - \hat{x}_3\hat{q}_2)^T.$$

Because $(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T \in RT^k(\widehat{T})$, it holds that

$$I_{\widehat{T}}^{RT^k}\hat{w} = I_{\widehat{T}}^{RT^k}\hat{v} - I_{\widehat{T}}^{RT^k}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T = I_{\widehat{T}}^{RT^k}\hat{v} - (0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T,$$

and thus, $(I_{\widehat{T}}^{RT^k}\hat{w})_1 = (I_{\widehat{T}}^{RT^k}\hat{v})_1$, therefore $(I_{\widehat{T}}^{RT^k}\hat{u})_1 = (I_{\widehat{T}}^{RT^k}\hat{w})_1$.

Because $\hat{w}_2|_{\widehat{E}_2} = 0$, $\hat{w}_3|_{\widehat{E}_3} = 0$, and (9.3.5), $I_{\widehat{T}}^{RT^k}\hat{w}$ is determined as follows:

$$\int_{\widehat{E}_1} (I_{\widehat{T}}^{RT^k}\hat{w})_1 \hat{p}_k d\hat{s} = \int_{\widehat{E}_1} \hat{w}_1 \hat{p}_k d\hat{s} \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_1), \tag{9.3.6a}$$

$$\int_{\widehat{E}_2} (I_{\widehat{T}}^{RT^k}\hat{w})_2 \hat{p}_k d\hat{s} = 0 \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_2), \tag{9.3.6b}$$

$$\int_{\widehat{E}_3} (I_{\widehat{T}}^{RT^k}\hat{w})_3 \hat{p}_k d\hat{s} = 0 \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_3), \tag{9.3.6c}$$

$$\begin{aligned} &\int_{\widehat{E}_4} \{(I_{\widehat{T}}^{RT^k}\hat{w})_1 + (I_{\widehat{T}}^{RT^k}\hat{w})_2 + (I_{\widehat{T}}^{RT^k}\hat{w})_3\} \hat{p}_k d\hat{s} \\ &\qquad = \int_{\widehat{E}_4} (\hat{w}_1 + \hat{w}_2 + \hat{w}_3)\hat{p}_k d\hat{s} \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_4), \end{aligned} \tag{9.3.6d}$$

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k}\hat{w})_1 \hat{p}_{k-1} d\hat{x} = \int_{\widehat{T}} \hat{w}_1 \hat{p}_{k-1} d\hat{x} \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}), \tag{9.3.6e}$$

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k}\hat{w})_2 \hat{p}_{k-1} d\hat{x} = 0 \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}), \tag{9.3.6f}$$

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k}\hat{w})_3 \hat{p}_{k-1} d\hat{x} = 0 \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}). \tag{9.3.6g}$$

For $\hat{r}_k \in \mathcal{P}^k(\widehat{T})$, using (9.3.5), the Green formula and the definition of $\hat{w}$ yields

$$\begin{aligned} \int_{\widehat{T}} \widehat{\operatorname{div}} \hat{w} \hat{r}_k d\hat{x} &= \int_{\partial \widehat{T}} (\hat{w} \cdot \hat{n}) \hat{r}_k d\hat{s} - \int_{\widehat{T}} (\hat{w} \cdot \widehat{\nabla}) \hat{r}_k d\hat{x} \\ &= \frac{1}{\sqrt{3}} \int_{\widehat{E}_4} (\hat{w}_1 + \hat{w}_2 + \hat{w}_3) \hat{r}_k d\hat{s} - \int_{\widehat{E}_1} \hat{w}_1 \hat{r}_k d\hat{s} - \int_{\widehat{T}} \hat{w}_1 \frac{\partial \hat{r}_k}{\partial \hat{x}_1} d\hat{x}. \end{aligned} \tag{9.3.7}$$

Furthermore, we have

$$\widehat{\mathrm{div}}\hat{w} = \widehat{\mathrm{div}}\hat{v} - \widehat{\mathrm{div}}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T = \widehat{\mathrm{div}}\hat{u} - \widehat{\mathrm{div}}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T. \tag{9.3.8}$$

Because $\hat{u}_1 = \hat{w}_1$, $(I_{\widehat{T}}^{RT^k}\hat{u})_1 = (I_{\widehat{T}}^{RT^k}\hat{w})_1$, (9.3.6), (9.3.7), (9.3.8), the definition of the Raviart–Thomas interpolation, and the trace theorem, we have

$$\begin{aligned}
\|(I_{\widehat{T}}^{RT^k}\hat{u})_1\|_{L^p(\widehat{T})} &= \|(I_{\widehat{T}}^{RT^k}\hat{w})_1\|_{L^p(\widehat{T})} \\
&\le c\left(\left| \int_{\widehat{E}_1} \hat{w}_1 d\hat{s} + \frac{1}{\sqrt{3}} \int_{\widehat{E}_4} (\hat{w}_1 + \hat{w}_2 + \hat{w}_3) d\hat{s} \right| + \left| \int_{\widehat{T}} \hat{w}_1 d\hat{x} \right| \right) \\
&\le c\left(\|\hat{u}_1\|_{W^{1,p}(\widehat{T})} + \|\widehat{\mathrm{div}}\hat{u}\|_{L^p(\widehat{T})} + \|\widehat{\mathrm{div}}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T\|_{L^p(\widehat{T})} \right).
\end{aligned}$$

To show (9.3.2) for $i = 1$ is that there exists c such that

$$\|\widehat{\mathrm{div}}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T\|_{L^p(\widehat{T})} \le c\left(\|\hat{u}_1\|_{W^{1,p}(\widehat{T})} + \|\widehat{\mathrm{div}}\hat{u}\|_{L^p(\widehat{T})} \right). \tag{9.3.9}$$

For any $\hat{s}_k \in \mathcal{P}^k(\widehat{T})$, from (9.3.5) and the Green formula, we have

$$\begin{aligned}
0 &= \int_{\widehat{T}} \{(0, \hat{v}_2 - \hat{x}_2\hat{q}_1, \hat{v}_3 - \hat{x}_3\hat{q}_2)^T \cdot \widehat{\nabla}\} \hat{s}_k d\hat{x} \\
&= \int_{\partial\widehat{T}} \{(0, \hat{v}_2 - \hat{x}_2\hat{q}_1, \hat{v}_3 - \hat{x}_3\hat{q}_2)^T \cdot \hat{n}\} \hat{s}_k d\hat{s} \\
&\quad - \int_{\widehat{T}} \widehat{\mathrm{div}}(0, \hat{v}_2 - \hat{x}_2\hat{q}_1, \hat{v}_3 - \hat{x}_3\hat{q}_2)^T \hat{s}_k d\hat{x}.
\end{aligned}$$

Setting

$$\hat{s}_k(\hat{x}) := (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3)\hat{t}_{k-1}, \quad \hat{t}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}),$$

it holds that

$$\int_{\widehat{T}} (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3) \widehat{\mathrm{div}}(0, \hat{v}_2 - \hat{x}_2\hat{q}_1, \hat{v}_3 - \hat{x}_3\hat{q}_2)^T \hat{t}_{k-1} d\hat{x} = 0,$$

that is,

$$\begin{aligned}
&\int_{\widehat{T}} (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3) \widehat{\mathrm{div}}(0, \hat{v}_2, \hat{v}_3)^T \hat{t}_{k-1} d\hat{x} \\
&\quad = \int_{\widehat{T}} (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3) \widehat{\mathrm{div}}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T \hat{t}_{k-1} d\hat{x},
\end{aligned}$$

because $\hat{s}_k = 0$ on $\widehat{E}_4$, $(\hat{v}_2 - \hat{x}_2\hat{q}_1)\hat{n}_2 = 0$ on $\partial\widehat{T}\backslash\widehat{E}_4$, and $(\hat{v}_3 - \hat{x}_3\hat{q}_2)\hat{n}_3 = 0$ on $\partial\widehat{T}\backslash\widehat{E}_4$. Therefore, setting $\hat{t}_{k-1} := \widehat{\mathrm{div}}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T$, and using the Hölder inequality, we have

$$\left\|\widehat{\mathrm{div}}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T\right\|^2_{L^2(\widehat{T})} \leq c\int_{\widehat{T}}(1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3)|\widehat{\mathrm{div}}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T|^2 d\hat{x}$$
$$\leq c\|1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3\|_{L^\infty(\widehat{T})}\|\widehat{\mathrm{div}}(0, \hat{v}_2, \hat{v}_3)^T\|_{L^p(\widehat{T})}\|\widehat{\mathrm{div}}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T\|_{L^{p'}(\widehat{T})}.$$

Note that $1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3 \geq 0$ in $\widehat{T}$. Because all the norms on $\mathcal{P}^{k-1}(\widehat{T})$ are equivalent, we obtain

$$\left\|\widehat{\mathrm{div}}(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T\right\|_{L^p(\widehat{T})} \leq c\|\widehat{\mathrm{div}}(0, \hat{v}_2, \hat{v}_3)^T\|_{L^p(\widehat{T})}. \tag{9.3.10}$$

Observing that $\widehat{\mathrm{div}}(0, \hat{v}_2, \hat{v}_3)^T = \widehat{\mathrm{div}}(0, \hat{u}_2, \hat{u}_3)^T$, and

$$\|\widehat{\mathrm{div}}(0, \hat{u}_2, \hat{u}_3)^T\|_{L^p(\widehat{T})} \leq \|\widehat{\mathrm{div}}\hat{u}\|_{L^p(\widehat{T})} + \|\widehat{\mathrm{div}}(\hat{u}_1, 0, 0)^T\|_{L^p(\widehat{T})}$$
$$\leq \|\widehat{\mathrm{div}}\hat{u}\|_{L^p(\widehat{T})} + \|\hat{u}_1\|_{W^{1,p}(\widehat{T})},$$

the target estimate (9.3.9) follows from (9.3.10).

By analogous argument, the estimates for $(I_{\widehat{T}}^{RT^k}\hat{u})_i$, $i = 2, 3$, can be proved. □

9.3.3 Three-dimensional case: Type ii

Let $\widehat{T} \subset \mathbb{R}^3$ be the reference triangle with vertices $\widehat{A}_1 := (1, 0, 0)^T$, $\widehat{A}_2 := (1, 1, 0)^T$, $\widehat{A}_3 := (0, 0, 1)^T$, and $\widehat{A}_4 := (0, 0, 0)^T$ with $\widehat{N}_1 := \frac{1}{\sqrt{2}}(-1, 1, 0)^T$, $\widehat{N}_2 := (0, -1, 0)^T$, $\widehat{N}_3 := (0, 0, -1)^T$, and $\widehat{N}_4 := \frac{1}{\sqrt{2}}(1, 0, 1)^T$. For $1 \leq i \leq 4$, let $\widehat{E}_i$ be the edge of $\widehat{T}$ opposite to $\widehat{A}_i$ and with $\overline{E}_1$ the projection of $\widehat{E}_1$ onto the plane given by $\hat{x}_1 = 0$.

Lemma 9.3.5. *Let $k \in \mathbb{N}_0$. Let $\hat{f}_1 \in L^p(\overline{E}_1)$, and $\hat{f}_i \in L^p(\widehat{E}_i)$, $i \in \{2, 3\}$. If*

$$\hat{u}(\hat{x}) = (\hat{f}_1(\hat{x}_2, \hat{x}_3), 0, 0)^T, \quad \hat{v}(\hat{x}) = (0, \hat{f}_2(\hat{x}_1, \hat{x}_3), 0)^T,$$
$$\hat{w}(\hat{x}) = (0, 0, \hat{f}_3(\hat{x}_1, \hat{x}_2))^T,$$

then there exist polynomials $\hat{q}_1 \in \mathcal{P}^k(\overline{E}_1)$, and $\hat{q}_i \in \mathcal{P}^k(\widehat{E}_i)$, $i \in \{2, 3\}$, such that

$$I_{\widehat{T}}^{RT^k}\hat{u} = (\hat{q}_1(\hat{x}_2, \hat{x}_3), 0, 0)^T, \quad I_{\widehat{T}}^{RT^k}\hat{v} = (0, \hat{q}_2(\hat{x}_1, \hat{x}_3), 0)^T,$$
$$I_{\widehat{T}}^{RT^k}\hat{w} = (0, 0, \hat{q}_3(\hat{x}_1, \hat{x}_2))^T.$$

Proof. We follow [1, Lemma 4.2]. The proof is similar to that of Lemma 9.3.3. We prove the second equality. The other two follow in an analogous argument.

Because $\widehat{\mathrm{div}}\hat{v} = 0$, from the definition of the Raviart–Thomas interpolation and the Green's formula, we have $\widehat{\mathrm{div}}(I_{\widehat{T}}^{RT^k}\hat{v}) = 0$. Therefore, form the property of the Raviart–Thomas interpolation, $I_{\widehat{T}}^{RT^k}\hat{v} \in \mathcal{P}^k(\widehat{T})^3$. Using (9.1.4) for $i = 3$, and $\hat{v}_3 = 0$, we have

$$\int_{\widehat{E}_3} (I_{\widehat{T}}^{RT^k}\hat{v})_3 \hat{p}_k d\hat{s} = 0 \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_3).$$

Setting $\hat{p}_k := (I_{\widehat{T}}^{RT^k}\hat{v})_3$, we obtain that $(I_{\widehat{T}}^{RT^k}\hat{v})_3|_{\widehat{E}_3} = 0$.

Let $k = 0$. Because $I_{\widehat{T}}^{RT^0}\hat{v} \in \mathcal{P}^0(\widehat{T})^3$ and $(I_{\widehat{T}}^{RT^0}\hat{v})_3|_{\widehat{E}_3} = 0$, it holds that $(I_{\widehat{T}}^{RT^0}\hat{v})_3 = 0$ in $\widehat{T}$. Using (9.1.4) for $i = 4$, and $\hat{v}_1 = \hat{v}_3 = 0$, we have

$$\int_{\widehat{E}_4} \{(I_{\widehat{T}}^{RT^0}\hat{v})_1 + (I_{\widehat{T}}^{RT^0}\hat{v})_3\} d\hat{s} = 0,$$

which leads to $(I_{\widehat{T}}^{RT^0}\hat{v})_1|_{\widehat{E}_4} = 0$. It then holds that that $(I_{\widehat{T}}^{RT^0}\hat{v})_1 = 0$ in $\widehat{T}$. This implies that the second result holds.

Let $k \geq 1$. As in the proof of Lemma 9.3.3, we obtain that $(I_{\widehat{T}}^{RT^k}\hat{v})_3 = 0$ in $\widehat{T}$. Using (9.1.4) for $i = 4$, and $\hat{v}_1 = \hat{v}_3 = 0$, we have

$$\int_{\widehat{E}_4} \{(I_{\widehat{T}}^{RT^k}\hat{v})_1 + (I_{\widehat{T}}^{RT^k}\hat{v})_3\} \hat{p}_k d\hat{s} = 0 \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_4),$$

which implies that $\{(I_{\widehat{T}}^{RT^k}\hat{v})_1 + (I_{\widehat{T}}^{RT^k}\hat{v})_3\}|_{\widehat{E}_4} = 0$, and hence

$$(I_{\widehat{T}}^{RT^k}\hat{v})_1 + (I_{\widehat{T}}^{RT^k}\hat{v})_3 = (1 - \hat{x}_1 - \hat{x}_3)\hat{r}$$

for some $\hat{r} \in \mathcal{P}^{k-1}(\widehat{T})$. Using (9.1.5) and $\hat{v}_1 = \hat{v}_3 = 0$, we have

$$\int_{\widehat{T}} \{(I_{\widehat{T}}^{RT^k}\hat{v})_1 + (I_{\widehat{T}}^{RT^k}\hat{v})_3\} \hat{r} d\hat{x} = 0. \quad \text{as } \hat{q}_{k-1} := (\hat{r}, 0, \hat{r})^T \text{ in (9.1.5)},$$

which leads to

$$\begin{aligned}\int_{\widehat{T}} \{(I_{\widehat{T}}^{RT^k}\hat{v})_1 + (I_{\widehat{T}}^{RT^k}\hat{v})_3\}^2 d\hat{x} &= \int_{\widehat{T}} (1 - \hat{x}_1 - \hat{x}_3)^2 \hat{r}^2 d\hat{x} \\ &\leq \|1 - \hat{x}_1 - \hat{x}_3\|_{L^\infty(\widehat{T})} \int_{\widehat{T}} (1 - \hat{x}_1 - \hat{x}_3) \hat{r}^2 d\hat{x} = 0.\end{aligned}$$

Note that $1 - \hat{x}_1 - \hat{x}_3 \geq 0$ in $\widehat{T}$. We hence have $(I_{\widehat{T}}^{RT^k}\hat{v})_1 + (I_{\widehat{T}}^{RT^k}\hat{v})_3 = 0$ in $\widehat{T}$. Because we know $(I_{\widehat{T}}^{RT^k}\hat{v})_3 = 0$ in $\widehat{T}$, we conclude that $(I_{\widehat{T}}^{RT^k}\hat{v})_1 = 0$ in $\widehat{T}$.

Because $\widehat{\operatorname{div}}(I_{\widehat{T}}^{RT^k}\hat{v}) = 0$, it follows that

$$\frac{\partial (I_{\widehat{T}}^{RT^k}\hat{v})_2}{\partial \hat{x}_2} = 0.$$

This means that $(I_{\widehat{T}}^{RT^k}\hat{v})_2$ is independent of $\hat{x}_2$. □

Lemma 9.3.6. *For $k \in \mathbb{N}_0$, there exists a constant c such that, for all $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)^T \in W^{1,p}(\widehat{T})^3$,*

$$\|(I_{\widehat{T}}^{R^kT}\hat{u})_i\|_{L^p(\widehat{T})} \leq c\left(\|\hat{u}_1\|_{W^{1,p}(\widehat{T})} + \left\|\frac{\partial \hat{u}_2}{\partial \hat{x}_2}\right\|_{L^p(\widehat{T})} + \left\|\frac{\partial \hat{u}_3}{\partial \hat{x}_3}\right\|_{L^p(\widehat{T})}\right), \quad (9.3.11a)$$

$$\|(I_{\widehat{T}}^{RT^k}\hat{u})_i\|_{L^p(\widehat{T})} \leq c\left(\|\hat{u}_i\|_{W^{1,p}(\widehat{T})} + \|\operatorname{div}\hat{u}\|_{L^p(\widehat{T})}\right), \quad i = 2, 3. \quad (9.3.11b)$$

In particular,

$$\|(I_{\widehat{T}}^{RT^k}\hat{u})_i\|_{L^p(\widehat{T})} \leq c\left(\|\hat{u}_i\|_{W^{1,p}(\widehat{T})} + \sum_{j=1, j\neq i}^{3}\left\|\frac{\partial \hat{u}_j}{\partial \hat{x}_j}\right\|_{L^p(\widehat{T})}\right), \quad i = 1, 2, 3. \quad (9.3.12)$$

Proof. We follow [1, Lemma 4.3]. We prove the estimates for $i = 1, 2$. The other one follows in an analogous argument.

Case for $i = 1$

From Lemma 9.3.5, if

$$\hat{v} := (\hat{u}_1, \hat{u}_2 - \hat{u}_2(\hat{x}_1, 0, \hat{x}_3), \hat{u}_3 - \hat{u}_3(\hat{x}_1, \hat{x}_2, 0))^T,$$

it holds that

$$I_{\widehat{T}}^{RT^k}\hat{v} = I_{\widehat{T}}^{RT^k}\hat{u} - I_{\widehat{T}}^{RT^k}(0, \hat{u}_2(\hat{x}_1, 0, \hat{x}_3), 0)^T - I_{\widehat{T}}^{RT^k}(0, 0, \hat{u}_3(\hat{x}_1, \hat{x}_2, 0))^T,$$

and thus, $(I_{\widehat{T}}^{RT^k}\hat{v})_1 = (I_{\widehat{T}}^{RT^k}\hat{u})_1$.

Let $k = 0$. Because $\hat{v}_2|_{\widehat{E}_2} = 0$ and $\hat{v}_3|_{\widehat{E}_3} = 0$, $I_{\widehat{T}}^{RT^0}\hat{v}$ is determined by the equations

$$\int_{\widehat{E}_1} \{-(I_{\widehat{T}}^{RT^0}\hat{v})_1 + (I_{\widehat{T}}^{RT^0}\hat{v})_2\} d\hat{s} = \int_{\widehat{E}_1} (-\hat{v}_1 + \hat{v}_2) d\hat{s}, \tag{9.3.13a}$$

$$\int_{\widehat{E}_2} (I_{\widehat{T}}^{RT^0}\hat{v})_2 d\hat{s} = 0, \tag{9.3.13b}$$

$$\int_{\widehat{E}_3} (I_{\widehat{T}}^{RT^0}\hat{v})_3 d\hat{s} = 0, \tag{9.3.13c}$$

$$\int_{\widehat{E}_4} \{(I_{\widehat{T}}^{RT^0}\hat{v})_1 + (I_{\widehat{T}}^{RT^0}\hat{v})_3\} d\hat{s} = \int_{\widehat{E}_4} (\hat{v}_1 + \hat{v}_3) d\hat{s}. \tag{9.3.13d}$$

From the divergence formula and the definition of $\hat{v}$, we have

$$\begin{aligned} \int_{\widehat{T}} \widehat{\operatorname{div}}(\hat{v}_1, \hat{v}_2, 0)^T d\hat{x} &= \int_{\partial\widehat{T}} (\hat{v}_1, \hat{v}_2, 0)^T \cdot \hat{n} d\hat{s} \\ &= \frac{1}{\sqrt{2}} \int_{\widehat{E}_1} (-\hat{v}_1 + \hat{v}_2) d\hat{s} + \frac{1}{\sqrt{2}} \int_{\widehat{E}_4} \hat{v}_1 d\hat{s}, \end{aligned} \tag{9.3.14}$$

and

$$\begin{aligned} \int_{\widehat{T}} \widehat{\operatorname{div}}(\hat{v}_1, 0, \hat{v}_3)^T d\hat{x} &= \int_{\partial\widehat{T}} (\hat{v}_1, 0, \hat{v}_3)^T \cdot \hat{n} d\hat{s} \\ &= \frac{1}{\sqrt{2}} \int_{\widehat{E}_1} (-\hat{v}_1) d\hat{s} + \frac{1}{\sqrt{2}} \int_{\widehat{E}_4} (\hat{v}_1 + \hat{v}_3) d\hat{s}, \end{aligned} \tag{9.3.15}$$

Because $\hat{u}_1 = \hat{v}_1$, $\frac{\partial \hat{u}_i}{\partial \hat{x}_i} = \frac{\partial \hat{v}_i}{\partial \hat{x}_i}$ for $i = 2, 3$, $(I_{\widehat{T}}^{RT^0}\hat{u})_1 = (I_{\widehat{T}}^{RT^0}\hat{v})_1$, (9.3.13), (9.3.14), (9.3.15), the definition of the Raviart–Thomas interpolation, and the trace theorem, we have

$$\begin{aligned} \|(I_{\widehat{T}}^{RT^0}\hat{u})_1\|_{L^p(\widehat{T})} &= \|(I_{\widehat{T}}^{RT^0}\hat{v})_1\|_{L^p(\widehat{T})} \\ &\le \sum_{i=1}^{4} \left| \int_{\widehat{E}_i} \hat{v} \cdot \widehat{N}_i d\hat{s} \right| \|(\hat{\theta}_i)_1\|_{L^p(\widehat{T})} \\ &\le c \left| \frac{1}{\sqrt{2}} \int_{\widehat{E}_1} (-\hat{v}_1 + \hat{v}_2) d\hat{s} + \frac{1}{\sqrt{2}} \int_{\widehat{E}_4} (\hat{v}_1 + \hat{v}_3) d\hat{s} \right| \\ &\le c \left(\|\hat{u}_1\|_{W^{1,p}(\widehat{T})} + \left\| \frac{\partial \hat{u}_2}{\partial \hat{x}_2} \right\|_{L^p(\widehat{T})} + \left\| \frac{\partial \hat{u}_3}{\partial \hat{x}_3} \right\|_{L^p(\widehat{T})} \right), \end{aligned}$$

which is the desired result for $k = 0$.

Let $k \geq 1$. Let $\hat{q}_i \in \mathcal{P}^{k-1}(\widehat{T})$, $i = 1, 2$, be such that

$$\int_{\widehat{T}} (\hat{v}_{i+1} - \hat{x}_{i+1}\hat{q}_i)\hat{p}_{k-1} d\hat{x} = 0 \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}), \quad i = 1, 2. \tag{9.3.16}$$

We set

$$\hat{w} := (\hat{v}_1, \hat{v}_2 - \hat{x}_2\hat{q}_1, \hat{v}_3 - \hat{x}_3\hat{q}_2)^T.$$

Because $(0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T \in RT^k(\widehat{T})$, it holds that

$$I_{\widehat{T}}^{RT^k} \hat{w} = I_{\widehat{T}}^{RT^k} \hat{v} - I_{\widehat{T}}^{RT^k} (0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T = I_{\widehat{T}}^{RT^k} \hat{v} - (0, \hat{x}_2\hat{q}_1, \hat{x}_3\hat{q}_2)^T,$$

and thus, $(I_{\widehat{T}}^{RT^k} \hat{w})_1 = (I_{\widehat{T}}^{RT^k} \hat{v})_1$, therefore $(I_{\widehat{T}}^{RT^k} \hat{u})_1 = (I_{\widehat{T}}^{RT^k} \hat{w})_1$.

Because $\hat{w}_2|_{\widehat{E}_2} = 0$, $\hat{w}_3|_{\widehat{E}_3} = 0$, and (9.3.16), $I_{\widehat{T}}^{RT^k} \hat{w}$ is determined as follows:

$$\int_{\widehat{E}_1} \{-(I_{\widehat{T}}^{RT^k} \hat{v})_1 + (I_{\widehat{T}}^{RT^k} \hat{v})_2\} \hat{p}_k d\hat{s} = \int_{\widehat{E}_1} (-\hat{w}_1 + \hat{w}_2)\hat{p}_k d\hat{s} \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_1), \tag{9.3.17a}$$

$$\int_{\widehat{E}_2} (I_{\widehat{T}}^{RT^k} \hat{w})_2 \hat{p}_k d\hat{s} = 0 \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_2), \tag{9.3.17b}$$

$$\int_{\widehat{E}_3} (I_{\widehat{T}}^{RT^k} \hat{w})_3 \hat{p}_k d\hat{s} = 0 \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_3), \tag{9.3.17c}$$

$$\int_{\widehat{E}_4} \{(I_{\widehat{T}}^{RT^k} \hat{v})_1 + (I_{\widehat{T}}^{RT^k} \hat{v})_3\} \hat{p}_k d\hat{s} = \int_{\widehat{E}_4} (\hat{w}_1 + \hat{w}_3)\hat{p}_k d\hat{s} \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_4), \tag{9.3.17d}$$

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k} \hat{w})_1 \hat{p}_{k-1} d\hat{x} = \int_{\widehat{T}} \hat{w}_1 \hat{p}_{k-1} d\hat{x} \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}), \tag{9.3.17e}$$

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k} \hat{w})_2 \hat{p}_{k-1} d\hat{x} = 0 \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}), \tag{9.3.17f}$$

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k} \hat{w})_3 \hat{p}_{k-1} d\hat{x} = 0 \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}). \tag{9.3.17g}$$

For $\hat{r}_k \in \mathcal{P}^k(\widehat{T})$, using (9.3.16), the Green formula and the definition of $\hat{w}$ yields

$$\begin{aligned} \int_{\widehat{T}} \widehat{\mathrm{div}} (\hat{w}_1, \hat{w}_2, 0)^T \hat{r}_k d\hat{x} &= \int_{\partial \widehat{T}} (\hat{w}_1, \hat{w}_2, 0)^T \cdot \hat{n} \hat{r}_k d\hat{s} - \int_{\widehat{T}} ((\hat{w}_1, \hat{w}_2, 0)^T \cdot \widehat{\nabla}) \hat{r}_k d\hat{x} \\ &= \frac{1}{\sqrt{2}} \int_{\widehat{E}_1} (-\hat{w}_1 + \hat{w}_2)\hat{r}_k d\hat{s} + \frac{1}{\sqrt{2}} \int_{\widehat{E}_4} \hat{w}_1 \hat{r}_k d\hat{s} - \int_{\widehat{T}} \hat{w}_1 \frac{\partial \hat{r}_k}{\partial \hat{x}_1} d\hat{x}, \end{aligned} \tag{9.3.18}$$

and

$$\int_{\widehat{T}} \widehat{\operatorname{div}}(\hat{w}_1, 0, \hat{w}_3)^T \hat{r}_k d\hat{x} = \int_{\partial\widehat{T}} (\hat{w}_1, 0, \hat{w}_3)^T \cdot \hat{n}\hat{r}_k d\hat{s} - \int_{\widehat{T}} ((\hat{w}_1, 0, \hat{w}_3)^T \cdot \widehat{\nabla})\hat{r}_k d\hat{x}$$
$$= \frac{1}{\sqrt{2}} \int_{\widehat{E}_1} (-\hat{w}_1)\hat{r}_k d\hat{s} + \frac{1}{\sqrt{2}} \int_{\widehat{E}_4} (\hat{w}_1 + \hat{w}_3)\hat{r}_k d\hat{s} - \int_{\widehat{T}} \hat{w}_1 \frac{\partial \hat{r}_k}{\partial \hat{x}_1} d\hat{x}, \tag{9.3.19}$$

Because $\hat{u}_1 = \hat{w}_1$, $(I_{\widehat{T}}^{RT^k} \hat{u})_1 = (I_{\widehat{T}}^{RT^k} \hat{w})_1$, (9.3.17), (9.3.18), (9.3.19), the definition of the Raviart–Thomas interpolation, the trace theorem, and

$$\frac{\partial}{\partial \hat{x}_{i+1}} (\hat{v}_{i+1} - \hat{x}_{i+1}\hat{q}_i) = \frac{\partial \hat{v}_{i+1}}{\partial \hat{x}_{i+1}} - \frac{\partial (\hat{x}_{i+1}\hat{q}_i)}{\partial \hat{x}_{i+1}}, \quad i = 1, 2,$$

we have

$$\begin{aligned} \|(I_{\widehat{T}}^{RT^k} \hat{u})_1\|_{L^p(\widehat{T})} &= \|(I_{\widehat{T}}^{RT^k} \hat{w})_1\|_{L^p(\widehat{T})} \\ &\leq c \left| \frac{1}{\sqrt{2}} \int_{\widehat{E}_1} (-\hat{w}_1 + \hat{w}_2) d\hat{s} + \frac{1}{\sqrt{2}} \int_{\widehat{E}_4} (\hat{w}_1 + \hat{w}_3) d\hat{s} \right| \\ &\leq c \Bigg(\|\hat{u}_1\|_{W^{1,p}(\widehat{T})} + \left\| \frac{\partial \hat{u}_2}{\partial \hat{x}_2} \right\|_{L^p(\widehat{T})} + \left\| \frac{\partial \hat{u}_3}{\partial \hat{x}_3} \right\|_{L^p(\widehat{T})} \\ &\qquad + \left\| \frac{\partial (\hat{x}_2 \hat{q}_1)}{\partial \hat{x}_2} \right\|_{L^p(\widehat{T})} + \left\| \frac{\partial (\hat{x}_3 \hat{q}_2)}{\partial \hat{x}_3} \right\|_{L^p(\widehat{T})} \Bigg). \end{aligned} \tag{9.3.20}$$

From the definition of $\hat{w}$ and (9.3.16), we have, for any $\hat{t}_k \in \mathcal{P}^k(\widehat{T})$,

$$0 = \int_{\widehat{T}} \hat{w}_3 \frac{\partial \hat{t}_k}{\partial \hat{x}_3} d\hat{x} = \int_{\partial\widehat{T}} \hat{w}_3 \hat{n}_3 \hat{t}_k d\hat{s} - \int_{\widehat{T}} \frac{\partial \hat{w}_3}{\partial \hat{x}_3} \hat{t}_k d\hat{x}.$$

If we set $\hat{t}_k := (1 - \hat{x}_1 - \hat{x}_3)\hat{t}_{k-1}$ with $\hat{t}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T})$,

$$\int_{\partial\widehat{T}} \hat{w}_3 \hat{n}_3 \hat{t}_k d\hat{s} = \frac{1}{\sqrt{2}} \int_{\widehat{E}_4} \hat{w}_3 (1 - \hat{x}_1 - \hat{x}_3)\hat{t}_{k-1} d\hat{s} - \int_{\widehat{E}_3} \hat{w}_3 \hat{t}_k d\hat{s} = 0.$$

Because $\frac{\partial \hat{u}_3}{\partial \hat{x}_3} = \frac{\partial \hat{v}_3}{\partial \hat{x}_3} = \frac{\partial \hat{w}_3}{\partial \hat{x}_3}$, We have

$$\begin{aligned} 0 &= \int_{\widehat{T}} \frac{\partial \hat{w}_3}{\partial \hat{x}_3} (1 - \hat{x}_1 - \hat{x}_3)\hat{t}_{k-1} d\hat{x} \\ &= \int_{\widehat{T}} \frac{\partial \hat{u}_3}{\partial \hat{x}_3} (1 - \hat{x}_1 - \hat{x}_3)\hat{t}_{k-1} d\hat{x} - \int_{\widehat{T}} \frac{\partial (\hat{x}_3 \hat{q}_2)}{\partial \hat{x}_3} (1 - \hat{x}_1 - \hat{x}_3)\hat{t}_{k-1} d\hat{x}. \end{aligned}$$

Setting $\hat{t}_{k-1} := \frac{\partial(\hat{x}_3\hat{q}_2)}{\partial\hat{x}_3}$ yields

$$\begin{aligned}\left\|\frac{\partial(\hat{x}_3\hat{q}_2)}{\partial\hat{x}_3}\right\|^2_{L^2(\widehat{T})} &\le c\int_{\widehat{T}}\left(\frac{\partial(\hat{x}_3\hat{q}_2)}{\partial\hat{x}_3}\right)^2(1-\hat{x}_1-\hat{x}_3)d\hat{x}\\ &= c\int_{\widehat{T}}\frac{\partial\hat{u}_3}{\partial\hat{x}_3}\frac{\partial(\hat{x}_3\hat{q}_2)}{\partial\hat{x}_3}(1-\hat{x}_1-\hat{x}_3)d\hat{x}\\ &\le c\|1-\hat{x}_1-\hat{x}_3\|_{L^\infty(\widehat{T})}\left\|\frac{\partial\hat{u}_3}{\partial\hat{x}_3}\right\|_{L^p(\widehat{T})}\left\|\frac{\partial(\hat{x}_3\hat{q}_2)}{\partial\hat{x}_3}\right\|_{L^{p'}(\widehat{T})}.\end{aligned}$$

Because all the norms on $\mathcal{P}^{k-1}(\widehat{T})$ are equivalent, we obtain

$$\left\|\frac{\partial(\hat{x}_3\hat{q}_2)}{\partial\hat{x}_3}\right\|_{L^p(\widehat{T})} \le c\left\|\frac{\partial\hat{u}_3}{\partial\hat{x}_3}\right\|_{L^p(\widehat{T})}. \tag{9.3.21}$$

By the analogous argument, we can prove

$$\left\|\frac{\partial(\hat{x}_2\hat{q}_1)}{\partial\hat{x}_3}\right\|_{L^p(\widehat{T})} \le c\left\|\frac{\partial\hat{u}_2}{\partial\hat{x}_2}\right\|_{L^p(\widehat{T})}. \tag{9.3.22}$$

The desired result follows from (9.3.20), (9.3.21) and (9.3.22).

Case for $i = 2$

From Lemma 9.3.5, if

$$\hat{v} := (\hat{u}_1 - \hat{u}_1(\hat{x}_2,\hat{x}_2,\hat{x}_3), \hat{u}_2, \hat{u}_3 - \hat{u}_3(\hat{x}_1,\hat{x}_2,0))^T,$$

it holds that

$$I_{\widehat{T}}^{RT^k}\hat{v} = I_{\widehat{T}}^{RT^k}\hat{u} - I_{\widehat{T}}^{RT^k}(\hat{u}_1(\hat{x}_2,\hat{x}_2,\hat{x}_3),0,0)^T - I_{\widehat{T}}^{RT^k}(0,0,\hat{u}_3(\hat{x}_1,\hat{x}_2,0))^T,$$

and thus, $(I_{\widehat{T}}^{RT^k}\hat{v})_2 = (I_{\widehat{T}}^{RT^k}\hat{u})_2$.

Let $k = 0$. Because $\hat{v}_1|_{\widehat{E}_1} = 0$ and $\hat{v}_3|_{\widehat{E}_3} = 0$, $I_{\widehat{T}}^{RT^0}\hat{v}$ is determined by the equations

$$\int_{\widehat{E}_1}\{-(I_{\widehat{T}}^{RT^0}\hat{v})_1 + (I_{\widehat{T}}^{RT^0}\hat{v})_2\}d\hat{s} = \int_{\widehat{E}_1}\hat{v}_2 d\hat{s}, \tag{9.3.23a}$$

$$\int_{\widehat{E}_2}(I_{\widehat{T}}^{RT^0}\hat{v})_2 d\hat{s} = \int_{\widehat{E}_2}\hat{v}_2 d\hat{s}, \tag{9.3.23b}$$

$$\int_{\widehat{E}_3}(I_{\widehat{T}}^{RT^0}\hat{v})_3 d\hat{s} = 0, \tag{9.3.23c}$$

$$\int_{\widehat{E}_4}\{(I_{\widehat{T}}^{RT^0}\hat{v})_1 + (I_{\widehat{T}}^{RT^0}\hat{v})_3\}d\hat{s} = \int_{\widehat{E}_4}(\hat{v}_1+\hat{v}_3)d\hat{s}. \tag{9.3.23d}$$

From the divergence formula and the definition of $\hat{v}$, we have

$$\int_{\widehat{T}} \widehat{\mathrm{div}}(\hat{v}_1, \hat{v}_2, \hat{v}_3)^T d\hat{x} = \int_{\partial \widehat{T}} (\hat{v}_1, \hat{v}_2, \hat{v}_3)^T \cdot \hat{n} d\hat{s}$$
$$= \frac{1}{\sqrt{2}} \int_{\widehat{E}_1} \hat{v}_2 d\hat{s} - \int_{\widehat{E}_2} \hat{v}_2 d\hat{s} + \frac{1}{\sqrt{2}} \int_{\widehat{E}_4} (\hat{v}_1 + \hat{v}_3) d\hat{s}, \tag{9.3.24}$$

Because $\hat{u}_2 = \hat{v}_2$, $\frac{\partial \hat{u}_i}{\partial \hat{x}_i} = \frac{\partial \hat{v}_i}{\partial \hat{x}_i}$ for $i = 1, 3$, $(I_{\widehat{T}}^{RT} \hat{u})_2 = (I_{\widehat{T}}^{RT} \hat{v})_2$, (9.3.23), (9.3.24), the definition of the Raviart–Thomas interpolation, and the trace theorem, we have

$$\begin{aligned}
\|(I_{\widehat{T}}^{RT^0} \hat{u})_2\|_{L^p(\widehat{T})} &= \|(I_{\widehat{T}}^{RT^0} \hat{v})_2\|_{L^p(\widehat{T})} \\
&\le \sum_{i=1}^{4} \left| \int_{\widehat{E}_i} \hat{v} \cdot \widehat{N}_i d\hat{s} \right| \|(\hat{\theta}_i)_2\|_{L^p(\widehat{T})} \\
&\le c \left| \frac{1}{\sqrt{2}} \int_{\widehat{E}_1} \hat{v}_2 d\hat{s} + \int_{\widehat{E}_2} \hat{v}_2 d\hat{s} + \frac{1}{\sqrt{2}} \int_{\widehat{E}_4} (\hat{v}_1 + \hat{v}_3) d\hat{s} \right| \\
&\le c \left(\|\hat{u}_2\|_{W^{1,p}(\widehat{T})} + \left\| \widehat{\mathrm{div}} \hat{u} \right\|_{L^p(\widehat{T})} \right),
\end{aligned}$$

which is the desired result for $k = 0$.

Let $k \ge 1$. Let $\hat{q}_i \in \mathcal{P}^{k-1}(\widehat{T})$, $i = 1, 3$, be such that, for any $\hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T})$,

$$\int_{\widehat{T}} \{\hat{v}_1 - (\hat{x}_2 - \hat{x}_1)\hat{q}_1\} \hat{p}_{k-1} d\hat{x} = 0, \tag{9.3.25a}$$

$$\int_{\widehat{T}} (\hat{v}_3 - \hat{x}_3 \hat{q}_3) \hat{p}_{k-1} d\hat{x} = 0. \tag{9.3.25b}$$

We set

$$\hat{w} := (\hat{v}_1 - (\hat{x}_2 - \hat{x}_1)\hat{q}_1, \hat{v}_2, \hat{v}_3 - \hat{x}_3 \hat{q}_3)^T.$$

Because $((\hat{x}_2 - \hat{x}_1)\hat{q}_1, 0, \hat{x}_3 \hat{q}_3)^T \in RT^k(\widehat{T})$, it holds that

$$I_{\widehat{T}}^{RT^k} \hat{w} = I_{\widehat{T}}^{RT^k} \hat{v} - I_{\widehat{T}}^{RT^k} ((\hat{x}_2 - \hat{x}_1)\hat{q}_1, 0, \hat{x}_3 \hat{q}_3)^T = I_{\widehat{T}}^{RT^k} \hat{v} - ((\hat{x}_2 - \hat{x}_1)\hat{q}_1, 0, \hat{x}_3 \hat{q}_3)^T,$$

and thus, $(I_{\widehat{T}}^{RT^k} \hat{w})_2 = (I_{\widehat{T}}^{RT^k} \hat{v})_2$, therefore $(I_{\widehat{T}}^{RT^k} \hat{u})_2 = (I_{\widehat{T}}^{RT^k} \hat{w})_2$.

Because $\hat{w}_1|_{\widehat{E}_1} = 0$, $\hat{w}_3|_{\widehat{E}_3} = 0$, and (9.3.25), $I_{\widehat{T}}^{RT^k}\hat{w}$ is determined as follows:

$$\int_{\widehat{E}_1} \{-(I_{\widehat{T}}^{RT^k}\hat{v})_1 + (I_{\widehat{T}}^{RT^k}\hat{v})_2\}\hat{p}_k d\hat{s} = \int_{\widehat{E}_1} \hat{w}_2\hat{p}_k d\hat{s} \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_1), \quad (9.3.26a)$$

$$\int_{\widehat{E}_2} (I_{\widehat{T}}^{RT^k}\hat{w})_2\hat{p}_k d\hat{s} = \int_{\widehat{E}_2} \hat{w}_2\hat{p}_k d\hat{s} \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_2), \quad (9.3.26b)$$

$$\int_{\widehat{E}_3} (I_{\widehat{T}}^{RT^k}\hat{w})_3\hat{p}_k d\hat{s} = 0 \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_3), \quad (9.3.26c)$$

$$\int_{\widehat{E}_4} \{(I_{\widehat{T}}^{RT^k}\hat{v})_1 + (I_{\widehat{T}}^{RT^k}\hat{v})_3\}\hat{p}_k d\hat{s} = \int_{\widehat{E}_4} (\hat{w}_1 + \hat{w}_3)\hat{p}_k d\hat{s} \quad \forall \hat{p}_k \in \mathcal{P}^k(\widehat{E}_4), \quad (9.3.26d)$$

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k}\hat{w})_1\hat{p}_{k-1} d\hat{x} = 0 \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}), \quad (9.3.26e)$$

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k}\hat{w})_2\hat{p}_{k-1} d\hat{x} = \int_{\widehat{T}} \hat{w}_2\hat{p}_{k-1} d\hat{x} \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}), \quad (9.3.26f)$$

$$\int_{\widehat{T}} (I_{\widehat{T}}^{RT^k}\hat{w})_3\hat{p}_{k-1} d\hat{x} = 0 \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}). \quad (9.3.26g)$$

For $\hat{r}_k \in \mathcal{P}^k(\widehat{T})$, using (9.3.25), the Green formula and the definition of $\hat{w}$ yields

$$\begin{aligned}
&\int_{\widehat{T}} \widehat{\text{div}}(\hat{w}_1, \hat{w}_2, \hat{w}_3)^T \hat{r}_k d\hat{x} \\
&= \int_{\partial\widehat{T}} (\hat{w}_1, \hat{w}_2, \hat{w}_3)^T \cdot \hat{n}\hat{r}_k d\hat{s} - \int_{\widehat{T}} ((\hat{w}_1, \hat{w}_2, \hat{w}_3)^T \cdot \widehat{\nabla})\hat{r}_k d\hat{x} \\
&= \frac{1}{\sqrt{2}} \int_{\widehat{E}_1} \hat{w}_2\hat{r}_k d\hat{s} - \int_{\widehat{E}_2} \hat{w}_2\hat{r}_k d\hat{s} + \frac{1}{\sqrt{2}} \int_{\widehat{E}_4} (\hat{w}_1 + \hat{w}_3)\hat{r}_k d\hat{s} - \int_{\widehat{T}} \hat{w}_2 \frac{\partial \hat{r}_k}{\partial \hat{x}_2} d\hat{x},
\end{aligned} \quad (9.3.27)$$

Furthermore, we have

$$\widehat{\text{div}}\hat{w} = \widehat{\text{div}}\hat{v} - \widehat{\text{div}}((\hat{x}_2 - \hat{x}_1)\hat{q}_1, 0, \hat{x}_3\hat{q}_3)^T = \widehat{\text{div}}\hat{u} - \widehat{\text{div}}((\hat{x}_2 - \hat{x}_1)\hat{q}_1, 0, \hat{x}_3\hat{q}_3)^T. \quad (9.3.28)$$

Because $\hat{u}_2 = \hat{w}_2$, $\frac{\partial \hat{u}_i}{\partial \hat{x}_i} = \frac{\partial \hat{v}_i}{\partial \hat{x}_i}$ for $i = 1, 3$, $(I_{\widehat{T}}^{RT^k}\hat{u})_2 = (I_{\widehat{T}}^{RT^k}\hat{w})_2$, (9.3.26), (9.3.27), (9.3.28), the definition of the Raviart–Thomas interpolation, and the trace theorem, we have

$$\begin{aligned}
&\|(I_{\widehat{T}}^{RT^k}\hat{u})_2\|_{L^p(\widehat{T})} = \|(I_{\widehat{T}}^{RT^k}\hat{w})_2\|_{L^p(\widehat{T})} \\
&\quad \le c\left| \frac{1}{\sqrt{2}}\int_{\widehat{E}_1}\hat{w}_2 d\hat{s} + \int_{\widehat{E}_2}\hat{w}_2 d\hat{s} + \frac{1}{\sqrt{2}}\int_{\widehat{E}_4}(\hat{w}_1+\hat{w}_3)d\hat{s}\right| \\
&\quad \le c\left(\|\hat{u}_2\|_{W^{1,p}(\widehat{T})} + \left\|\widehat{\mathrm{div}}\hat{u}\right\|_{L^p(\widehat{T})} + \left\|\widehat{\mathrm{div}}((\hat{x}_2-\hat{x}_1)\hat{q}_1, 0, \hat{x}_3\hat{q}_3)^T\right\|_{L^p(\widehat{T})}\right).
\end{aligned} \tag{9.3.29}$$

For any $\hat{s}_k \in \mathcal{P}^k(\widehat{T})$, from (9.3.25) and the Green formula, we have

$$\begin{aligned}
0 &= \int_{\widehat{T}} \{(\hat{v}_1 - (\hat{x}_2-\hat{x}_1)\hat{q}_1, 0, \hat{v}_3 - \hat{x}_3\hat{q}_3)^T \cdot \widehat{\nabla}\}\hat{s}_k d\hat{x} \\
&= \int_{\partial\widehat{T}} \{(\hat{v}_1 - (\hat{x}_2-\hat{x}_1)\hat{q}_1, 0, \hat{v}_3 - \hat{x}_3\hat{q}_3)^T \cdot \hat{n}\}\hat{s}_k d\hat{s} \\
&\quad - \int_{\widehat{T}} \widehat{\mathrm{div}}(\hat{v}_1 - (\hat{x}_2-\hat{x}_1)\hat{q}_1, 0, \hat{v}_3 - \hat{x}_3\hat{q}_3)^T \hat{s}_k d\hat{x}.
\end{aligned}$$

Setting

$$\hat{s}_k(\hat{x}) := (1-\hat{x}_1-\hat{x}_3)\hat{t}_{k-1}, \quad \hat{t}_{k-1} \in \mathcal{P}^{k-1}(\widehat{T}),$$

it holds that

$$\begin{aligned}
&\int_{\widehat{T}} (1-\hat{x}_1-\hat{x}_3)\widehat{\mathrm{div}}(\hat{v}_1, 0, \hat{v}_3)^T \hat{t}_{k-1} d\hat{x} \\
&\quad = \int_{\widehat{T}} (1-\hat{x}_1-\hat{x}_3)\widehat{\mathrm{div}}((\hat{x}_2-\hat{x}_1)\hat{q}_1, 0, \hat{x}_3\hat{q}_3)^T \hat{t}_{k-1} d\hat{x},
\end{aligned}$$

because $\hat{s}_k = 0$ on $\widehat{E}_4$, $\{\hat{v}_1 - (\hat{x}_2-\hat{x}_1)\hat{q}_1\}\hat{n}_1 = 0$ on $\partial\widehat{T}\backslash\widehat{E}_4$, and $(\hat{v}_3 - \hat{x}_3\hat{q}_3)\hat{n}_3 = 0$ on $\partial\widehat{T}\backslash\widehat{E}_4$.

Therefore, setting $\hat{t}_{k-1} := \widehat{\mathrm{div}}((\hat{x}_2-\hat{x}_1)\hat{q}_1, 0, \hat{x}_3\hat{q}_3)^T$, and using the Hölder inequality, we have

$$\begin{aligned}
&\left\|\widehat{\mathrm{div}}((\hat{x}_2-\hat{x}_1)\hat{q}_1, 0, \hat{x}_3\hat{q}_3)^T\right\|_{L^2(\widehat{T})}^2 \\
&\le c\int_{\widehat{T}} (1-\hat{x}_1-\hat{x}_3)\left|\widehat{\mathrm{div}}((\hat{x}_2-\hat{x}_1)\hat{q}_1, 0, \hat{x}_3\hat{q}_3)^T\right|^2 d\hat{x} \\
&\le c\|1-\hat{x}_1-\hat{x}_3\|_{L^\infty(\widehat{T})}\left\|\widehat{\mathrm{div}}(\hat{v}_1, 0, \hat{v}_3)^T\right\|_{L^p(\widehat{T})}\left\|\widehat{\mathrm{div}}((\hat{x}_2-\hat{x}_1)\hat{q}_1, 0, \hat{x}_3\hat{q}_3)^T\right\|_{L^{p'}(\widehat{T})}.
\end{aligned}$$

Because all the norms on $\mathcal{P}^{k-1}(\widehat{T})$ are equivalent, we obtain

$$\left\|\widehat{\mathrm{div}}((\hat{x}_2-\hat{x}_1)\hat{q}_1, 0, \hat{x}_3\hat{q}_3)^T\right\|_{L^p(\widehat{T})} \le c\|\widehat{\mathrm{div}}(\hat{v}_1, 0, \hat{v}_3)^T\|_{L^p(\widehat{T})}. \tag{9.3.30}$$

Observing that $\widehat{\text{div}}(\hat{v}_1, 0, \hat{v}_3)^T = \widehat{\text{div}}(\hat{u}_1, 0, \hat{u}_3)^T$, and

$$\begin{aligned}\|\widehat{\text{div}}(\hat{u}_1, 0, \hat{u}_3)^T\|_{L^p(\widehat{T})} &\le \|\widehat{\text{div}}\hat{u}\|_{L^p(\widehat{T})} + \|\widehat{\text{div}}(0, \hat{u}_2, 0)^T\|_{L^p(\widehat{T})} \\ &\le \|\widehat{\text{div}}\hat{u}\|_{L^p(\widehat{T})} + \|\hat{u}_2\|_{W^{1,p}(\widehat{T})},\end{aligned}$$

the target estimate (9.3.11b) follows from (9.3.30). □

9.4 Scaling Argument

We present estimates related to the scaling argument corresponding to [30, Lemma 1.113].

Note 9.4.1. We use the following calculations in (9.4.2). Let $\hat{v} \in \mathcal{C}^{\ell+1}(\widehat{T})^d$ with $\tilde{v} = \widehat{\Psi}\hat{v}$, $v^s = \widetilde{\Psi}\tilde{v}$ and $v = \Psi_{T^s} v^s$.

Using the definition of Piola transformations (Definition 3.4.2) yields, for $1 \le i, k \le d$,

$$\begin{aligned}\frac{\partial \hat{v}_k}{\partial \hat{x}_i} &= \det(\widehat{\mathcal{A}}^{(d)}) h_k^{-1} h_i \frac{\partial \tilde{v}_k}{\partial \tilde{x}_i}, \\ \frac{\partial \tilde{v}_k}{\partial \tilde{x}_i} &= \det(\widetilde{\mathcal{A}}) \sum_{\eta=1}^d [\widetilde{\mathcal{A}}^{-1}]_{k\eta} \sum_{i_1^{(1)}=1}^d \frac{\partial v_\eta^s}{\partial x_{i_1^{(1)}}^s} \frac{\partial x_{i_1^{(1)}}^s}{\partial \tilde{x}_i} = \det(\widetilde{\mathcal{A}}) \sum_{\eta=1}^d [\widetilde{\mathcal{A}}^{-1}]_{k\eta} \sum_{i_1^{(1)}=1}^d \frac{\partial v_\eta^s}{\partial x_{i_1^{(1)}}^s} [\widetilde{\mathcal{A}}]_{i_1^{(1)} i},\end{aligned}$$

$$\frac{\partial v_\eta^s}{\partial x_{i_1^{(1)}}^s} = \det(\mathcal{A}_T) \sum_{\nu=1}^d [\mathcal{A}_T^{-1}]_{\eta\nu} \sum_{i_1^{(0,1)}=1}^d \frac{\partial v_\nu}{\partial x_{i_1^{(0,1)}}} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}},$$

which leads to

$$\frac{\partial \hat{v}_k}{\partial \hat{x}_i} = \det(\mathcal{A}^s) \det(\mathcal{A}_T) h_k^{-1} \sum_{\eta,\nu=1}^d [\widetilde{\mathcal{A}}^{-1}]_{k\eta} [\mathcal{A}_T^{-1}]_{\eta\nu} \sum_{i_1^{(1)}, i_1^{(0,1)}=1}^d h_i [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} \frac{\partial v_\nu}{\partial x_{i_1^{(0,1)}}}.$$

By an analogous calculation, for $1 \le i, j, k \le d$,

$$\begin{aligned}\frac{\partial^2 \hat{v}_k}{\partial \hat{x}_i \partial \hat{x}_j} &= \det(\widehat{\mathcal{A}}^{(d)}) h_k^{-1} h_i h_j \frac{\partial^2 \tilde{v}_k}{\partial \tilde{x}_i \partial \tilde{x}_j}, \\ \frac{\partial^2 \tilde{v}_k}{\partial \tilde{x}_i \partial \tilde{x}_j} &= \det(\widetilde{\mathcal{A}}) \sum_{\eta=1}^d [\widetilde{\mathcal{A}}^{-1}]_{k\eta} \sum_{i_1^{(1)}, j_1^{(1)}=1}^d \frac{\partial^2 v_\eta^s}{\partial x_{i_1^{(1)}}^s \partial x_{j_1^{(1)}}^s} [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} [\widetilde{\mathcal{A}}]_{j_1^{(1)} j},\end{aligned}$$

$$\frac{\partial^2 v_\eta^s}{\partial x_{i_1^{(1)}}^s \partial x_{j_1^{(1)}}^s} = \det(\mathcal{A}_T) \sum_{\nu=1}^d [\mathcal{A}_T^{-1}]_{\eta\nu} \sum_{i_1^{(0,1)}, j_1^{(0,1)}=1}^d \frac{\partial^2 v_\nu}{\partial x_{i_1^{(0,1)}} \partial x_{j_1^{(0,1)}}} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} [\mathcal{A}_T]_{j_1^{(0,1)} j_1^{(1)}},$$

which leads to

$$\frac{\partial^2 \hat{v}_k}{\partial \hat{x}_i \partial \hat{x}_j} = \det(\mathcal{A}^s)\det(\mathcal{A}_T) h_k^{-1} \sum_{\eta,\nu=1}^{d} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}[\mathcal{A}_T^{-1}]_{\eta\nu}$$

$$\sum_{i_1^{(1)}, i_1^{(0,1)}=1}^{d} h_i [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} \sum_{j_1^{(1)}, j_1^{(0,1)}=1}^{d} h_j [\widetilde{\mathcal{A}}]_{j_1^{(1)} j} [\mathcal{A}_T]_{j_1^{(0,1)} j_1^{(1)}} \frac{\partial^2 v_\nu}{\partial x_{i_1^{(0,1)}} \partial x_{j_1^{(0,1)}}}.$$

For any multi-indices β and γ, for $1 \le k \le d$,

$$\partial_{\hat{x}}^{\beta+\gamma} \hat{v}_k = \frac{\partial^{|\beta|+|\gamma|}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d} \partial \hat{x}_1^{\gamma_1} \cdots \partial \hat{x}_d^{\gamma_d}} \hat{v}_k$$

$$= \det(\mathcal{A}^s)\det(\mathcal{A}_T) h_k^{-1} \sum_{\eta,\nu=1}^{d} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}[\mathcal{A}_T^{-1}]_{\eta\nu}$$

$$\underbrace{\sum_{i_1^{(1)}, i_1^{(0,1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_1^{(1)} 1} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} \cdots \sum_{i_{\beta_1}^{(1)}, i_{\beta_1}^{(0,1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_{\beta_1}^{(1)} 1} [\mathcal{A}_T]_{i_{\beta_1}^{(0,1)} i_{\beta_1}^{(1)}}}_{\beta_1 \text{times}} \cdots$$

$$\underbrace{\sum_{i_1^{(d)}, i_1^{(0,d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_1^{(d)} d} [\mathcal{A}_T]_{i_1^{(0,d)} i_1^{(d)}} \cdots \sum_{i_{\beta_d}^{(d)}, i_{\beta_d}^{(0,d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_{\beta_d}^{(d)} d} [\mathcal{A}_T]_{i_{\beta_d}^{(0,d)} i_{\beta_d}^{(d)}}}_{\beta_d \text{times}}$$

$$\underbrace{\sum_{j_1^{(1)}, j_1^{(0,1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{j_1^{(1)} 1} [\mathcal{A}_T]_{j_1^{(0,1)} j_1^{(1)}} \cdots \sum_{j_{\gamma_1}^{(1)}, j_{\gamma_1}^{(0,1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{j_{\gamma_1}^{(1)} 1} [\mathcal{A}_T]_{j_{\gamma_1}^{(0,1)} j_{\gamma_1}^{(1)}}}_{\gamma_1 \text{times}} \cdots$$

$$\underbrace{\sum_{j_1^{(d)}, j_1^{(0,d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{j_1^{(d)} d} [\mathcal{A}_T]_{j_1^{(0,d)} j_1^{(d)}} \cdots \sum_{j_{\gamma_d}^{(d)}, j_{\gamma_d}^{(0,d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{j_{\gamma_d}^{(d)} d} [\mathcal{A}_T]_{j_{\gamma_d}^{(0,d)} j_{\gamma_d}^{(d)}}}_{\gamma_d \text{times}}$$

$$\underbrace{\frac{\partial^{\beta_1}}{\partial x_{i_1^{(0,1)}} \cdots \partial x_{i_{\beta_1}^{(0,1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_d}}{\partial x_{i_1^{(0,d)}} \cdots \partial x_{i_{\beta_d}^{(0,d)}}}}_{\beta_d \text{times}} \underbrace{\frac{\partial^{\gamma_1}}{\partial x_{j_1^{(0,1)}} \cdots \partial x_{j_{\gamma_1}^{(0,1)}}}}_{\gamma_1 \text{times}} \cdots \underbrace{\frac{\partial^{\gamma_d}}{\partial x_{j_1^{(0,d)}} \cdots \partial x_{j_{\gamma_d}^{(0,d)}}}}_{\gamma_d \text{times}} v_\nu.$$

Using (1.3.1), (3.6.1c) and (3.6.2), we have, for $1 \le i, k \le d$,

$$\left|\frac{\partial \hat{v}_k}{\partial \hat{x}_i}\right| \le |\det(\mathcal{A}^s)||\det(\mathcal{A}_T)| h_k^{-1} \sum_{\eta,\nu=1}^{d} |[\widetilde{\mathcal{A}}^{-1}]_{k\eta}||[\mathcal{A}_T^{-1}]_{\eta\nu}|$$

$$h_i \left| \sum_{i_1^{(1)}, i_1^{(0,1)}=1}^{d} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} (r_i)_{i_1^{(1)}} \frac{\partial v_\nu}{\partial x_{i_1^{(0,1)}}} \right|$$
$$\leq |\det(\mathcal{A}^s)| h_k^{-1} \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{\nu=1}^{d} h_i \left| \frac{\partial v_\nu}{\partial r_i} \right|,$$

and, for $1 \leq i, j, k \leq d$,

$$\left| \frac{\partial^2 \hat{v}_k}{\partial \hat{x}_i \partial \hat{x}_j} \right| \leq |\det(\mathcal{A}^s)| |\det(\mathcal{A}_T)| h_k^{-1} \sum_{\eta,\nu=1}^{d} |[\widetilde{\mathcal{A}}^{-1}]_{k\eta}| |[\mathcal{A}_T^{-1}]_{\eta\nu}|$$
$$h_i h_j \left| \sum_{i_1^{(1)}, i_1^{(0,1)}=1}^{d} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} (r_i)_{i_1^{(1)}} \sum_{j_1^{(1)}, j_1^{(0,1)}=1}^{d} [\mathcal{A}_T]_{j_1^{(0,1)} j_1^{(1)}} (r_j)_{j_1^{(1)}} \frac{\partial^2 v_\nu}{\partial x_{i_1^{(0,1)}} \partial x_{j_1^{(0,1)}}} \right|$$
$$\leq |\det(\mathcal{A}^s)| h_k^{-1} \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{\nu=1}^{d} h_i h_j \left| \frac{\partial^2 v_\nu}{\partial r_i \partial r_j} \right|,$$

Note 9.4.2. We use the following calculations in (9.4.3). Let $\hat{v} \in \mathcal{C}^{\ell+1}(\widehat{T})^d$ with $\tilde{v} = \widehat{\Psi}\hat{v}$ and $v^s = \widetilde{\Psi}\tilde{v}$.

Using the definition of Piola transformations (Definition 3.4.2) yields, for $1 \leq i, k \leq d$,

$$\frac{\partial \hat{v}_k}{\partial \hat{x}_i} = \det(\widehat{\mathcal{A}}^{(d)}) h_k^{-1} h_i \frac{\partial \tilde{v}_k}{\partial \tilde{x}_i},$$
$$\frac{\partial \tilde{v}_k}{\partial \tilde{x}_i} = \det(\widetilde{\mathcal{A}}) \sum_{\eta=1}^{d} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} \sum_{i_1^{(1)}=1}^{d} \frac{\partial v_\eta^s}{\partial x_{i_1^{(1)}}^s} [\widetilde{\mathcal{A}}]_{i_1^{(1)} i},$$

which leads to

$$\frac{\partial \hat{v}_k}{\partial \hat{x}_i} = \det(\mathcal{A}^s) h_k^{-1} \sum_{\eta=1}^{d} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} \sum_{i_1^{(1)}=1}^{d} h_i [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} \frac{\partial v_\eta^s}{\partial x_{i_1^{(1)}}^s}.$$

By an analogous calculation, for $1 \leq i, j, k \leq d$,

$$\frac{\partial^2 \hat{v}_k}{\partial \hat{x}_i \partial \hat{x}_j} = \det(\widehat{\mathcal{A}}^{(d)}) h_k^{-1} h_i h_j \frac{\partial^2 \tilde{v}_k}{\partial \tilde{x}_i \partial \tilde{x}_j},$$
$$\frac{\partial^2 \tilde{v}_k}{\partial \tilde{x}_i \partial \tilde{x}_j} = \det(\widetilde{\mathcal{A}}) \sum_{\eta=1}^{d} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} \sum_{i_1^{(1)}, j_1^{(1)}=1}^{d} \frac{\partial^2 v_\eta^s}{\partial x_{i_1^{(1)}}^s \partial x_{j_1^{(1)}}^s} [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} [\widetilde{\mathcal{A}}]_{j_1^{(1)} j},$$

which leads to

$$\frac{\partial^2 \hat{v}_k}{\partial \hat{x}_i \partial \hat{x}_j} = \det(\mathcal{A}^s) h_k^{-1} \sum_{\eta=1}^{d} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} \sum_{i_1^{(1)}=1}^{d} h_i [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} \sum_{j_1^{(1)}=1}^{d} h_j [\widetilde{\mathcal{A}}]_{j_1^{(1)} j} \frac{\partial^2 v_\eta^s}{\partial x_{i_1^{(1)}} \partial x_{j_1^{(1)}}}.$$

For any multi-indices β and γ, for $1 \le k \le d$,

$$\partial_{\hat{x}}^{\beta+\gamma} \hat{v}_k = \frac{\partial^{|\beta|+|\gamma|}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d} \partial \hat{x}_1^{\gamma_1} \cdots \partial \hat{x}_d^{\gamma_d}} \hat{v}_k$$

$$= \det(\mathcal{A}^s) h_k^{-1} \sum_{\eta=1}^{d} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}$$

$$\underbrace{\sum_{i_1^{(1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_1^{(1)} 1} \cdots \sum_{i_{\beta_1}^{(1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_{\beta_1}^{(1)} 1}}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_1^{(d)} d} \cdots \sum_{i_{\beta_d}^{(d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_{\beta_d}^{(d)} d}}_{\beta_d \text{times}}$$

$$\underbrace{\sum_{j_1^{(1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{j_1^{(1)} 1} \cdots \sum_{j_{\gamma_1}^{(1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{j_{\gamma_1}^{(1)} 1}}_{\gamma_1 \text{times}} \cdots \underbrace{\sum_{j_1^{(d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{j_1^{(d)} d} \cdots \sum_{j_{\gamma_d}^{(d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{j_{\gamma_d}^{(d)} d}}_{\gamma_d \text{times}}$$

$$\frac{\partial^{\beta_1}}{\underbrace{\partial x^s_{i_1^{(1)}} \cdots \partial x^s_{i_{\beta_1}^{(1)}}}_{\beta_1 \text{times}}} \cdots \frac{\partial^{\beta_d}}{\underbrace{\partial x^s_{i_1^{(d)}} \cdots \partial x^s_{i_{\beta_d}^{(d)}}}_{\beta_d \text{times}}} \frac{\partial^{\gamma_1}}{\underbrace{\partial x^s_{j_1^{(1)}} \cdots \partial x^s_{j_{\gamma_1}^{(1)}}}_{\gamma_1 \text{times}}} \cdots \frac{\partial^{\gamma_d}}{\underbrace{\partial x^s_{j_1^{(d)}} \cdots \partial x^s_{j_{\gamma_d}^{(d)}}}_{\gamma_d \text{times}}} v_\eta^s.$$

Using (1.3.1), we have, for $1 \le i, k \le d$,

$$\left| \frac{\partial \hat{v}_k}{\partial \hat{x}_i} \right| \le |\det(\mathcal{A}^s)| h_k^{-1} \sum_{\eta=1}^{d} |[\widetilde{\mathcal{A}}^{-1}]_{k\eta}| \sum_{i_1^{(1)}=1}^{d} h_i |[\widetilde{\mathcal{A}}]_{i_1^{(1)} i}| \left| \frac{\partial v_\eta^s}{\partial x^s_{i_1^{(1)}}} \right|$$

$$\le c |\det(\mathcal{A}^s)| h_k^{-1} \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{\eta=1}^{d} \sum_{i_1^{(1)}=1}^{d} \mathscr{H}_{i_1^{(1)}} \left| \frac{\partial v_\eta^s}{\partial x^s_{i_1^{(1)}}} \right|,$$

and, for $1 \le i, j, k \le d$,

$$\left| \frac{\partial^2 \hat{v}_k}{\partial \hat{x}_i \partial \hat{x}_j} \right| \le |\det(\mathcal{A}^s)| h_k^{-1} \sum_{\eta=1}^{d} |[\widetilde{\mathcal{A}}^{-1}]_{k\eta}| \sum_{i_1^{(1)}=1}^{d} h_i |[\widetilde{\mathcal{A}}]_{i_1^{(1)} i}| \sum_{j_1^{(1)}=1}^{d} h_j |[\widetilde{\mathcal{A}}]_{j_1^{(1)} j}| \left| \frac{\partial^2 v_\eta^s}{\partial x_{i_1^{(1)}} \partial x_{j_1^{(1)}}} \right|$$

$$\le c |\det(\mathcal{A}^s)| h_k^{-1} \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{\eta=1}^{d} \sum_{i_1^{(1)}, j_1^{(1)}=1}^{d} \mathscr{H}_{i_1^{(1)}} \mathscr{H}_{j_1^{(1)}} \left| \frac{\partial^2 v_\eta^s}{\partial x_{i_1^{(1)}} \partial x_{j_1^{(1)}}} \right|.$$

Lemma 9.4.3. *Let $p \in [1,\infty)$. Let $T^s \in \mathfrak{T}^{(d)}$ satisfy Condition 3.2.1 or Condition 3.2.2. Let $T \subset \mathbb{R}^d$ be a simplex such that $T^s = \Phi_{T^s}^{-1}(T)$. It holds that, for any $\hat{v} = (\hat{v}_1,\ldots,\hat{v}_d)^T \in L^p(\widehat{T})^d$ with $\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_d)^T := \widehat{\Psi}\hat{v}$ and $v^s = (v_1^s,\ldots,v_d^s)^T := \widetilde{\Psi}\tilde{v}$,*

$$\|v^s\|_{L^p(T^s)^d} \leq c|\det(\mathcal{A}^s)|^{\frac{1-p}{p}} \|\widetilde{\mathcal{A}}\|_2 \left(\sum_{j=1}^d h_j^p \|\hat{v}_j\|^p_{L^p(\widehat{T})} \right)^{1/p}. \tag{9.4.1}$$

Let $\ell, m \in \mathbb{N}_0$ and $k \in \mathbb{N}$ with $1 \leq k \leq d$. Let $\beta := (\beta_1,\ldots,\beta_d) \in \mathbb{N}_0^d$ and $\gamma := (\gamma_1,\ldots,\gamma_d) \in \mathbb{N}_0^d$ be multi-indices with $|\beta| = \ell$ and $|\gamma| = m$, respectively. It then holds that, for any $\hat{v} = (\hat{v}_1,\ldots,\hat{v}_d)^T \in W^{|\beta|+|\gamma|,p}(\widehat{T})^d$ with $\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_d)^T := \widehat{\Psi}\hat{v}$, $v^s = (v_1^s,\ldots,v_d^s)^T := \widetilde{\Psi}\tilde{v}$ and $v = (v_1,\ldots,v_d)^T := \Psi_{T^s} v^s$,

$$\left\| \partial_{\hat{x}}^{\beta} \partial_{\hat{x}}^{\gamma} \hat{v}_k \right\|_{L^p(\widehat{T})} \leq c|\det(\mathcal{A}^s)|^{\frac{p-1}{p}} h_k^{-1} \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{|\varepsilon|=\ell+m} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^d}. \tag{9.4.2}$$

If Condition 3.3.1 is imposed, it holds that, for any $\hat{v} = (\hat{v}_1,\ldots,\hat{v}_d)^T \in W^{|\beta|+|\gamma|,p}(\widehat{T})^d$ with $\tilde{v} = (\tilde{v}_1,\ldots,\tilde{v}_d)^T := \widehat{\Psi}\hat{v}$, $v^s = (v_1^s,\ldots,v_d^s)^T := \widetilde{\Psi}\tilde{v}$ and $v = (v_1,\ldots,v_d)^T := \Psi_{T^s} v^s$,

$$\left\| \partial_{\hat{x}}^{\beta} \partial_{\hat{x}}^{\gamma} \hat{v}_k \right\|_{L^p(\widehat{T})} \leq c|\det(\mathcal{A}^s)|^{\frac{p-1}{p}} h_k^{-1} \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{|\varepsilon|=\ell+m} \mathscr{H}^{\varepsilon} \|\partial_{x^s}^{\varepsilon} (\Psi_{T^s}^{-1} v)\|_{L^p(\Phi_{T^s}^{-1}(T))^d}. \tag{9.4.3}$$

Proof. We divide the proof into three parts.

Proof of (9.4.1)

Because the space $\mathcal{C}(\widehat{T})^d$ is dense in the space $L^p(\widehat{T})^d$, we show (9.4.1) for $\hat{v} \in \mathcal{C}(\widehat{T})^d$ with $\tilde{v} = \widehat{\Psi}\hat{v}$ and $v^s = \widetilde{\Psi}\tilde{v}$. From the definition of the Piola transformation, for $i = 1,\ldots,d$,

$$\tilde{v}_i(\tilde{x}) = \frac{1}{\det(\widehat{\mathcal{A}}^{(d)})} h_i \hat{v}_i(\hat{x}), \quad v_i^s(x) = \frac{1}{\det(\widetilde{\mathcal{A}})} \sum_{j=1}^d \widetilde{\mathcal{A}}_{ij} \tilde{v}_j(\tilde{x}),$$

where $\hat{x}_j = h_j^{-1}\tilde{x}_j$. This leads to

$$v_i^s(x^s) = \frac{1}{\det(\widetilde{\mathcal{A}}) \det(\widehat{\mathcal{A}}^{(d)})} \sum_{j=1}^d \widetilde{\mathcal{A}}_{ij} h_j \hat{v}_j(\hat{x}).$$

If $1 \le p < \infty$, for $i = 1, \dots, d$,

$$\|v^s\|^p_{L^p(T^s)^d} = \sum_{i=1}^d \|v_i^s\|^p_{L^p(T^s)} \le c|\det(\mathcal{A}^s)|^{1-p} \|\widetilde{\mathcal{A}}\|_2^p \sum_{j=1}^d h_j^p \|\hat{v}_j\|^p_{L^p(\widehat{T})},$$

which leads to (9.4.1) together with (1.3.1) and (3.6.2).

Proof of (9.4.3)

Because the space $\mathcal{C}^{\ell+m}(\widehat{T})^d$ is dense in the space $W^{\ell+m,p}(\widehat{T})^d$, we show (9.4.3) for $\hat{v} \in \mathcal{C}^{\ell+m}(\widehat{T})^d$ with $\tilde{v} = \widehat{\Psi}\hat{v}$, $v^s = \widetilde{\Psi}\tilde{v}$ and $v = \Psi_{T^s}\Psi v^s$. Using (1.3.1), through a simple calculation, we have, for $1 \le k \le d$,

$$\begin{aligned}
|\partial_{\hat{x}}^{\beta+\gamma}\hat{v}_k| &= \left| \frac{\partial^{|\beta|+|\gamma|}}{\partial \hat{x}_1^{\beta_1}\cdots\partial\hat{x}_d^{\beta_d}\partial\hat{x}_1^{\gamma_1}\cdots\partial\hat{x}_d^{\gamma_d}} \hat{v}_k \right| \\
&\le c|\det(\mathcal{A}^s)| \|\widetilde{\mathcal{A}}^{-1}\|_2 h_k^{-1} \\
&\quad \sum_{\eta=1}^d \underbrace{\sum_{i_1^{(1)}=1}^d \cdots \sum_{i_{\beta_1}^{(1)}=1}^d}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(d)}=1}^d \cdots \sum_{i_{\beta_d}^{(d)}=1}^d}_{\beta_d \text{times}} \underbrace{\sum_{j_1^{(1)}=1}^d \cdots \sum_{j_{\gamma_1}^{(1)}=1}^d}_{\gamma_1 \text{times}} \cdots \underbrace{\sum_{j_1^{(d)}=1}^d \cdots \sum_{j_{\gamma_d}^{(d)}=1}^d}_{\gamma_d \text{times}} \\
&\quad \underbrace{\mathscr{H}_{i_1^{(1)}} \cdots \mathscr{H}_{i_{\varepsilon_1}^{(1)}}}_{\beta_1 \text{times}} \cdots \underbrace{\mathscr{H}_{i_1^{(d)}} \cdots \mathscr{H}_{i_{\varepsilon_d}^{(d)}}}_{\beta_d \text{times}} \underbrace{\mathscr{H}_{j_1^{(1)}} \cdots \mathscr{H}_{j_{\varepsilon_1}^{(1)}}}_{\gamma_1 \text{times}} \cdots \underbrace{\mathscr{H}_{j_1^{(d)}} \cdots \mathscr{H}_{j_{\varepsilon_d}^{(d)}}}_{\gamma_d \text{times}} \\
&\quad \left| \underbrace{\frac{\partial^{\beta_1}}{\partial x^s_{i_1^{(1)}} \cdots \partial x^s_{i_{\beta_1}^{(1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_d}}{\partial x^s_{i_1^{(d)}} \cdots \partial x^s_{i_{\beta_d}^{(d)}}}}_{\beta_d \text{times}} \underbrace{\frac{\partial^{\gamma_1}}{\partial x^s_{j_1^{(1)}} \cdots \partial x^s_{j_{\gamma_1}^{(1)}}}}_{\gamma_1 \text{times}} \cdots \underbrace{\frac{\partial^{\gamma_d}}{\partial x^s_{j_1^{(d)}} \cdots \partial x^s_{j_{\gamma_d}^{(d)}}}}_{\gamma_d \text{times}} v_\eta^s \right| \\
&\le c|\det(\mathcal{A}^s)| \|\widetilde{\mathcal{A}}^{-1}\|_2 h_k^{-1} \sum_{\eta=1}^d \sum_{|\varepsilon|=|\beta|+|\gamma|} \mathscr{H}^{\varepsilon} |\partial^{\varepsilon}_{x^s} v^s_\eta|.
\end{aligned}$$

Because $1 \le p < \infty$, it holds that, for $1 \le k \le d$,

$$\left\| \partial_{\hat{x}}^{\beta} \partial_{\hat{x}}^{\gamma} \hat{v}_k \right\|^p_{L^p(\widehat{T})} \le c|\det(\mathcal{A}^s)|^{p-1} \|\widetilde{\mathcal{A}}^{-1}\|_2^p h_k^{-p} \sum_{|\varepsilon|=|\beta|+|\gamma|} \mathscr{H}^{\varepsilon p} \int_{T^s} |\partial^{\varepsilon}_{x^s} v^s|^p dx^s,$$

which leads to (9.4.3) together with (1.6.1).

Proof of (9.4.2)

Because the space $\mathcal{C}^{\ell+m}(\widehat{T})^d$ is dense in the space $W^{\ell+m,p}(\widehat{T})^d$, we show (9.4.3) for $\hat{v} \in \mathcal{C}^{\ell+m}(\widehat{T})^d$ with $\tilde{v} = \widehat{\Psi}\hat{v}$, $v^s = \widetilde{\Psi}\tilde{v}$ and $v = \Psi_{T^s} v^s$. Using (1.3.1), (3.6.1c) and (3.6.2), through a simple calculation, we have, for $1 \le k \le d$,

$$|\partial_{\hat{x}}^{\beta+\gamma}\hat{v}_k| = \left| \frac{\partial^{|\beta|+|\gamma|}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d} \partial \hat{x}_1^{\gamma_1} \cdots \partial \hat{x}_d^{\gamma_d}} \hat{v}_k \right|$$

$$\le c|\det(\mathcal{A}^s)| \|\widetilde{\mathcal{A}}^{-1}\|_2 h_k^{-1} \sum_{\nu=1}^{d} h^\beta h^\gamma$$

$$\Bigg| \underbrace{\sum_{i_1^{(1)}, i_1^{(0,1)}=1}^{d} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} (r_1)_{i_1^{(1)}} \cdots \sum_{i_{\beta_1}^{(1)}, i_{\beta_1}^{(0,1)}=1}^{d} [\mathcal{A}_T]_{i_{\beta_1}^{(0,1)} i_{\beta_1}^{(1)}} (r_1)_{i_{\beta_1}^{(1)}}}_{\beta_1 \text{times}} \cdots$$

$$\underbrace{\sum_{i_1^{(d)}, i_1^{(0,d)}=1}^{d} [\mathcal{A}_T]_{i_1^{(0,d)} i_1^{(d)}} (r_d)_{i_1^{(d)}} \cdots \sum_{i_{\beta_d}^{(d)}, i_{\beta_d}^{(0,d)}=1}^{d} [\mathcal{A}_T]_{i_{\beta_d}^{(0,d)} i_{\beta_d}^{(d)}} (r_d)_{i_{\beta_d}^{(d)}}}_{\beta_d \text{times}}$$

$$\underbrace{\sum_{j_1^{(1)}, j_1^{(0,1)}=1}^{d} [\mathcal{A}_T]_{j_1^{(0,1)} j_1^{(1)}} (r_1)_{j_1^{(1)}} \cdots \sum_{j_{\gamma_1}^{(1)}, j_{\gamma_1}^{(0,1)}=1}^{d} [\mathcal{A}_T]_{j_{\gamma_1}^{(0,1)} j_{\gamma_1}^{(1)}} (r_1)_{j_{\gamma_1}^{(1)}}}_{\gamma_1 \text{times}} \cdots$$

$$\underbrace{\sum_{j_1^{(d)}, j_1^{(0,d)}=1}^{d} [\mathcal{A}_T]_{j_1^{(0,d)} j_1^{(d)}} (r_d)_{j_1^{(d)}} \cdots \sum_{j_{\gamma_d}^{(d)}, j_{\gamma_d}^{(0,d)}=1}^{d} [\mathcal{A}_T]_{j_{\gamma_d}^{(0,d)} j_{\gamma_d}^{(d)}} (r_d)_{j_{\gamma_d}^{(d)}}}_{\gamma_d \text{times}}$$

$$\underbrace{\frac{\partial^{\beta_1}}{\partial x_{i_1^{(0,1)}} \cdots \partial x_{i_{\beta_1}^{(0,1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_d}}{\partial x_{i_1^{(0,d)}} \cdots \partial x_{i_{\beta_d}^{(0,d)}}}}_{\beta_d \text{times}} \underbrace{\frac{\partial^{\gamma_1}}{\partial x_{j_1^{(0,1)}} \cdots \partial x_{j_{\gamma_1}^{(0,1)}}}}_{\gamma_1 \text{times}} \cdots \underbrace{\frac{\partial^{\gamma_d}}{\partial x_{j_1^{(0,d)}} \cdots \partial x_{j_{\gamma_d}^{(0,d)}}}}_{\gamma_d \text{times}} v_\nu \Bigg|$$

$$\le c|\det(\mathcal{A}^s)| \|\widetilde{\mathcal{A}}^{-1}\|_2 h_k^{-1} \sum_{\nu=1}^{d} \sum_{|\varepsilon|=|\beta|+|\gamma|} h^\varepsilon |\partial_r^\varepsilon v_\nu|.$$

Because $1 \le p < \infty$, it holds that, for $1 \le k \le d$,

$$\left\| \partial_{\hat{x}}^\beta \partial_{\hat{x}}^\gamma \hat{v}_k \right\|_{L^p(\widehat{T})}^p \le c|\det(\mathcal{A}^s)|^{p-1} \|\widetilde{\mathcal{A}}^{-1}\|_2^p h_k^{-p} \sum_{|\varepsilon|=|\beta|+|\gamma|} h^{\varepsilon p} \int_T |\partial_r^\varepsilon v|^p dx,$$

which leads to (9.4.2) together with (1.6.1). □

Remark 9.4.4. In inequality (9.4.3), it is possible to obtain the estimates in T by specifically determining the matrix $\mathcal{A}_T$.

Let $\hat{v} \in \mathcal{C}^1(\widehat{T})^d$ with $\tilde{v} = \widehat{\Psi}\hat{v}$, $v^s = \widetilde{\Psi}\tilde{v}$ and $v = \Psi_{T^s} v^s$. Using (1.3.1), (3.6.1c), (3.6.2) and the definition of Piola transformations (Definition 3.4.2), we have, for $1 \leq i, k \leq d$,

$$\left| \frac{\partial \hat{v}_k}{\partial \hat{x}_i} \right| \leq c |\det(\mathcal{A}^s)| \|\widetilde{\mathcal{A}}^{-1}\|_2 h_k^{-1} \sum_{\nu=1}^{d} \sum_{i_1^{(1)}, i_1^{(0,1)}=1}^{d} \mathscr{H}_{i_1^{(1)}} |[\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}}| \left| \frac{\partial v_\nu}{\partial x_{i_1^{(0,1)}}} \right|.$$

Let $d = 3$. We define the matrix $\mathcal{A}_T$ as

$$\mathcal{A}_T := \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then have

$$\left| \frac{\partial \hat{v}_k}{\partial \hat{x}_i} \right| \leq c |\det(\mathcal{A}^s)| \|\widetilde{\mathcal{A}}^{-1}\|_2 h_k^{-1} \sum_{\nu=1}^{3} \left(\mathscr{H}_1 \left| \frac{\partial v_\nu}{\partial x_2} \right| + \mathscr{H}_2 \left| \frac{\partial v_\nu}{\partial x_1} \right| + \mathscr{H}_3 \left| \frac{\partial v_\nu}{\partial x_3} \right| \right).$$

Because $1 \leq p < \infty$, it holds that, for $1 \leq i, k \leq 3$,

$$\begin{aligned} \left\| \frac{\partial \hat{v}_k}{\partial \hat{x}_i} \right\|_{L^p(\widehat{T})}^p &\leq c |\det(\mathcal{A}^s)|^{p-1} \|\widetilde{\mathcal{A}}^{-1}\|_2^p h_k^{-p} \\ &\quad \times \left(\mathscr{H}_1^p \left\| \frac{\partial v}{\partial x_2} \right\|_{L^p(T)}^p + \mathscr{H}_2^p \left\| \frac{\partial v}{\partial x_1} \right\|_{L^p(T)}^p + \mathscr{H}_3^p \left\| \frac{\partial v}{\partial x_3} \right\|_{L^p(T)}^p \right). \end{aligned}$$

The following two lemmata are divided into the element on $\mathfrak{T}^{(2)}$ or $\mathfrak{T}_1^{(3)}$ and the element on $\mathfrak{T}_2^{(3)}$. Because $T_i = T^s$, we denote $\Phi_{T_i} = \Phi_{T^s}$ and $\Psi_{T_i} = \Psi_{T^s}$ for $i = 1, 2$.

Lemma 9.4.5. *Let $T_1 \in \mathfrak{T}^{(2)}$ or $T_1 \in \mathfrak{T}_1^{(3)}$ satisfy Condition 3.2.1 or Condition 3.2.2, respectively. Let $T \subset \mathbb{R}^d$ be a simplex such that $T_1 = \Phi_{T_1}^{-1}(T)$. Let $\beta := (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$ be a multi-index with $|\beta| = \ell$. Let $p \in [0, \infty)$. It then holds that, for any $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_d)^T \in W^{\ell+1,p}(\widehat{T}_1)^d$ with $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_d)^T := \widehat{\Psi}\hat{v}$, $v^s = (v_1^s, \ldots, v_d^s)^T := \widetilde{\Psi}\tilde{v}$ and $v = (v_1, \ldots, v_d)^T := \psi_{T_1} v^s$,*

$$\left\| \partial_{\hat{x}}^\beta \nabla_{\hat{x}} \cdot \hat{v} \right\|_{L^p(\widehat{T}_1)} \leq c |\det(\mathcal{A}^s)|^{\frac{p-1}{p}} \sum_{|\varepsilon|=\ell} h^\varepsilon \left\| \partial_r^\varepsilon \nabla \cdot v \right\|_{L^p(T)}. \tag{9.4.4}$$

If Condition 3.3.1 is imposed, it holds that

$$\left\| \partial_{\hat{x}}^\beta \nabla_{\hat{x}} \cdot \hat{v} \right\|_{L^p(\widehat{T}_1)} \leq c |\det(\mathcal{A}^s)|^{\frac{p-1}{p}} \sum_{|\varepsilon|=\ell} \mathscr{H}^\varepsilon \left\| \partial_{x^s}^\varepsilon \nabla_{x^s} \cdot (\Psi_{T_1}^{-1} v) \right\|_{L^p(\Phi_{T_1}^{-1}(T))}. \tag{9.4.5}$$

Proof. Because the space $\mathcal{C}^{\ell+1}(\widehat{T})^d$ is dense in the space $W^{\ell+1,p}(\widehat{T})^d$, we show (9.4.4) and (9.4.5) for $\hat{v} \in \mathcal{C}^{\ell+1}(\widehat{T})^d$ with $\tilde{v} = \widehat{\Psi}\hat{v}$, $v^s = \widetilde{\Psi}\tilde{v}$ and $v = \Psi_{T^s} v^s$.

By a simple calculation, from Note 9.4.1,

$$\begin{aligned}
\nabla_{\hat{x}} \cdot \hat{v} &= \sum_{k=1}^{d} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \\
&= \det(\mathcal{A}^s) \det(\mathcal{A}_T) \sum_{k,\eta,\nu,i_1^{(1)},i_1^{(0,1)}=1}^{d} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} [\widetilde{\mathcal{A}}]_{i_1^{(1)} k} [\mathcal{A}_T^{-1}]_{\eta\nu} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} \frac{\partial v_\nu}{\partial x_{i_1^{(0,1)}}} \\
&= \det(\mathcal{A}^s) \det(\mathcal{A}_T) \nabla \cdot v, \\
\frac{\partial}{\partial \hat{x}_i} \nabla_{\hat{x}} \cdot \hat{v} &= \sum_{k=1}^{d} \frac{\partial^2 \hat{v}_k}{\partial \hat{x}_i \partial \hat{x}_k} \\
&= \det(\mathcal{A}^s) \det(\mathcal{A}_T) h_i \sum_{i_1^{(1)},i_1^{(0,1)}=1}^{d} [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} \\
&\qquad \sum_{k,\eta,\nu,j_1^{(1)},j_1^{(0,1)}=1}^{d} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} [\widetilde{\mathcal{A}}]_{j_1^{(1)} k} [\mathcal{A}_T^{-1}]_{\eta\nu} [\mathcal{A}_T]_{j_1^{(0,1)} j_1^{(1)}} \frac{\partial^2 v_\nu}{\partial x_{i_1^{(0,1)}} \partial x_{j_1^{(0,1)}}} \\
&= \det(\mathcal{A}^s) \det(\mathcal{A}_T) h_i \sum_{i_1^{(1)},i_1^{(0,1)}=1}^{d} [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} \frac{\partial (\nabla \cdot v)}{\partial x_{i_1^{(0,1)}}}.
\end{aligned}$$

For a general derivative $\partial_{\hat{x}}^{\beta} \nabla_{\hat{x}} \cdot \hat{v}$ with order $|\beta| = \ell$, we obtain

$$\begin{aligned}
&\partial_{\hat{x}}^{\beta} \nabla_{\hat{x}} \cdot \hat{v} = \frac{\partial^{|\beta|}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d}} \nabla_{\hat{x}} \cdot \hat{v} \\
&= \det(\mathcal{A}^s) \det(\mathcal{A}_T) \\
&\quad \underbrace{\sum_{i_1^{(1)},i_1^{(0,1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_1^{(1)} 1} [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} \cdots \sum_{i_{\beta_1}^{(1)},i_{\beta_1}^{(0,1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_{\beta_1}^{(1)} 1} [\mathcal{A}_T]_{i_{\beta_1}^{(0,1)} i_{\beta_1}^{(1)}}}_{\beta_1 \text{times}} \cdots \\
&\quad \underbrace{\sum_{i_1^{(d)},i_1^{(0,d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_1^{(d)} d} [\mathcal{A}_T]_{i_1^{(0,d)} i_1^{(d)}} \cdots \sum_{i_{\beta_d}^{(d)},i_{\beta_d}^{(0,d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_{\beta_d}^{(d)} d} [\mathcal{A}_T]_{i_{\beta_d}^{(0,d)} i_{\beta_d}^{(d)}}}_{\beta_d \text{times}}
\end{aligned}$$

$$\underbrace{\frac{\partial^{\beta_1}}{\partial x_{i_1^{(0,1)}} \cdots \partial x_{i_{\beta_1}^{(0,1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_d}}{\partial x_{i_1^{(0,d)}} \cdots \partial x_{i_{\beta_d}^{(0,d)}}}}_{\beta_d \text{times}} \nabla \cdot v$$

$$= \det(\mathcal{A}^s) \det(\mathcal{A}_T)$$

$$\underbrace{\sum_{i_1^{(1)}, i_1^{(0,1)}=1}^{d} h_1 [\mathcal{A}_T]_{i_1^{(0,1)} i_1^{(1)}} (r_1)_{i_1^{(1)}} \cdots \sum_{i_{\beta_1}^{(1)}, i_{\beta_1}^{(0,1)}=1}^{d} h_1 [\mathcal{A}_T]_{i_{\beta_1}^{(0,1)} i_{\beta_1}^{(1)}} (r_1)_{i_{\beta_1}^{(1)}} \cdots}_{\beta_1 \text{times}}$$

$$\underbrace{\sum_{i_1^{(d)}, i_1^{(0,d)}=1}^{d} h_d [\mathcal{A}_T]_{i_1^{(0,d)} i_1^{(d)}} (r_d)_{i_1^{(d)}} \cdots \sum_{i_{\beta_d}^{(d)}, i_{\beta_d}^{(0,d)}=1}^{d} h_d [\mathcal{A}_T]_{i_{\beta_d}^{(0,d)} i_{\beta_d}^{(d)}} (r_d)_{i_{\beta_d}^{(d)}}}_{\beta_d \text{times}}$$

$$\underbrace{\frac{\partial^{\beta_1}}{\partial x_{i_1^{(0,1)}} \cdots \partial x_{i_{\beta_1}^{(0,1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_d}}{\partial x_{i_1^{(0,d)}} \cdots \partial x_{i_{\beta_d}^{(0,d)}}}}_{\beta_d \text{times}} \nabla \cdot v.$$

It then holds that, using (3.6.2) and (1.6.1),

$$|\partial_{\hat{x}}^{\beta} \nabla_{\hat{x}} \cdot \hat{v}| \le c |\det(\mathcal{A}^s)| \sum_{|\varepsilon|=\ell} h^{\varepsilon} |\partial_r^{\varepsilon} \nabla \cdot v|,$$

which leads to

$$\|\partial_{\hat{x}}^{\beta} \nabla_{\hat{x}} \cdot \hat{v}\|_{L^p(\widehat{T}_1)} \le c |\det(\mathcal{A}^s)|^{\frac{p-1}{p}} \sum_{|\varepsilon|=\ell} h^{\varepsilon} \|\partial_r^{\varepsilon} \nabla \cdot v\|_{L^p(T)}.$$

Using an analogous argument, if Condition 3.3.1 is imposed, for a general derivative $\partial_{\hat{x}}^{\beta} \nabla_{\hat{x}} \cdot \hat{v}$ with order $|\beta| = \ell$, we obtain

$$\partial_{\hat{x}}^{\beta} \nabla_{\hat{x}} \cdot \hat{v} = \frac{\partial^{|\beta|}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d}} \nabla_{\hat{x}} \cdot \hat{v}$$

$$= \det(\mathcal{A}^s)$$

$$\underbrace{\sum_{i_1^{(1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_1^{(1)} 1} \cdots \sum_{i_{\beta_1}^{(1)}=1}^{d} h_1 [\widetilde{\mathcal{A}}]_{i_{\beta_1}^{(1)} 1}}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_1^{(d)} d} \cdots \sum_{i_{\beta_d}^{(d)}=1}^{d} h_d [\widetilde{\mathcal{A}}]_{i_{\beta_d}^{(d)} d}}_{\beta_d \text{times}}$$

$$\underbrace{\frac{\partial^{\beta_1}}{\partial x^s_{i_1^{(1)}} \cdots \partial x^s_{i_{\beta_1}^{(1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_d}}{\partial x^s_{i_1^{(d)}} \cdots \partial x^s_{i_{\beta_d}^{(d)}}}}_{\beta_d \text{times}} \nabla_{x^s} \cdot v^s.$$

It then holds that

$$
\begin{aligned}
&\left|\partial_{\hat{x}}^{\beta} \nabla_{\hat{x}} \cdot \hat{v}\right| \\
&\quad \leq c|\det(\mathcal{A}^s)| \\
&\qquad \underbrace{\sum_{i_1^{(1)}=1}^{d} h_1 |[\widetilde{\mathcal{A}}]_{i_1^{(1)}1}| \cdots \sum_{i_{\beta_1}^{(1)}=1}^{d} h_1 |[\widetilde{\mathcal{A}}]_{i_{\beta_1}^{(1)}1}|}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(d)}=1}^{d} h_d |[\widetilde{\mathcal{A}}]_{i_1^{(d)}d}| \cdots \sum_{i_{\beta_d}^{(d)}=1}^{d} h_d |[\widetilde{\mathcal{A}}]_{i_{\beta_d}^{(d)}d}|}_{\beta_d \text{times}} \\
&\qquad \left| \underbrace{\frac{\partial^{\beta_1}}{\partial x^s_{i_1^{(1)}} \cdots \partial x^s_{i_{\beta_1}^{(1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_d}}{\partial x^s_{i_1^{(d)}} \cdots \partial x^s_{i_{\beta_d}^{(d)}}}}_{\beta_d \text{times}} \nabla_{x^s} \cdot v^s \right| \\
&\quad \leq c|\det(\mathcal{A}^s)| \sum_{|\varepsilon|=\ell} \mathscr{H}^{\varepsilon} |\partial_{x^s}^{\varepsilon} \nabla_{x^s} \cdot v^s|,
\end{aligned}
$$

which leads to

$$
\|\partial_{\hat{x}}^{\beta} \nabla_{\hat{x}} \cdot \hat{v}\|_{L^p(\widehat{T}_1)} \leq c|\det(\mathcal{A}^s)|^{\frac{p-1}{p}} \sum_{|\varepsilon|=\ell} \mathscr{H}^{\varepsilon} \|\partial_{x^s}^{\varepsilon} \nabla_{x^s} \cdot v^s\|_{L^p(T_1)}.
$$

□

Lemma 9.4.6. *Let $d = 3$. Let $T_2 \in \mathfrak{T}_2^{(3)}$ satisfy Condition 3.2.2. Let $T \subset \mathbb{R}^3$ be a simplex such that $T_2 = \Phi_{T_2}^{-1}(T)$. Let $\ell \in \mathbb{N}_0$ and $k \in \mathbb{N}$ with $1 \leq k \leq 3$. Let $\beta := (\beta_1, \beta_2, \beta_3) \in \mathbb{N}_0^3$ be a multi-index with $|\beta| = \ell$. Let $p \in [0, \infty)$. It holds that, for any $\hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)^T \in W^{\ell+1,p}(\widehat{T}_2)^d$ with $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T := \widehat{\Psi}\hat{v}$, $v^s = (v_1^s, v_2^s, v_3^s)^T := \widetilde{\Psi}\tilde{v}$ and $v = (v_1, v_2, v_3)^T := \Psi_{T_2} v^s$,*

$$
\left\| \partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right\|_{L^p(\widehat{T}_2)} \leq c|\det(\mathcal{A}^s)|^{\frac{p-1}{p}} \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{|\varepsilon|=\ell} h^{\varepsilon} \left\| \partial_r^{\varepsilon} \frac{\partial v}{\partial r_k} \right\|_{L^p(T)^3}. \tag{9.4.6}
$$

If Condition 3.3.1 is imposed, it holds that

$$
\left\| \partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right\|_{L^p(\widehat{T}_2)} \leq c|\det(\mathcal{A}^s)|^{\frac{p-1}{p}} \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{|\varepsilon|=\ell} \mathscr{H}^{\varepsilon} \left\| \partial_{x^s}^{\varepsilon} \frac{\partial (\Psi_{T_2}^{-1} v)}{\partial r_k^s} \right\|_{L^p(\Phi_{T_2}^{-1}(T))^3}. \tag{9.4.7}
$$

Proof. Because the space $\mathcal{C}^{\ell+1}(\widehat{T})^3$ is dense in the space $W^{\ell+1,p}(\widehat{T})^3$, we show (9.4.6) and (9.4.7) for $\hat{v} \in \mathcal{C}^{\ell+1}(\widehat{T})^3$ with $\tilde{v} = \widehat{\Psi}\hat{v}$, $v^s = \widetilde{\Psi}\tilde{v}$ and $v = \Psi_{T^s} v^s$.

By a simple calculation, from Note 9.4.1, for $1 \le i, k \le 3$,

$$\frac{\partial \hat{v}_k}{\partial \hat{x}_k} = \det(\mathcal{A}^s)\det(\mathcal{A}_T) \sum_{\eta,\nu,i_1^{(1)},i_1^{(0,1)}=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}[\widetilde{\mathcal{A}}]_{i_1^{(1)}k}[\mathcal{A}_T^{-1}]_{\eta\nu}[\mathcal{A}_T]_{i_1^{(0,1)}i_1^{(1)}} \frac{\partial v_\nu}{\partial x_{i_1^{(0,1)}}}$$

$$= \det(\mathcal{A}^s)\det(\mathcal{A}_T) \sum_{\eta,\nu,i_1^{(1)},i_1^{(0,1)}=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}[\mathcal{A}_T^{-1}]_{\eta\nu}[\mathcal{A}_T]_{i_1^{(0,1)}i_1^{(1)}}(r_k)_{i_1^{(1)}} \frac{\partial v_\nu}{\partial x_{i_1^{(0,1)}}}$$

$$= \det(\mathcal{A}^s)\det(\mathcal{A}_T) \sum_{\eta,\nu=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}[\mathcal{A}_T^{-1}]_{\eta\nu} \frac{\partial v_\nu}{\partial r_k},$$

$$\frac{\partial^2 \hat{v}_k}{\partial \hat{x}_i \partial \hat{x}_k} = \det(\mathcal{A}^s)\det(\mathcal{A}_T) \sum_{\eta,\nu,i_1^{(1)},i_1^{(0,1)},j_1^{(1)},j_1^{(0,1)}=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}[\mathcal{A}_T^{-1}]_{\eta\nu}$$

$$h_i[\widetilde{\mathcal{A}}]_{i_1^{(1)}i}[\widetilde{\mathcal{A}}]_{j_1^{(1)}k}[\mathcal{A}_T]_{i_1^{(0,1)}i_1^{(1)}}[\mathcal{A}_T]_{j_1^{(0,1)}j_1^{(1)}} \frac{\partial^2 v_\nu}{\partial x_{i_1^{(0,1)}}\partial x_{j_1^{(0,1)}}}$$

$$= \det(\mathcal{A}^s)\det(\mathcal{A}_T) \sum_{\eta,\nu,i_1^{(1)},i_1^{(0,1)},j_1^{(1)},j_1^{(0,1)}=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}[\mathcal{A}_T^{-1}]_{\eta\nu}$$

$$h_i[\mathcal{A}_T]_{i_1^{(0,1)}i_1^{(1)}}(r_i)_{i_1^{(1)}}[\mathcal{A}_T]_{j_1^{(0,1)}j_1^{(1)}}(r_k)_{j_1^{(1)}} \frac{\partial^2 v_\nu}{\partial x_{i_1^{(0,1)}}\partial x_{j_1^{(0,1)}}}$$

$$= \det(\mathcal{A}^s)\det(\mathcal{A}_T) \sum_{\eta,\nu=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}[\mathcal{A}_T^{-1}]_{\eta\nu} h_i \frac{\partial^2 v_\nu}{\partial r_i \partial r_k}.$$

For a general derivative $\partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k}$ $(1 \le k \le 3)$ with order $|\beta| = \ell$, we obtain

$$\partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} = \frac{\partial^{|\beta|}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d}} \frac{\partial \hat{v}_k}{\partial \hat{x}_k}$$

$$= \det(\mathcal{A}^s)\det(\mathcal{A}_T) \sum_{\eta,\nu=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}[\mathcal{A}_T^{-1}]_{\eta\nu}$$

$$\underbrace{\sum_{i_1^{(1)},i_1^{(0,1)}=1}^{3} h_1[\mathcal{A}_T]_{i_1^{(0,1)}i_1^{(1)}}(r_1)_{i_1^{(1)}} \cdots \sum_{i_{\beta_1}^{(1)},i_{\beta_1}^{(0,1)}=1}^{3} h_1[\mathcal{A}_T]_{i_{\beta_1}^{(0,1)}i_{\beta_1}^{(1)}}(r_1)_{i_{\beta_1}^{(1)}}}_{\beta_1 \text{times}} \cdots$$

$$\underbrace{\sum_{i_1^{(3)},i_1^{(0,3)}=1}^{3} h_3[\mathcal{A}_T]_{i_1^{(0,3)}i_1^{(3)}}(r_3)_{i_1^{(3)}} \cdots \sum_{i_{\beta_3}^{(3)},i_{\beta_3}^{(0,3)}=1}^{3} h_3[\mathcal{A}_T]_{i_{\beta_3}^{(0,3)}i_{\beta_3}^{(3)}}(r_3)_{i_{\beta_3}^{(3)}}}_{\beta_d \text{times}}$$

$$\underbrace{\frac{\partial^{\beta_1}}{\partial x_{i_1^{(0,1)}} \cdots \partial x_{i_{\beta_1}^{(0,1)}}}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_3}}{\partial x_{i_1^{(0,3)}} \cdots \partial x_{i_{\beta_d}^{(0,d)}}}}_{\beta_d \text{times}} \frac{\partial v_\nu}{\partial r_k}$$

$$= \det(\mathcal{A}^s) \det(\mathcal{A}_T) \sum_{\eta,\nu=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} [\mathcal{A}_T^{-1}]_{\eta\nu} h^\beta \underbrace{\frac{\partial^{\beta_1}}{\partial r_1 \cdots \partial r_1}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_3}}{\partial r_3 \cdots \partial r_3}}_{\beta_3 \text{times}} \frac{\partial v_\nu}{\partial r_k}.$$

It then holds that, using (1.3.1), (3.6.1c) and (3.6.2),

$$\left| \partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right| \leq c |\det(\mathcal{A}^s)| \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{\nu=1}^{3} \sum_{|\varepsilon|=|\beta|} h^\varepsilon \left| \partial_r^\varepsilon \frac{\partial v_\nu}{\partial r_k} \right|,$$

which leads to, using (1.6.1),

$$\left\| \partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right\|_{L^p(\widehat{T}_2)} \leq c |\det(\mathcal{A}^s)|^{\frac{p-1}{p}} \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{|\varepsilon|=|\beta|} h^\varepsilon \left\| \partial_r^\varepsilon \frac{\partial v}{\partial r_k} \right\|_{L^p(T)^3}.$$

If Condition 3.3.1 is imposed, by a simple calculation, from Note 9.4.2, for $1 \leq i, k \leq 3$,

$$\begin{aligned}
\frac{\partial \hat{v}_k}{\partial \hat{x}_k} &= \det(\mathcal{A}^s) \sum_{\eta=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} \sum_{i_1^{(1)}=1}^{3} [\widetilde{\mathcal{A}}]_{i_1^{(1)} k} \frac{\partial v_\eta^s}{\partial x_{i_1^{(1)}}^s} \\
&= \det(\mathcal{A}^s) \sum_{\eta=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} \sum_{i_1^{(1)}=1}^{3} (r_k)_{i_1^{(1)}} \frac{\partial v_\eta^s}{\partial x_{i_1^{(1)}}^s} \\
&= \det(\mathcal{A}^s) \sum_{\eta=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} \frac{\partial v_\eta^s}{\partial r_k^s}, \\
\frac{\partial^2 \hat{v}_k}{\partial \hat{x}_i \partial \hat{x}_k} &= \det(\mathcal{A}^s) \sum_{\eta, i_1^{(1)}=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} h_i [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} \sum_{j_1^{(1)}=1}^{3} [\widetilde{\mathcal{A}}]_{j_1^{(1)} k} \frac{\partial^2 v_\eta^s}{\partial x_{i_1^{(1)}} \partial x_{j_1^{(1)}}} \\
&= \det(\mathcal{A}^s) \sum_{\eta, i_1^{(1)}=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} h_i [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} \sum_{j_1^{(1)}=1}^{3} (r_k)_{j_1^{(1)}} \frac{\partial^2 v_\eta^s}{\partial x_{i_1^{(1)}} \partial x_{j_1^{(1)}}} \\
&= \det(\mathcal{A}^s) \sum_{\eta, i_1^{(1)}=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta} h_i [\widetilde{\mathcal{A}}]_{i_1^{(1)} i} \frac{\partial^2 v_\eta^s}{\partial x_{i_1^{(1)}} \partial r_k^s}.
\end{aligned}$$

For a general derivative $\partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k}$ $(1 \leq k \leq 3)$ with order $|\beta| = \ell$, we obtain

$$\partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} = \frac{\partial^{|\beta|}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d}} \frac{\partial \hat{v}_k}{\partial \hat{x}_k}$$

$$= \det(\mathcal{A}^s) \sum_{\eta=1}^{3} [\widetilde{\mathcal{A}}^{-1}]_{k\eta}$$

$$\underbrace{\sum_{i_1^{(1)}=1}^{3} h_1 [\widetilde{\mathcal{A}}]_{i_1^{(1)}1} \cdots \sum_{i_{\beta_1}^{(1)}=1}^{3} h_1 [\widetilde{\mathcal{A}}]_{i_{\beta_1}^{(1)}1}}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(3)}=1}^{3} h_3 [\widetilde{\mathcal{A}}]_{i_1^{(3)}3} \cdots \sum_{i_{\beta_3}^{(3)}=1}^{3} h_3 [\widetilde{\mathcal{A}}]_{i_{\beta_3}^{(3)}3}}_{\beta_3 \text{times}}$$

$$\underbrace{\frac{\partial^{\beta_1}}{\partial x_{i_1^{(1)}}^s \cdots \partial x_{i_{\beta_1}^{(1)}}^s}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_3}}{\partial x_{i_1^{(3)}}^s \cdots \partial x_{i_{\beta_3}^{(3)}}^s}}_{\beta_3 \text{times}} \frac{\partial v_\eta^s}{\partial r_k^s}.$$

It then holds that, using (1.3.1),

$$\left| \partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right|$$

$$\leq |\det(\mathcal{A}^s)| \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{\eta=1}^{3}$$

$$\underbrace{\sum_{i_1^{(1)}=1}^{3} h_1 |[\widetilde{\mathcal{A}}]_{i_1^{(1)}1}| \cdots \sum_{i_{\beta_1}^{(1)}=1}^{3} h_1 |[\widetilde{\mathcal{A}}]_{i_{\beta_1}^{(1)}1}|}_{\beta_1 \text{times}} \cdots \underbrace{\sum_{i_1^{(3)}=1}^{3} h_3 |[\widetilde{\mathcal{A}}]_{i_1^{(3)}3}| \cdots \sum_{i_{\beta_3}^{(3)}=1}^{3} h_3 |[\widetilde{\mathcal{A}}]_{i_{\beta_d}^{(3)}3}|}_{\beta_3 \text{times}}$$

$$\left| \underbrace{\frac{\partial^{\beta_1}}{\partial x_{i_1^{(1)}}^s \cdots \partial x_{i_{\beta_1}^{(1)}}^s}}_{\beta_1 \text{times}} \cdots \underbrace{\frac{\partial^{\beta_3}}{\partial x_{i_1^{(3)}}^s \cdots \partial x_{i_{\beta_3}^{(3)}}^s}}_{\beta_3 \text{times}} \frac{\partial v_\eta^s}{\partial r_k^s} \right|$$

$$\leq c |\det(\mathcal{A}^s)| \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{\eta=1}^{3} \sum_{|\varepsilon|=|\beta|} \mathscr{H}^{\varepsilon} \left| \partial_{x^s}^{\varepsilon} \frac{\partial v_\eta^s}{\partial r_k^s} \right|,$$

which leads to, using (1.6.1),

$$\left\| \partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right\|_{L^p(\widehat{T}_2)} \leq c |\det(\mathcal{A}^s)|^{\frac{p-1}{p}} \|\widetilde{\mathcal{A}}^{-1}\|_2 \sum_{|\varepsilon|=|\beta|} \mathscr{H}^{\varepsilon} \left\| \partial_{x^s}^{\varepsilon} \frac{\partial v^s}{\partial r_k^s} \right\|_{L^p(T^s)^3}.$$

□

Lemma 9.4.7. *For any $T \in \mathbb{T}_h$, let $\Psi_{T^s} : V(T^s) \to V(T)$ be the Piola transformation defined in* (3.4.5)*. Let $s \geq 0$ and $p \in [0, \infty)$. There exists positive constants c_1 and c_2 such that, for all $T \in \mathbb{T}_h$ and $v \in W^{s,p}(T)^d$,*

$$c_1 |v|_{W^{s,p}(T)^d} \leq |v^s|_{W^{s,p}(T^s)^d} \leq c_2 |v|_{W^{s,p}(T)^d}, \tag{9.4.8}$$

with $v^s = \Psi_{T^s}^{-1} v$.

Proof. The following inequalities are found in [30, Lemma 1.113]. There exists a positive constant c such that, for all $T \in \mathbb{T}_h$ and $v \in W^{s,p}(T)^d$,

$$|v^s|_{W^{s,p}(T^s)^d} \leq c \|\mathcal{A}_T\|_2^s \|\mathcal{A}_T^{-1}\|_2 |\det(\mathcal{A}_T)|^{\frac{1}{p'}} |v|_{W^{s,p}(T)^d}, \tag{9.4.9}$$

$$|v|_{W^{s,p}(T)^d} \leq c \|\mathcal{A}_T^{-1}\|_2^s \|\mathcal{A}_T\|_2 |\det(\mathcal{A}_T)|^{-\frac{1}{p'}} |v^s|_{W^{s,p}(T^s)^d}, \tag{9.4.10}$$

with $v^s = \Psi_{T^s}^{-1} v$. Because the length of all edges of a simplex and measure of the simplex are not changed by a rotation and mirror imaging matrix and $\mathcal{A}_T, \mathcal{A}_T^{-1} \in O(d)$,

$$|\det(\mathcal{A}_T)| = \frac{|T|}{|T^s|} = 1, \quad \|\mathcal{A}_T\|_2 = 1, \quad \|\mathcal{A}_T^{-1}\|_2 = 1. \tag{9.4.11}$$

From (9.7.2), (9.7.3), and (12.4.6), we obtain the desired inequality (9.4.8). □

9.5 Stability of the local Raviart–Thomas interpolation

The following two lemmata are divided into the element on $\mathfrak{T}^{(2)}$ or $\mathfrak{T}_1^{(3)}$ and the element on $\mathfrak{T}_2^{(3)}$.

Lemma 9.5.1. *Let $p \in [0, \infty)$. Let $T \subset \mathbb{R}^d$ be a simplex such that $T_1 = \Phi_{T_1}^{-1}(T)$. Let $T_1 \in \mathfrak{T}^{(2)}$ or $T_1 \in \mathfrak{T}_1^{(3)}$ satisfy Condition 3.2.1 or Condition 3.2.2, respectively. It holds that, for any $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_d)^T \in W^{1,p}(\widehat{T}_1)^d$ with $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_d)^T := \widehat{\Psi} \hat{v}$, $v^s = (v_1^s, \ldots, v_d^s)^T := \widetilde{\Psi} \tilde{v}$ and $v = (v_1, \ldots, v_d)^T := \psi_{T_1} v^s$,*

$$\|I_T^{RT^k} v\|_{L^p(T)^d} \leq c \left[\frac{H_T}{h_T} \left(\|v\|_{L^p(T)^d} + \sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^d} \right) + h_T \|\nabla \cdot v\|_{L^p(T)} \right]. \tag{9.5.1}$$

Proof. From (9.4.1) and (9.4.8),

$$\|I_T^{RT^k} v\|_{L^p(T)^d} \leq c\|I_{T_1}^{RT^k} v^s\|_{L^p(T_1)^d}$$
$$\leq c|\det(\mathcal{A}^s)|^{-\frac{p-1}{p}} \|\widetilde{\mathcal{A}}\|_2 \left(\sum_{j=1}^{d} h_j^p \|(I_{\widehat{T}_1}^{RT^k} \hat{v})_j\|^p_{L^p(\widehat{T}_1)} \right)^{1/p}. \quad (9.5.2)$$

The component-wise stability (9.3.1) for $2d$ or (9.3.2) for $3d$ yields

$$\sum_{j=1}^{d} h_j^p \|(I_{\widehat{T}_1}^{RT^k} \hat{v})_j\|^p_{L^p(\widehat{T}_1)} \leq c \sum_{j=1}^{d} h_j^p \left(\|\hat{v}_j\|^p_{W^{1,p}(\widehat{T}_1)} + \|\nabla_{\hat{x}} \cdot \hat{v}\|^p_{L^p(\widehat{T}_1)} \right). \quad (9.5.3)$$

From (9.4.2) with $\ell = 0$ and $m \in \{0, 1\}$,

$$\|\hat{v}_j\|^p_{W^{1,p}(\widehat{T}_1)}$$
$$= \|\hat{v}_j\|^p_{L^p(\widehat{T}_1)} + \sum_{k=1}^{d} \left\| \frac{\partial \hat{v}_j}{\partial \hat{x}_k} \right\|^p_{L^p(\widehat{T}_1)}$$
$$\leq c|\det(\mathcal{A}^s)|^{p-1} \|\widetilde{\mathcal{A}}^{-1}\|_2^p h_j^{-p} \left[\|v\|^p_{L^p(T)^d} + \left(\sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^d} \right)^p \right]. \quad (9.5.4)$$

From (9.4.4) with $\ell = 0$,

$$\|\nabla_{\hat{x}} \cdot \hat{v}\|_{L^p(\widehat{T}_1)} \leq c|\det(\mathcal{A}^s)|^{\frac{p-1}{p}} \|\nabla \cdot v\|_{L^p(T)}. \quad (9.5.5)$$

Combining the above inequalities (9.5.2), (9.5.3), (9.5.4), and (9.5.5) with (3.6.1b) and (1.6.1) yields

$$\|I_T^{RT^k} v\|_{L^p(T)^d}$$
$$\leq c \left[\frac{H_T}{h_T} \left(\|v\|_{L^p(T)^d} + \sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^d} \right) + h_T \|\nabla \cdot v\|_{L^p(T)} \right],$$

which is the desired estimate. □

Lemma 9.5.2. *Let $d = 3$ and $p \in [0, \infty)$. Let $T \subset \mathbb{R}^3$ be a simplex such that $T_2 = \Phi_{T_2}^{-1}(T)$. Let $T_2 \in \mathfrak{T}_2^{(3)}$ satisfy Condition 3.2.2. It holds that, for any $\hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)^T \in W^{1,p}(\widehat{T}_2)^3$ with $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T := \widehat{\Psi}\hat{v}$, $v^s = (v_1^s, v_2^s, v_3^s)^T := \widetilde{\Psi}\tilde{v}$ and $v = (v_1, v_2, v_3)^T := \psi_{T_2} v^s$,*

$$\|I_T^{RT^k} v\|_{L^p(T)^3} \leq c \frac{H_T}{h_T} \left[\|v\|_{L^p(T)^3} + h_T \sum_{k=1}^{3} \left\| \frac{\partial v}{\partial r_k} \right\|_{L^p(T)^3} \right]. \quad (9.5.6)$$

Proof. The component-wise stability (9.3.11) yields

$$\sum_{j=1}^{3} h_j^p \|(I_{\widehat{T}_2}^{RT^k} \hat{v})_j\|_{L^p(\widehat{T}_2)}^p \leq c \sum_{j=1}^{3} h_j^p \left(\|\hat{v}_j\|_{W^{1,p}(\widehat{T}_2)}^p + \sum_{k=1, k \neq j}^{3} \left\| \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right\|_{L^p(\widehat{T}_2)}^p \right). \tag{9.5.7}$$

From (9.4.6) with $\ell = 0$,

$$\left\| \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right\|_{L^p(\widehat{T}_2)} \leq c |\det(\mathcal{A}^s)|^{\frac{p-1}{p}} \|\widetilde{\mathcal{A}}^{-1}\|_2 \left\| \frac{\partial v}{\partial r_k} \right\|_{L^p(T)^3}. \tag{9.5.8}$$

By analogous argument in Lemma 9.5.1,

$$\begin{aligned} &\|\hat{v}_j\|_{W^{1,p}(\widehat{T}_2)}^p \\ &\quad = \|\hat{v}_j\|_{L^p(\widehat{T}_2)}^p + \sum_{k=1}^{3} \left\| \frac{\partial \hat{v}_j}{\partial \hat{x}_k} \right\|_{L^p(\widehat{T}_2)}^p \\ &\quad \leq c |\det(\mathcal{A}^s)|^{p-1} \|\widetilde{\mathcal{A}}^{-1}\|_2^p h_j^{-p} \left[\|v\|_{L^p(T)^3}^p + \left(\sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^3} \right)^p \right]. \end{aligned} \tag{9.5.9}$$

Combining the above inequalities (9.5.2), (9.5.7), (9.5.8), and (9.5.9) with (3.6.1b) and (1.6.1) yields

$$\begin{aligned} &\|I_T^{RT^k} v\|_{L^p(T)^3} \leq c \|I_{T_2}^{RT^k} v^s\|_{L^p(T_2)^3} \\ &\quad \leq c |\det(\mathcal{A}^s)|^{-\frac{p-1}{p}} \|\widetilde{\mathcal{A}}\|_2 \left(\sum_{j=1}^{d} h_j^p \|(I_{\widehat{T}_2}^{RT^k} \hat{v})_j\|_{L^p(\widehat{T}_1)}^p \right)^{1/p} \\ &\quad \leq c \frac{H_T}{h_T} \left[\|v\|_{L^p(T)^3} + \sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^3} + \sum_{j=1}^{3} h_j \sum_{k=1, k \neq j}^{3} \left\| \frac{\partial v}{\partial r_k} \right\|_{L^p(T)^3} \right], \end{aligned}$$

which is the desired result. □

9.6 Local Interpolation Error Estimates

The following two theorems are divided into the element on $\mathfrak{T}^{(2)}$ or $\mathfrak{T}_1^{(3)}$ and the element on $\mathfrak{T}_2^{(3)}$.

Theorem 9.6.1. *Let $p \in [0, \infty)$. Let $T \subset \mathbb{R}^d$ be a simplex such that $T_1 = \Phi_{T_1}^{-1}(T)$. Let $T_1 \in \mathfrak{T}^{(2)}$ or $T_1 \in \mathfrak{T}_1^{(3)}$ satisfy Condition 3.2.1 or Condition 3.2.2, respectively. For $k \in \mathbb{N}_0$, let $\{T, RT^k(T), \Sigma\}$ be the Raviart–Thomas finite element and $I_T^{RT^k}$ the local interpolation operator defined in* (9.1.12). *Let ℓ be such that $0 \leq \ell \leq k$. For any $\hat{v} \in W^{\ell+1,p}(\widehat{T}_1)^d$ with $\tilde{v} = \widehat{\Psi}\hat{v}$, $v^s = \widetilde{\Psi}\tilde{v}$ and $v = \psi_{T_1} v^s$, it holds that*

$$\|I_T^{RT^k} v - v\|_{L^p(T)^d} \leq c \left(\frac{H_T}{h_T} \sum_{|\varepsilon|=\ell+1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^d} + h_T \sum_{|\beta|=\ell} h^{\beta} \|\partial_r^{\beta} \nabla \cdot v\|_{L^p(T)} \right). \tag{9.6.1}$$

If Condition 3.3.1 is imposed, it holds that

$$\begin{aligned} &\|I_T^{RT^k} v - v\|_{L^p(T)^d} \\ &\leq c \Bigg(\frac{H_T}{h_T} \sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial_{x^s}^{\varepsilon} (\Psi_{T_1}^{-1} v)\|_{L^p(\Phi_{T_1}^{-1}(T))^d} \\ &\quad + h_T \sum_{|\beta|=\ell} \mathscr{H}^{\beta} \|\partial_{x^s}^{\beta} \nabla_{x^s} \cdot (\Psi_{T_1}^{-1} v)\|_{L^p(\Phi_{T_1}^{-1}(T))} \Bigg). \end{aligned} \tag{9.6.2}$$

Proof. Let $\hat{v} \in W^{\ell+1,p}(\widehat{T}_1)^d$. Let $I_{\widehat{T}_1}^{RT^k}$ be the local interpolation operators on $\widehat{T}_1$ defined by (9.1.4) and (9.1.5). If $q \in \mathcal{P}^{\ell}(T)^d \subset RT^k(T)$, then $I_T^{RT} q = q$.

We set $\mathfrak{Q}^{(\ell+1)} v := (Q^{(\ell+1)} v_1, \ldots, Q^{(\ell+1)} v_d)^T \in \mathcal{P}^{\ell}(T_1)^d$, where $Q^{(\ell+1)} v_j$ is defined by (1.6.8) for any j. We then obtain

$$\|I_T^{RT^k} v - v\|_{L^p(T)^d} \leq \|I_T^{RT^k} (v - \mathfrak{Q}^{(\ell+1)} v)\|_{L^p(T)^d} + \|\mathfrak{Q}^{(\ell+1)} v - v\|_{L^p(T)^d}. \tag{9.6.3}$$

The inequalities (9.4.1) and (9.4.8) for the first term on the right-hand side of (9.6.3) yield

$$\begin{aligned} &\|I_T^{RT^k} (v - \mathfrak{Q}^{(\ell+1)} v)\|_{L^p(T)^d} \\ &\quad \leq c |\det(\mathcal{A}^s)|^{\frac{1-p}{p}} \|\widetilde{\mathcal{A}}\|_2 \left(\sum_{j=1}^{d} h_j^p \|\{I_{\widehat{T}_1}^{RT^k} (\hat{v} - \widehat{\mathfrak{Q}}^{(\ell+1)} \hat{v})\}_j\|_{L^p(\widehat{T}_1)}^p \right)^{1/p}. \end{aligned} \tag{9.6.4}$$

The component-wise stability (9.3.1) for $2d$ or (9.3.2) for $3d$ yields

$$\sum_{j=1}^{d} h_j^p \|\{I_{\widehat{T}_1}^{RT^k}(\hat{v} - \widehat{\mathfrak{Q}}^{(\ell+1)}\hat{v})\}_j\|_{L^p(\widehat{T}_1)}^p$$
$$\leq c \sum_{j=1}^{d} h_j^p \left(\|\hat{v}_j - \widehat{Q}^{(\ell+1)}\hat{v}_j\|_{W^{1,p}(\widehat{T}_1)}^p + \|\nabla_{\hat{x}} \cdot (\hat{v} - \widehat{\mathfrak{Q}}^{(\ell+1)}\hat{v})\|_{L^p(\widehat{T}_1)}^p \right). \tag{9.6.5}$$

The inequalities (9.4.1) and (9.4.8) for the second term on the right-hand side of (9.6.3) yields

$$\|\mathfrak{Q}^{(\ell+1)}v - v\|_{L^p(T)^d}$$
$$\leq c|\det(\mathcal{A}^s)|^{\frac{1-p}{p}} \|\widetilde{\mathcal{A}}\|_2 \left(\sum_{j=1}^{d} h_j^p \|\widehat{Q}^{(\ell+1)}\hat{v}_j - \hat{v}_j\|_{L^p(\widehat{T}_1)}^p \right)^{1/p}. \tag{9.6.6}$$

Case in which Condition 3.3.1 is not imposed

The Bramble–Hilbert-type lemma (Lemma 1.6.9) and (9.4.2),

$$\|\hat{v}_j - \widehat{Q}^{(\ell+1)}\hat{v}_j\|_{W^{1,p}(\widehat{T}_1)}^p$$
$$= \|\hat{v}_j - \widehat{Q}^{(\ell+1)}\hat{v}_j\|_{L^p(\widehat{T}_1)}^p + \sum_{k=1}^{d} \left\| \frac{\partial}{\partial \hat{x}_k}(\hat{v}_j - \widehat{Q}^{(\ell+1)}\hat{v}_j) \right\|_{L^p(\widehat{T}_1)}^p$$
$$\leq c \left(\sum_{|\gamma|=\ell+1} \|\partial_{\hat{x}}^{\gamma}\hat{v}_j\|_{L^p(\widehat{T}_1)}^p + \sum_{k=1}^{d} \sum_{|\beta|=\ell} \left\| \partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_j}{\partial \hat{x}_k} \right\|_{L^p(\widehat{T}_1)}^p \right)$$
$$\leq c|\det(\mathcal{A}^s)|^{p-1} h_j^{-p} \|\widetilde{\mathcal{A}}^{-1}\|_2^p \left(\sum_{|\varepsilon|=\ell+1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^d} \right)^p. \tag{9.6.7}$$

Because from [22, Proposition 4.1.17] it holds that

$$\widehat{\operatorname{div}}(\widehat{\mathfrak{Q}}^{(\ell+1)}\hat{v}) = \widehat{Q}^{\ell}(\widehat{\operatorname{div}}\hat{v}), \tag{9.6.8}$$

from Lemma 1.6.9 and (9.4.4),

$$\begin{aligned}
\|\nabla_{\hat{x}} \cdot (\hat{v} - \widehat{\mathfrak{Q}}^{(\ell+1)}\hat{v}))\|^p_{L^p(\widehat{T}_1)} &= \|\nabla_{\hat{x}} \cdot \hat{v} - \widehat{Q}^{\ell}(\nabla_{\hat{x}} \cdot \hat{v})\|^p_{L^p(\widehat{T}_1)} \\
&\leq \|\nabla_{\hat{x}} \cdot \hat{v} - \widehat{Q}^{\ell}(\nabla_{\hat{x}} \cdot \hat{v})\|^p_{W^{\ell,p}(\widehat{T}_1)} \\
&\leq c|\nabla_{\hat{x}} \cdot \hat{v}|^p_{W^{\ell,p}(\widehat{T}_1)} = c \sum_{|\beta|=\ell} \|\partial^{\beta}\nabla_{\hat{x}} \cdot \hat{v}\|^p_{L^p(\widehat{T}_1)} \\
&\leq c|\det(\mathcal{A}^s)|^{p-1} \left(\sum_{|\varepsilon|=\ell} h^{\varepsilon} \left\| \partial^{\varepsilon}_r \nabla \cdot v \right\|_{L^p(T)} \right)^p .
\end{aligned} \tag{9.6.9}$$

Combining (9.6.4), (9.6.5), (9.6.7), and (9.6.9) with (3.6.1b) yields

$$\begin{aligned}
&\|I_T^{RT^k}(v - \mathfrak{Q}^{(\ell+1)}v)\|_{L^p(T)^d} \\
&\quad \leq c \left(\frac{H_T}{h_T} \sum_{|\varepsilon|=\ell+1} h^{\varepsilon} \left\| \partial^{\varepsilon}_r v \right\|_{L^p(T)^d} + h_T \sum_{|\beta|=\ell} h^{\beta} \|\partial^{\beta}_r \nabla \cdot v\|_{L^p(T)} \right).
\end{aligned} \tag{9.6.10}$$

Furthermore, using a similar argument, from Lemma 1.6.9, (9.4.2), and (9.6.6) together with (3.6.1b),

$$\|\mathfrak{Q}^{(\ell+1)}v - v\|_{L^p(T)^d} \leq c \frac{H_T}{h_T} \sum_{|\varepsilon|=\ell+1} h^{\varepsilon} \left\| \partial^{\varepsilon}_r v \right\|_{L^p(T)^d} . \tag{9.6.11}$$

Therefore, from (9.6.3), (9.6.10), and (9.6.11), we have (9.6.1).

Case in which Condition 3.3.1 is imposed

From Lemma 1.6.9 and (9.4.3),

$$\begin{aligned}
&\|\hat{v}_j - \widehat{Q}^{(\ell+1)}\hat{v}_j\|^p_{W^{1,p}(\widehat{T}_1)} \\
&\quad \leq c|\hat{v}_j|^p_{W^{\ell+1,p}(\widehat{T}_1)} + c \sum_{k=1}^{d} \left| \frac{\partial \hat{v}_j}{\partial \hat{x}_k} \right|^p_{W^{\ell,p}(\widehat{T}_1)} \\
&\quad = c \left(\sum_{|\gamma|=\ell+1} \|\partial^{\gamma}_{\hat{x}} \hat{v}_j\|^p_{L^p(\widehat{T}_1)} + \sum_{k=1}^{d} \sum_{|\beta|=\ell} \left\| \partial^{\beta}_{\hat{x}} \frac{\partial \hat{v}_j}{\partial \hat{x}_k} \right\|^p_{L^p(\widehat{T}_1)} \right) \\
&\quad \leq c|\det(\mathcal{A}^s)|^{p-1} h_j^{-p} \|\widetilde{\mathcal{A}}^{-1}\|_2^p \left(\sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon}_{x^s} v^s\|_{L^p(T_1)^d} \right)^p .
\end{aligned} \tag{9.6.12}$$

Because (9.6.8), from Lemma 1.6.9 and (9.4.5),

$$\|\nabla_{\hat{x}} \cdot (\hat{v} - \widehat{\mathfrak{Q}}^{(\ell+1)}\hat{v}))\|^p_{L^p(\widehat{T}_1)} \leq c|\det(\mathcal{A}^s)|^{p-1} \left(\sum_{|\varepsilon|=\ell} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon}_{x^s} \nabla_{x^s} \cdot v^s\|_{L^p(T_1)} \right)^p . \tag{9.6.13}$$

Combining (9.6.4), (9.6.5), (9.6.12), and (9.6.13) with (3.6.1b) yields

$$\begin{aligned} &\|I_T^{RT^k}(v - \mathfrak{Q}^{(\ell+1)}v)\|_{L^p(T)^d} \\ &\quad \leq c \left(\frac{H_T}{h_T} \sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon}_{x^s} v^s\|_{L^p(T_1)^d} + h_T \sum_{|\beta|=\ell} \mathscr{H}^{\beta} \|\partial^{\beta}_{x^s} \nabla_{x^s} \cdot v^s\|_{L^p(T_1)} \right). \end{aligned} \tag{9.6.14}$$

Furthermore, using a similar argument, from Lemma 1.6.9, (9.4.3), and (9.6.6) together with (3.6.1b),

$$\|\mathfrak{Q}^{(\ell+1)}v - v\|_{L^p(T)^d} \leq c \frac{H_T}{h_T} \sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon}_{x^s} v^s\|_{L^p(T_1)^d}, \tag{9.6.15}$$

Therefore, from (9.6.3), (9.6.14), and (9.6.15), we have (9.6.2). □

Theorem 9.6.2. *Let $d = 3$ and $p \in [0, \infty)$. Let $T \subset \mathbb{R}^3$ be a simplex such that $T_2 = \Phi_{T_2}^{-1}(T)$. Let $T_2 \in \mathfrak{T}_2^{(3)}$ satisfy Condition 3.2.2. For $k \in \mathbb{N}_0$, let $\{T, RT^k(T), \Sigma\}$ be the Raviart–Thomas finite element and $I_T^{RT^k}$ the local interpolation operator defined in* (9.1.12). *Let ℓ be such that $0 \leq \ell \leq k$. For any $\hat{v} \in W^{\ell+1,p}(\widehat{T}_2)^d$ with $\tilde{v} = \widehat{\Psi}\hat{v}$, $v^s = \widetilde{\Psi}\tilde{v}$ and $v = \psi_{T_2} v^s$, it holds that*

$$\|I_T^{RT^k} v - v\|_{L^p(T)^3} \leq c \frac{H_T}{h_T} \left(h_T \sum_{k=1}^{3} \sum_{|\varepsilon|=\ell} h^{\varepsilon} \left\| \partial^{\varepsilon}_r \frac{\partial v}{\partial r_k} \right\|_{L^p(T)^3} \right). \tag{9.6.16}$$

If Condition 3.3.1 is imposed, it holds that

$$\begin{aligned} &\|I_T^{RT^k} v - v\|_{L^p(T)^3} \\ &\quad \leq c \frac{H_T}{h_T} \left(\sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon}_{x^s} (\Psi_{T_2}^{-1} v)\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right. \\ &\qquad \left. + h_T \sum_{k=1}^{3} \sum_{|\varepsilon|=\ell} \mathscr{H}^{\varepsilon} \left\| \partial^{\varepsilon}_{x^s} \frac{\partial (\Psi_{T_2}^{-1} v)}{\partial r_k^s} \right\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right). \end{aligned} \tag{9.6.17}$$

Proof. An analogous proof of Theorem 9.6.1 yields the desired result (9.6.16), where we use Lemma 9.3.6 instead of Lemma 9.3.4, and Lemma 9.4.6 instead of Lemma 9.4.5.

Let $\hat{v} \in W^{\ell+1,p}(\widehat{T}_2)^3$. Let $I_{\widehat{T}_2}^{RT^k}$ be the local interpolation operators on $\widehat{T}_2$ defined by (9.1.4) and (9.1.5). If $q \in \mathcal{P}^\ell(T)^3 \subset RT^k(T)$, then $I_T^{RT^k} q = q$.

We set $\mathfrak{Q}^{(\ell+1)} v := (Q^{(\ell+1)} v_1, Q^{(\ell+1)} v_2, Q^{(\ell+1)} v_3)^T \in \mathcal{P}^\ell(T)^3$, where $Q^{(\ell+1)} v_j^s$ is defined by (1.6.8) for any j. We then obtain

$$\| I_T^{RT^k} v - v \|_{L^p(T)^3} \leq \| I_T^{RT^k} (v - \mathfrak{Q}^{(\ell+1)} v) \|_{L^p(T)^3} + \| \mathfrak{Q}^{(\ell+1)} v - v \|_{L^p(T)^3}. \tag{9.6.18}$$

The inequalities (9.4.1) and (9.4.8) for the first term on the right-hand side of (9.6.18) yield

$$\begin{aligned} &\| I_T^{RT^k} (v - \mathfrak{Q}^{(\ell+1)} v) \|_{L^p(T)^3} \\ &\quad \leq c |\det(\mathcal{A}^s)|^{\frac{1-p}{p}} \| \widetilde{\mathcal{A}} \|_2 \left(\sum_{j=1}^3 h_j^p \| \{ I_{\widehat{T}_2}^{RT^k} (\hat{v} - \widehat{\mathfrak{Q}}^{(\ell+1)} \hat{v}) \}_j \|_{L^p(\widehat{T}_2)}^p \right)^{1/p}. \end{aligned} \tag{9.6.19}$$

The component-wise stability (9.3.12) for $3d$ yields

$$\begin{aligned} &\sum_{j=1}^3 h_j^p \| \{ I_{\widehat{T}_2}^{RT^k} (\hat{v} - \widehat{\mathfrak{Q}}^{(\ell+1)} \hat{v}) \}_j \|_{L^p(\widehat{T}_2)}^p \\ &\quad \leq c \sum_{j=1}^3 h_j^p \left(\| \hat{v}_j - \widehat{Q}^{(\ell+1)} \hat{v}_j \|_{W^{1,p}(\widehat{T}_2)}^p + \sum_{k=1, k \neq j}^3 \left\| \frac{\partial}{\partial \hat{x}_k} (\hat{v} - \widehat{\mathfrak{Q}}^{(\ell+1)} \hat{v})_k \right\|_{L^p(\widehat{T}_2)}^p \right). \end{aligned} \tag{9.6.20}$$

The inequalities (9.4.1) and (9.4.8) for the second term on the right-hand side of (9.6.18) yields

$$\begin{aligned} &\| \mathfrak{Q}^{(\ell+1)} v - v \|_{L^p(T)^3} \\ &\quad \leq c |\det(\mathcal{A}^s)|^{\frac{1-p}{p}} \| \widetilde{\mathcal{A}} \|_2 \left(\sum_{j=1}^d h_j^p \| \widehat{Q}^{(\ell+1)} \hat{v}_j - \hat{v}_j \|_{L^p(\widehat{T}_2)}^p \right)^{1/p}. \end{aligned} \tag{9.6.21}$$

Case in which Condition 3.3.1 is not imposed

From Lemma 1.6.9 and (9.4.2), we have

$$
\begin{aligned}
&\|\hat{v}_j - \widehat{Q}^{(\ell+1)}\hat{v}_j\|^p_{W^{1,p}(\widehat{T}_2)} \\
&\quad = \|\hat{v}_j - \widehat{Q}^{(\ell+1)}\hat{v}_j\|^p_{L^p(\widehat{T}_2)} + \sum_{k=1}^{d} \left\| \frac{\partial}{\partial \hat{x}_k}(\hat{v}_j - \widehat{Q}^{(\ell+1)}\hat{v}_j) \right\|^p_{L^p(\widehat{T}_2)} \\
&\quad \le c \left(\sum_{|\gamma|=\ell+1} \|\partial^\gamma_{\hat{x}} \hat{v}_j\|^p_{L^p(\widehat{T}_2)} + \sum_{k=1}^{3} \sum_{|\beta|=\ell} \left\| \partial^\beta_{\hat{x}} \frac{\partial \hat{v}_j}{\partial \hat{x}_k} \right\|^p_{L^p(\widehat{T}_2)} \right) \\
&\quad \le c|\det(\mathcal{A}^s)|^{p-1} h_j^{-p} \|\widetilde{\mathcal{A}}^{-1}\|_2^p \left(\sum_{|\varepsilon|=\ell+1} h^\varepsilon \|\partial^\varepsilon_r v\|_{L^p(T)^3} \right)^p . \qquad (9.6.22)
\end{aligned}
$$

From Lemma 1.6.9 and (9.4.6), we have

$$
\begin{aligned}
\sum_{k=1,k\neq j}^{3} \left\| \frac{\partial}{\partial \hat{x}_k}(\hat{v}_k - \widehat{Q}^{(\ell+1)}\hat{v}_k)) \right\|^p_{L^p(\widehat{T}_2)} &= \sum_{k=1,k\neq j}^{3} \left\| \frac{\partial \hat{v}_k}{\partial \hat{x}_k} - \widehat{Q}^{(\ell+1)} \left(\frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right) \right\|^p_{L^p(\widehat{T}_2)} \\
&\le c \sum_{k=1,k\neq j}^{3} \sum_{|\beta|=\ell} \left\| \partial^\beta_{\hat{x}} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right\|^p_{L^p(\widehat{T}_2)} \\
&\le c|\det(\mathcal{A}^s)|^{p-1} \|\widetilde{\mathcal{A}}^{-1}\|_2^p \sum_{k=1,k\neq j}^{3} \sum_{|\varepsilon|=\ell} h^{\varepsilon p} \left\| \partial^\varepsilon_r \frac{\partial v}{\partial r_k} \right\|^p_{L^p(T)^3} . \qquad (9.6.23)
\end{aligned}
$$

Gathering (9.6.19), (9.6.20), (9.6.22) and (9.6.23) together with (1.6.1) and (3.6.1b) yields

$$
\begin{aligned}
&\|I_T^{RT^k}(v - \mathfrak{Q}^{(\ell+1)}v)\|_{L^p(T)^3} \\
&\quad \le c \frac{H_T}{h_T} \left(\sum_{|\varepsilon|=\ell+1} h^\varepsilon \|\partial^\varepsilon_r v\|_{L^p(T)^3} + \sum_{j=1}^{3} h_j \sum_{k=1,k\neq j}^{3} \sum_{|\varepsilon|=\ell} h^\varepsilon \left\| \partial^\varepsilon_r \frac{\partial v}{\partial r_k} \right\|_{L^p(T)^3} \right) \\
&\quad \le c \frac{H_T}{h_T} \left(\sum_{|\varepsilon|=\ell+1} h^\varepsilon \|\partial^\varepsilon_r v\|_{L^p(T)^3} + h_T \sum_{k=1}^{3} \sum_{|\varepsilon|=\ell} h^\varepsilon \left\| \partial^\varepsilon_r \frac{\partial v}{\partial r_k} \right\|_{L^p(T)^3} \right). \qquad (9.6.24)
\end{aligned}
$$

From Lemma 1.6.9 and (9.4.2),

$$
\|\hat{v}_j - \widehat{Q}^{(\ell+1)}\hat{v}_j\|^p_{L^p(\widehat{T}_2)} \le c \sum_{|\gamma|=\ell+1} \|\partial^\gamma_{\hat{x}} \hat{v}_j\|^p_{L^p(\widehat{T}_2)}
$$

$$\leq c|\det(\mathcal{A}^s)|^{p-1} h_j^{-p} \|\widetilde{\mathcal{A}}^{-1}\|_2^p \left(\sum_{|\varepsilon|=\ell+1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^3} \right)^p . \tag{9.6.25}$$

From (9.6.21) and (9.6.25) together with (3.6.1b), we obtain

$$\|\mathfrak{Q}^{(\ell+1)} v - v\|_{L^p(T)^3} \leq c \frac{H_T}{h_T} \sum_{|\varepsilon|=\ell+1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^3} . \tag{9.6.26}$$

Therefore, from (9.6.18) and (9.6.24), (9.6.26), we have (9.6.16).

Case in which Condition 3.3.1 is imposed

From Lemma 1.6.9 and (9.4.3), we have

$$\begin{aligned}
&\|\hat{v}_j - \widehat{Q}^{(\ell+1)} \hat{v}_j\|_{W^{1,p}(\widehat{T}_2)}^p \\
&\quad = \|\hat{v}_j - \widehat{Q}^{(\ell+1)} \hat{v}_j\|_{L^p(\widehat{T}_2)}^p + \sum_{k=1}^{d} \left\| \frac{\partial}{\partial \hat{x}_k} (\hat{v}_j - \widehat{Q}^{(\ell+1)} \hat{v}_j) \right\|_{L^p(\widehat{T}_2)}^p \\
&\quad \leq c \left(\sum_{|\gamma|=\ell+1} \|\partial_{\hat{x}}^{\gamma} \hat{v}_j\|_{L^p(\widehat{T}_2)}^p + \sum_{k=1}^{3} \sum_{|\beta|=\ell} \left\| \partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_j}{\partial \hat{x}_k} \right\|_{L^p(\widehat{T}_2)}^p \right) \\
&\quad \leq c|\det(\mathcal{A}^s)|^{p-1} h_j^{-p} \|\widetilde{\mathcal{A}}^{-1}\|_2^p \left(\sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon} (\Psi_{T_2}^{-1} v)\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right)^p .
\end{aligned} \tag{9.6.27}$$

From Lemma 1.6.9 and (9.4.6), we have

$$\begin{aligned}
\sum_{k=1,k\neq j}^{3} \left\| \frac{\partial}{\partial \hat{x}_k} (\hat{v}_k - \widehat{Q}^{(\ell+1)} \hat{v}_k)) \right\|_{L^p(\widehat{T}_2)}^p &= \sum_{k=1,k\neq j}^{3} \left\| \frac{\partial \hat{v}_k}{\partial \hat{x}_k} - \widehat{Q}^{(\ell+1)} \left(\frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right) \right\|_{L^p(\widehat{T}_2)}^p \\
&\leq c \sum_{k=1,k\neq j}^{3} \sum_{|\beta|=\ell} \left\| \partial_{\hat{x}}^{\beta} \frac{\partial \hat{v}_k}{\partial \hat{x}_k} \right\|_{L^p(\widehat{T}_2)}^p \\
&\leq c|\det(\mathcal{A}^s)|^{p-1} \|\widetilde{\mathcal{A}}^{-1}\|_2^p \sum_{k=1,k\neq j}^{3} \sum_{|\varepsilon|=\ell} \mathscr{H}^{\varepsilon} \left\| \partial_{x^s}^{\varepsilon} \frac{\partial (\Psi_{T_2}^{-1} v)}{\partial r_k^s} \right\|_{L^p(\Phi_{T_2}^{-1}(T))^3} .
\end{aligned} \tag{9.6.28}$$

Gathering (9.6.19), (9.6.20), (9.6.27) and (9.6.28) together with (1.6.1) and (3.6.1b) yields

$$\begin{aligned}
&\|I_T^{RT^k}(v - \mathfrak{Q}^{(\ell+1)}v)\|_{L^p(T)^3} \\
&\quad \leq c\frac{H_T}{h_T}\left(\sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon}(\Psi_{T_2}^{-1}v)\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right. \\
&\qquad \left. + \sum_{j=1}^{3} h_j \sum_{k=1,k\neq j}^{3} \sum_{|\varepsilon|=\ell} \mathscr{H}^{\varepsilon} \left\| \partial_{x^s}^{\varepsilon} \frac{\partial(\Psi_{T_2}^{-1}v)}{\partial r_k^s} \right\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right) \\
&\quad \leq c\frac{H_T}{h_T}\left(\sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon}(\Psi_{T_2}^{-1}v)\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right. \\
&\qquad \left. + h_T \sum_{k=1}^{3} \sum_{|\varepsilon|=\ell} \mathscr{H}^{\varepsilon} \left\| \partial_{x^s}^{\varepsilon} \frac{\partial(\Psi_{T_2}^{-1}v)}{\partial r_k^s} \right\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right). \qquad (9.6.29)
\end{aligned}$$

From Lemma 1.6.9 and (9.4.3),

$$\begin{aligned}
&\|\hat{v}_j - \widehat{Q}^{(\ell+1)}\hat{v}_j\|_{L^p(\widehat{T}_2)}^p \\
&\quad \leq c \sum_{|\gamma|=\ell+1} \|\partial_{\hat{x}}^{\gamma}\hat{v}_j\|_{L^p(\widehat{T}_2)}^p \\
&\quad \leq c|\det(\mathcal{A}^s)|^{p-1} h_j^{-p} \|\widetilde{\mathcal{A}}^{-1}\|_2^p \left(\sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon}(\Psi_{T_2}^{-1}v)\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right)^p . \qquad (9.6.30)
\end{aligned}$$

From (9.6.21) and (9.6.30) together with (3.6.1b), we obtain

$$\|\mathfrak{Q}^{(\ell+1)}v - v\|_{L^p(T)^3} \leq c\frac{H_T}{h_T} \sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon}(\Psi_{T_2}^{-1}v)\|_{L^p(\Phi_{T_2}^{-1}(T))^3}. \qquad (9.6.31)$$

Therefore, from (9.6.18) and (9.6.29), (9.6.31), we have (9.6.17).

□

9.7 Global Interpolation Error Estimates

For any $T \in \mathbb{T}_h$, let Φ be the affine mapping defined in (3.4.3). Let $\Psi : V(\widehat{T}) \to V(T)$ be the Piola transformation defined in (3.4.6).

Let $\{\widehat{T}, \widehat{P} := RT^k(\widehat{T}), \widehat{\Sigma}\}$ be the Raviart–Thomas finite element with $k \in \mathbb{N}_0$ introduced in Section 9.1. We define a broken finite element space as

$$RT^k(\mathbb{T}_h) := \left\{ v_h \in L^1(\Omega)^d;\ \Psi^{-1}(v_h|_T) \in RT^k(\widehat{T})\ \forall T \in \mathbb{T}_h \right\}.$$

The corresponding (global) Raviart–Thomas finite element space is defined as

$$V_h^{RT^k} := \{ v_h \in RT^k(\mathbb{T}_h);\ [\![v_h \cdot n]\!]_F = 0,\ \forall F \in \mathcal{F}_h^i \}.$$

Setting $V(\Omega) := W^{s,p}(\Omega)^d$ with $sp > 1$, $p \in (1, \infty)$ or $s = 1$, $p = 1$, we define the global Raviart–Thomas interpolation $I_h^{RT^k} : V(\Omega) \to V_h^{RT^k}$ as

$$(I_h^{RT^k} v)|_T = I_T^{RT^k}(v|_T) \quad \forall T \in \mathbb{T}_h, \quad \forall v \in V(\Omega).$$

Corollary 9.7.1 (de Rham complex)**.** *Let $k \in \mathbb{N}_0$ and $p \in [1, \infty)$. Let $\Pi_h^k : L^1(\Omega) \to M_h^k := \left\{ \varphi_h \in L^1(\Omega);\ \varphi_h|_T \in \mathcal{P}^k(T)\ \forall T \in \mathbb{T}_h \right\}$ be the L^2-orthogonal projection defined as*

$$(\Pi_h^k \varphi)|_T := \Pi_T^k(\varphi|_T),$$

where $\Pi_{T_0}^k$ is defined in Lemma 9.1.2. Let $I_h^{RT^k} : W^{1,p}(\Omega)^d \to V_h^{RT^k}$ be the associated global Raviart–Thomas interpolation. Then, the following diagram commute:

$$\begin{array}{ccc} W^{1,p}(\Omega)^d & \xrightarrow{\operatorname{div}} & L^1(\Omega) \\ {\scriptstyle I_h^{RT^k}} \downarrow & & \downarrow {\scriptstyle \Pi_h^k} \\ V_h^{RT^k} & \xrightarrow[\operatorname{div}]{} & M_h^k \end{array}$$

In other words, it holds that

$$\operatorname{div}(I_h^{RT^k} v) = \Pi_h^k(\operatorname{div} v) \quad \forall v \in W^{1,p}(\Omega)^d. \tag{9.7.1}$$

Proof. Combine Lemma 9.1.2. □

Theorem 9.7.2 (Stability)**.** *We impose Condition 4.3.1 with $h \leq 1$. Let $p \in [1, \infty)$. It then holds that*

$$\|I_h^{RT^k} v\|_{L^p(\Omega)^d} \leq c \|v\|_{W^{1,p}(\Omega)^d} \quad \forall v \in W^{1,p}(\Omega)^d.$$

Proof. Using (9.4.8), Lemmata 9.5.1, and 9.5.2,

$$\|I_h^{RT^k} v\|^p_{L^p(\Omega)^d} = \sum_{T \in \mathbb{T}_h} \|I_T^{RT^k} v\|^p_{L^p(T)^d} \leq c \sum_{T \in \mathbb{T}_h} \|v\|^p_{W^{1,p}(T)^d} = c \|v\|^p_{W^{1,p}(\Omega)^d},$$

which leads to the desired result. □

Theorem 9.7.3. *We impose Condition 4.3.1. Let $p \in [1, \infty)$ and let ℓ be such that $0 \leq \ell \leq k$. It then holds that, for any $v \in W^{\ell+1,p}(\Omega)^d$;*

(I) *if all elements are composed of the type $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is not imposed,*

$$\|I_h^{RT^k} v - v\|_{L^p(\Omega)^d} \leq c \sum_{T \in \mathbb{T}_h} \sum_{|\varepsilon|=\ell+1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^d} + h \left(\sum_{T \in \mathbb{T}_h} \sum_{|\beta|=\ell} h^{\beta p} \|\partial_r^{\beta} \nabla \cdot v\|_{L^p(T)}^p \right)^{1/p}; \tag{9.7.2}$$

(II) *if all elements are composed of the type $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is imposed,*

$$\begin{aligned} \|I_h^{RT^k} v - v\|_{L^p(\Omega)^d} &\leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial_{x^s}^{\varepsilon} (\Psi_{T_1}^{-1} v)\|_{L^p(\Phi_{T_1}^{-1}(T))^d} \right. \\ &\quad \left. + h_T \sum_{|\beta|=\ell} \mathscr{H}^{\beta} \|\partial_{x^s}^{\beta} \nabla_{x^s} \cdot (\Psi_{T_1}^{-1} v)\|_{L^p(\Phi_{T_1}^{-1}(T))} \right); \end{aligned} \tag{9.7.3}$$

(III) *if all elements are composed of the type $T_2 \in \mathfrak{T}_2^{(3)}$ and Condition 3.3.1 is not imposed,*

$$\|I_h^{RT^k} v - v\|_{L^p(\Omega)^3} \leq ch \left(\sum_{T \in \mathbb{T}_h} \sum_{k=1}^{3} \sum_{|\varepsilon|=\ell} h^{\varepsilon p} \left\| \partial_r^{\varepsilon} \frac{\partial v}{\partial r_k} \right\|_{L^p(T)^3}^p \right)^{1/p}; \tag{9.7.4}$$

(IV) *if all elements are composed of the type $T_2 \in \mathfrak{T}_2^{(3)}$ and Condition 3.3.1 is imposed,*

$$\begin{aligned} \|I_h^{RT^k} v - v\|_{L^p(\Omega)^3} &\leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon|=\ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon} (\Psi_{T_2}^{-1} v)\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right. \\ &\quad \left. + h_T \sum_{k=1}^{3} \sum_{|\varepsilon|=\ell} \mathscr{H}^{\varepsilon} \left\| \partial_{x^s}^{\varepsilon} \frac{\partial (\Psi_{T_2}^{-1} v)}{\partial r_k^s} \right\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right). \end{aligned} \tag{9.7.5}$$

Proof. If $T_1^s \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is not imposed, using (9.6.1),

$$\begin{aligned}
\|I_h^{RT^k} v - v\|_{L^p(\Omega)^d}^p &= \sum_{T \in \mathbb{T}_h} \|I_T^{RT^k} v - v\|_{L^p(T)^d}^p \\
&\le c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon| = \ell+1} h^{\varepsilon} \|\partial_r^{\varepsilon} v\|_{L^p(T)^d} + h_T \sum_{|\beta| = \ell} h^{\beta} \|\partial_r^{\beta} \nabla \cdot v\|_{L^p(T)} \right)^p \\
&\le c \sum_{T \in \mathbb{T}_h} \sum_{|\varepsilon| = \ell+1} h^{\varepsilon p} \|\partial_r^{\varepsilon} v\|_{L^p(T)^d}^p + h_T^p \sum_{|\beta| = \ell} h^{\beta p} \|\partial_r^{\beta} \nabla \cdot v\|_{L^p(T)}^p
\end{aligned}$$

which leads to the desired result together with (1.6.1).

If $T_1^s \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is imposed, using (9.6.2),

$$\begin{aligned}
&\|I_h^{RT^k} v - v\|_{L^p(\Omega)^d}^p \\
&\quad \le c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon| = \ell+1} \mathscr{H}^{\varepsilon} \|\partial_{x^s}^{\varepsilon} (\Psi_{T^s}^{-1} v)\|_{L^p(\Phi_{T_1}^{-1}(T))^d} \right. \\
&\qquad \left. + h_T \sum_{|\beta| = \ell} \mathscr{H}^{\beta} \|\partial_{x^s}^{\beta} \nabla_{x^s} \cdot (\Psi_{T^s}^{-1} v)\|_{L^p(\Phi_{T_1}^{-1}(T))} \right)^p,
\end{aligned}$$

which leads to the desired result together with (1.6.1).

If $T_2^s \in \mathfrak{T}_2^{(3)}$ and Condition 3.3.1 is not imposed, using (9.6.16),

$$\|I_h^{RT^k} v - v\|_{L^p(\Omega)^3}^p \le c \sum_{T \in \mathbb{T}_h} h_T^p \sum_{k=1}^{3} \sum_{|\varepsilon| = \ell} h^{\varepsilon p} \left\| \partial_r^{\varepsilon} \frac{\partial v}{\partial r_k} \right\|_{L^p(T)^3}^p,$$

which leads to the desired result.

If $T_2^s \in \mathfrak{T}_2^{(3)}$ and Condition 3.3.1 is imposed, using (9.6.16),

$$\begin{aligned}
&\|I_h^{RT^k} v - v\|_{L^p(\Omega)^3} \\
&\quad \le c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon| = \ell+1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon} (\Psi_{T_2}^{-1} v)\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right. \\
&\qquad \left. + h_T \sum_{k=1}^{3} \sum_{|\varepsilon| = \ell} \mathscr{H}^{\varepsilon} \left\| \partial_{x^s}^{\varepsilon} \frac{\partial (\Psi_{T_2}^{-1} v)}{\partial r_k^s} \right\|_{L^p(\Phi_{T_2}^{-1}(T))^3} \right),
\end{aligned}$$

which leads to the desired result. □

Chapter 10

Inverse Inequalities on Anisotropic Meshes

This chapter presents some limited results of inverse inequalities.

10.1 Inverse Inequalities

Lemma 10.1.1. *Let $\widehat{P} := \mathcal{P}^k$ with $k \in \mathbb{N}$. Then, if $d = 2$, there exist positive constants $C_i^{IV,2}$, $i = 1, 2$, independent of h_T and T, such that, for all $\varphi_h \in P^s = \{\hat{\varphi}_h \circ (\Phi^s)^{-1};\ \hat{\varphi}_h \in \widehat{P}\}$,*

$$\left\| \frac{\partial \varphi_h}{\partial x_i} \right\|_{L^q(T^s)} \leq C_i^{IV,2} |T^s|^{\frac{1}{q} - \frac{1}{p}} \frac{1}{\mathscr{H}_i} \|\varphi_h\|_{L^p(T^s)}, \quad i = 1, 2. \tag{10.1.1}$$

If $d = 3$, there exist positive constants $C_i^{IV,3}$, $i = 1, 2, 3$, independent of h_T and T, such that, for all $\varphi_h \in P^s = \{\hat{\varphi}_h \circ (\Phi^s)^{-1};\ \hat{\varphi}_h \in \widehat{P}\}$,

$$\left\| \frac{\partial \varphi_h}{\partial x_i} \right\|_{L^q(T^s)} \leq C_i^{IV,3} |T^s|^{\frac{1}{q} - \frac{1}{p}} \frac{1}{\mathscr{H}_i} \|\varphi_h\|_{L^p(T^s)}, \quad i = 1, 2, \tag{10.1.2}$$

and, for $i = 3$,

$$\left\| \frac{\partial \varphi_h}{\partial x_3} \right\|_{L^q(T^s)} \leq C_3^{IV,3} |T^s|^{\frac{1}{q} - \frac{1}{p}} \frac{H_{T^s}}{h_{T^s}} \frac{1}{\mathscr{H}_2} \|\varphi_h\|_{L^p(T^s)}. \tag{10.1.3}$$

In particular, if Condition 3.3.1 is imposed, it holds that

$$\left\| \frac{\partial \varphi_h}{\partial x_3} \right\|_{L^q(T^s)} \leq C_3^{IV,3} |T^s|^{\frac{1}{q} - \frac{1}{p}} \frac{1}{\mathscr{H}_3} \|\varphi_h\|_{L^p(T^s)}. \tag{10.1.4}$$

Proof. Let $\varphi_h \in P^s$. From $\hat{x}_j = h_j^{-1}\tilde{x}_j$ and $\tilde{x} = \widetilde{\mathcal{A}}^{-1}x$, we have, for $i = 1,\dots,d$,

$$\frac{\partial \varphi_h}{\partial x_i} = \sum_{j=1}^{d} \frac{\partial \tilde{\varphi}_h}{\partial \tilde{x}_j}\frac{\partial \tilde{x}_j}{\partial x_i} = \sum_{j=1}^{d} \frac{\partial \tilde{\varphi}_h}{\partial \tilde{x}_j}\widetilde{\mathcal{A}}_{ji}^{-1} = \sum_{j=1}^{d} \frac{\partial \hat{\varphi}_h}{\partial \hat{x}_j} h_j^{-1}\widetilde{\mathcal{A}}_{ji}^{-1}.$$

It thus holds that, for $i = 1,\dots,d$,

$$\begin{aligned}\left\| \frac{\partial \varphi_h}{\partial x_i} \right\|_{L^q(T^s)}^q &= \int_{T^s} \left| \frac{\partial \varphi_h}{\partial x_i} \right|^q dx = |\det(\mathcal{A}^s)| \int_{\widehat{T}} \left| \widehat{\frac{\partial \varphi_h}{\partial x_i}} \right|^q d\hat{x} \\ &\leq c|\det(\mathcal{A}^s)| \sum_{j=1}^{d} h_j^{-q} |\widetilde{\mathcal{A}}_{ji}^{-1}|^q \left\| \frac{\partial \hat{\varphi}_h}{\partial \hat{x}_j} \right\|_{L^q(\widehat{T})}^q .\end{aligned}$$

Using (1.6.1), we have

$$\left\| \frac{\partial \varphi_h}{\partial x_i} \right\|_{L^q(T^s)} \leq c|\det(\mathcal{A}^s)|^{\frac{1}{q}} \sum_{j=1}^{d} h_j^{-1} |\widetilde{\mathcal{A}}_{ji}^{-1}| \left\| \frac{\partial \hat{\varphi}_h}{\partial \hat{x}_j} \right\|_{L^q(\widehat{T})} . \tag{10.1.5}$$

All the norms in $\widehat{P}$ are equivalent, that is, there exists a positive constant C^E depending on $\widehat{T}$ and s ($s \in \mathbb{N}_0$) such that

$$\|\hat{\varphi}_h\|_{W^{s,\infty}(\widehat{T})} \leq C^E \|\hat{\varphi}_h\|_{L^1(\widehat{T})} \quad \forall \hat{\varphi}_h \in \widehat{P}. \tag{10.1.6}$$

Together with (10.1.6) and (5.3.2) ($m = \ell = 0$), we have, for $j = 1,\dots,d$,

$$\left\| \frac{\partial \hat{\varphi}_h}{\partial \hat{x}_j} \right\|_{L^q(\widehat{T})} \leq \|\hat{\varphi}_h\|_{W^{1,q}(\widehat{T})} \leq c\|\hat{\varphi}_h\|_{L^p(\widehat{T})} \leq c|\det(\mathcal{A}^s)|^{-\frac{1}{p}} \|\varphi_h\|_{L^p(T^s)}. \tag{10.1.7}$$

Two-dimensional case

Because (3.1.4), $|s| \leq 1$ and $h_2 \leq h_1$, we have

$$\sum_{j=1}^{2} h_j^{-1} |\widetilde{\mathcal{A}}_{ji}^{-1}| = \begin{cases} \dfrac{1}{h_1} = \dfrac{1}{\mathscr{H}_1} & \text{if } i = 1, \\ \dfrac{|s|}{h_1 t} + \dfrac{1}{h_2 t} \leq \dfrac{2}{h_2 t} = \dfrac{2}{\mathscr{H}_2} & \text{if } i = 2. \end{cases} \tag{10.1.8}$$

From (10.1.5), (10.1.7) and (10.1.8), we have (10.1.1) for $i = 1, 2$.

Three-dimensional case

Because (3.1.6), $|s_1| \leq 1$ and $h_2 \leq h_3 \leq h_1$, we have, for $\widetilde{\mathcal{A}}^{-1} \in \{\widetilde{\mathcal{A}}_1^{-1}, \widetilde{\mathcal{A}}_2^{-1}\}$,

$$\sum_{j=1}^{3} h_j^{-1} |\widetilde{\mathcal{A}}_{ji}^{-1}| \leq \begin{cases} \dfrac{1}{h_1} = \dfrac{1}{\mathscr{H}_1} & \text{if } i = 1, \\ \dfrac{|s_1|}{h_1 t_1} + \dfrac{1}{h_2 t_1} \leq \dfrac{2}{h_2 t_1} = \dfrac{2}{\mathscr{H}_2} & \text{if } i = 2, \\ \dfrac{|s_1 s_{22}| + |t_1 s_{21}|}{h_1 t_1 t_2} + \dfrac{|s_{22}|}{h_2 t_1 t_2} + \dfrac{1}{h_3 t_2} & \text{if } i = 3. \end{cases} \tag{10.1.9}$$

Because $0 < t_1 \leq 1$, $|s_{21}| \leq 1$, and $|s_{22}| \leq 1$, if Condition 3.3.1 is not imposed, we have

$$\begin{aligned} &\frac{|s_1 s_{22}| + |t_1 s_{21}|}{h_1 t_1 t_2} + \frac{|s_{22}|}{h_2 t_1 t_2} + \frac{1}{h_3 t_2} \\ &\quad = \frac{\frac{1}{6}(|s_1 s_{22}| h_2 h_3 + |t_1 s_{21}| h_2 h_3 + |s_{22}| h_1 h_3 + h_1 h_2 t_1)}{\frac{1}{6} h_1 h_2 h_3 t_1 t_2} \\ &\quad \leq \frac{1}{6|T|} \left(\frac{2 h_1 h_2 h_3 t_1}{h_1 t_1} + \frac{h_1 h_2 h_3 t_1}{h_2 t_1} + \frac{h_1 h_2 h_3 t_1^2}{h_3 t_1} \right) \\ &\quad \leq \frac{2}{3} \frac{h_1 h_2 h_3}{|T^s|} \frac{1}{h_2 t_1} = \frac{2 H_T}{3 h_T} \frac{1}{\mathscr{H}_2}. \end{aligned} \tag{10.1.10}$$

If Condition 3.3.1 is imposed, that is, there exists a positive constant M independent of h_T such that $|s_{22}| \leq M \frac{h_2 t_1}{h_3}$, we have

$$\begin{aligned} \frac{|s_1 s_{22}| + |t_1 s_{21}|}{h_1 t_1 t_2} + \frac{|s_{22}|}{h_2 t_1 t_2} + \frac{1}{h_3 t_2} &\leq M \frac{h_2}{h_1 h_3 t_2} + \frac{1}{h_1 t_2} + M \frac{1}{h_3 t_2} + \frac{1}{h_3 t_2} \\ &\leq 2(M+1) \frac{1}{h_3 t_2} = 2(M+1) \frac{1}{\mathscr{H}_3}. \end{aligned} \tag{10.1.11}$$

From (10.1.5), (10.1.7), (10.1.9), (10.1.10), and (10.1.11), we have (10.1.2), (10.1.3) and (10.1.4) for $i = 1, 2, 3$. □

Theorem 10.1.2. *Let $\widehat{P} := \mathcal{P}^k$ with $k \in \mathbb{N}_0$. Let $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}_0^d$ be a multi-index such that $0 \leq |\gamma| \leq k$. If Condition 3.3.1 is imposed is imposed, there exists a positive constant C^{IVC}, independent of h_T and T, such that, for all $\varphi_h \in P^s = \{\hat{\varphi}_h \circ (\Phi^s)^{-1};\ \hat{\varphi}_h \in \widehat{P}\}$,*

$$\|\partial^\gamma \varphi_h\|_{L^q(T^s)} \leq C^{IVC} |T^s|^{\frac{1}{q} - \frac{1}{p}} \mathscr{H}^{-\gamma} \|\varphi_h\|_{L^p(T^s)}. \tag{10.1.12}$$

Proof. When $|\gamma| = 0$, the proof is standard (see, for example, [30, Lemma 1.138]).

Let $\varphi_h \in P^s$ and $|\gamma| > 0$. From $\hat{x}_j = h_j^{-1} \tilde{x}_j$ and $\tilde{x} = \widetilde{\mathcal{A}}^{-1} x$, we have, for $1 \leq i, k \leq d$,

$$\frac{\partial^2 \varphi_h}{\partial x_i \partial x_k} = \sum_{j,n=1}^{d} \frac{\partial^2 \hat{\varphi}_h}{\partial \hat{x}_j \partial \hat{x}_n} h_j^{-1} h_n^{-1} \widetilde{\mathcal{A}}_{ji}^{-1} \widetilde{\mathcal{A}}_{nk}^{-1}.$$

It thus holds that, for $1 \leq i, k \leq d$,

$$\left\| \frac{\partial^2 \varphi_h}{\partial x_i \partial x_k} \right\|_{L^q(T^s)} \leq c |\det(\mathcal{A}^s)|^{\frac{1}{q}} \sum_{j,n=1}^{d} h_j^{-1} h_n^{-1} |\widetilde{\mathcal{A}^s}_{ji}^{-1}| |\widetilde{\mathcal{A}}_{nk}^{-1}| \left\| \frac{\partial^2 \hat{\varphi}_h}{\partial \hat{x}_j \partial \hat{x}_n} \right\|_{L^q(\widehat{T})}. \tag{10.1.13}$$

Together with (10.1.6) and (5.3.2) ($m = \ell = 0$), we have, for $1 \leq j, n \leq d$,

$$\left\| \frac{\partial^2 \hat{\varphi}_h}{\partial \hat{x}_j \partial \hat{x}_n} \right\|_{L^q(\widehat{T})} \leq \|\hat{\varphi}_h\|_{W^{2,q}(\widehat{T})} \leq c \|\hat{\varphi}_h\|_{L^p(\widehat{T})} \leq c |\det(\mathcal{A}^s)|^{-\frac{1}{p}} \|\varphi_h\|_{L^p(T^s)}. \tag{10.1.14}$$

Furthermore, it holds that, for $1 \leq i, k \leq d$,

$$\sum_{j,n=1}^{d} h_j^{-1} h_n^{-1} |\widetilde{\mathcal{A}}_{ji}^{-1}| |\widetilde{\mathcal{A}}_{nk}^{-1}| \leq c \frac{1}{\mathscr{H}_i \mathscr{H}_k}. \tag{10.1.15}$$

From (10.1.13), (10.1.14) and (10.1.15) together with (3.6.2), we obtain

$$\left\| \frac{\partial^2 \varphi_h}{\partial x_i \partial x_k} \right\|_{L^q(T^s)} \leq c |T^s|^{\frac{1}{q} - \frac{1}{p}} \frac{1}{\mathscr{H}_i \mathscr{H}_k} \|\varphi_h\|_{L^p(T^s)}.$$

By repeating the above argument, we obtain, for a general derivative $\partial^\gamma \varphi_h$ with $|\gamma|$,

$$\|\partial^\gamma \varphi_h\|_{L^q(T^s)} \leq C^{IVC} |T^s|^{\frac{1}{q} - \frac{1}{p}} \mathscr{H}^{-\gamma} \|\varphi_h\|_{L^p(T^s)}.$$

□

Part IV

Applications

Chapter 11

Second-order Elliptic PDEs: Non-conforming Approximation

11.1 Continuous Problem

We consider the Poisson problem as follows. Find $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{11.1.1}$$

where $f \in L^2(\Omega)$ is a given function. The variational formulation for the Poisson problem (11.1.1) is then as follows. Find $u \in H_0^1(\Omega)$ such that

$$a_0(u, \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega), \tag{11.1.2}$$

where $a_0 : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ denotes a bilinear form defined by

$$a_0(u, \varphi) := (\nabla u, \nabla \varphi).$$

By the Lax–Milgram lemma, there exists a unique solution $u \in H_0^1(\Omega)$ for any $f \in L^2(\Omega)$ and it holds that

$$|u|_{H^1(\Omega)} \leq C_P(\Omega)\|f\|,$$

where $C_P(\Omega)$ is the Poincaré constant depending on Ω. Furthermore, if Ω is convex, then $u \in H^2(\Omega)$ and

$$|u|_{H^2(\Omega)} \leq \|\Delta u\|. \tag{11.1.3}$$

The proof can be found in, for example, [40, Theorem 3.1.1.2, Theorem 3.2.1.2].

11.2 Crouzeix–Raviart Finite Element Approximation

11.2.1 Finite Element Approximation

We introduce the following subspace of V_h^{CR} for the homogeneous Dirichlet boundary condition.

$$V_{h0}^{CR} := \left\{ \varphi_h \in V_h^{CR} : \int_F \varphi_h ds = 0 \ \forall F \in \mathcal{F}_h^{\partial} \right\}.$$

The Crouzeix–Raviart finite element problem is to find $u_h^{CR} \in V_{h0}^{CR}$ such that

$$a_h^{CR}(u_h^{CR}, \varphi_h) = \ell_h(\varphi_h) \quad \forall \varphi_h \in V_{h0}^{CR}, \tag{11.2.1}$$

where $a_h^{CR} : (V_{h0}^{CR} + H_0^1(\Omega)) \times (V_{h0}^{CR} + H_0^1(\Omega)) \to \mathbb{R}$ and $\ell_h : V_{h0}^{CR} \to \mathbb{R}$ are defined as

$$a_h^{CR}(\psi_h, \varphi_h) := \sum_{T \in \mathbb{T}_h} \int_T \nabla \psi_h \cdot \nabla \varphi_h dx = (\nabla_h \psi_h, \nabla_h \varphi_h),$$
$$\ell_h(\varphi_h) := \int_\Omega f \varphi_h dx.$$

This problem is nonconforming because $CR_{h0}^1 \not\subset H_0^1(\Omega)$.

11.2.2 Discrete Poincaré Inequality, Well-posedness, Stability

We propose the discrete Poincaré inequality on anisotropic meshes, c.f., [62].

Lemma 11.2.1 (Discrete Poincaré inequality on anisotropic meshes)**.** *Let $p \in (\frac{2d}{d+2}, 2]$ and Ω is $W^{2,p}$-regular domain in $\mathbb{R}^s$. We impose Condition 4.3.1 with $h \leq 1$. Then, there exists $C(\Omega)$, independent of h such that*

$$\|\varphi_h\|_{L^2(\Omega)} \leq C(\Omega) |\varphi_h|_{H^1(\mathbb{T}_h)} \quad \forall \varphi_h \in V_{h0}^{CR}. \tag{11.2.2}$$

Proof. Let $\varphi_h \in V_{h0}^{CR}$. We consider the dual problem. Let $z \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ be such that

$$-\Delta z = \varphi_h \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \partial\Omega,$$

with a priori estimates:

$$|z|_{H^1(\Omega)} \leq C_P \|\varphi_h\|_{L^2(\Omega)}, \quad \|z\|_{W^{2,p}(\Omega)} \leq c \|\varphi_h\|_{L^p(\Omega)} \leq c \|\varphi_h\|_{L^2(\Omega)},$$

where C_P is the Poincaré constant and we used the fact that all the norms in finite dimensional spaces are equivalent.

To obtain the target estimate, we first introduce simple calculations as follows.

$$
\begin{aligned}
-\int_\Omega \operatorname{div}(\nabla z)\varphi_h dx &= \int_\Omega (\Pi_h^0 \operatorname{div}(\nabla z) - \operatorname{div}(\nabla z))\varphi_h dx - \int_\Omega (\Pi_h^0 \operatorname{div}(\nabla z))\varphi_h dx \\
&= \int_\Omega (\Pi_h^0 \operatorname{div}(\nabla z) - \operatorname{div}(\nabla z))(\varphi_h - \Pi_h^0 \varphi_h) dx \\
&\quad - \int_\Omega (\operatorname{div} I_h^{RT^0}(\nabla z))\varphi_h dx \\
&= - \int_\Omega \operatorname{div}(\nabla z) \left(\varphi_h - \Pi_h^0 \varphi_h\right) dx \\
&\quad - \int_\Omega (\nabla z - I_h^{RT^0}(\nabla z)) \cdot \nabla_h \varphi_h dx + \int_\Omega \nabla z \cdot \nabla_h \varphi_h dx,
\end{aligned}
$$

where

$$
\begin{aligned}
\int_\Omega (\operatorname{div} I_h^{RT^0}(\nabla z))\varphi_h dx &= \sum_{T\in\mathbb{T}_h} \int_{\partial T} n_T \cdot I_h^{RT^0}(\nabla z)\varphi_h ds - \int_\Omega I_h^{RT^0}(\nabla z) \cdot \nabla_h \varphi_h dx \\
&= \int_\Omega (\nabla z - I_h^{RT^0}(\nabla z)) \cdot \nabla_h \varphi_h dx - \int_\Omega \nabla z \cdot \nabla_h \varphi_h dx.
\end{aligned}
$$

We use the duality argument to show the target inequality. That is to say, we have

$$
\begin{aligned}
\|\varphi_h\|_{L^2(\Omega)} &= \frac{1}{\|\varphi_h\|_{L^2(\Omega)}}(\varphi_h, \varphi_h) = (-\Delta z, \varphi_h) = (-\operatorname{div} \nabla z, \varphi_h) \\
&= (-\operatorname{div} \nabla z, \varphi_h - \Pi_h^0 \varphi_h) - (\nabla z - I_h^{RT^0}(\nabla z), \nabla_h \varphi_h) + (\nabla z, \nabla_h \varphi_h) \\
&\le \|\Delta z\| \|\varphi_h - \Pi_h^0 \varphi_h\| + \|\nabla z - I_h^{RT^0}(\nabla z)\|_{L^p(\Omega)} |\varphi_h|_{H^1(\mathbb{T}_h)} + |z|_{H^1(\Omega)} |\varphi_h|_{H^1(\mathbb{T}_h)} \\
&\le c\left(h\|\Delta z\| + h|z|_{W^{2,p}(\Omega)} + |z|_{H^1(\Omega)}\right) |\varphi_h|_{H^1(\mathbb{T}_h)} \\
&\le c\,(h + h + C_P)\,\|\varphi_h\|_{L^2(\Omega)} |\varphi_h|_{H^1(\mathbb{T}_h)},
\end{aligned}
$$

which leads to

$$
\|\varphi_h\|_{L^2(\Omega)} \le c(2 + C_p)|\varphi_h|_{H^1(\mathbb{T}_h)} \quad \text{if } h \le 1.
$$

□

Lemma 11.2.2 (Well-posedness, Stability). *The map*

$$
\varphi_h \mapsto |\varphi_h|_{H^1(\mathbb{T}_h)} \tag{11.2.3}
$$

is a norm on V_{h0}^{CR}. *For any* $f \in L^2(\Omega)$, *the discrete problem* (11.2.1) *is well-posedness. Furthermore, it holds that*

$$|u_h^{CR}|_{H^1(\mathbb{T}_h)} \leq C(\Omega)\|f\|_{L^2(\Omega)}. \tag{11.2.4}$$

Proof. The nontrivial property is to show that $|\varphi_h|_{H^1(\mathbb{T}_h)} = 0$ implies $\varphi_h = 0$ for any $\varphi_h \in V_{h0}^{CR}$. If $|\varphi_h|_{H^1(\mathbb{T}_h)} = 0$, φ_h is piecewise constant. The property $\int_F [\![\varphi_h]\!] ds = 0$ for any $F \in \mathcal{F}_h^i$ implies that φ_h is globally constant on Ω. Thus, $\varphi_h = 0$ follows from $\int_F \varphi_h ds = 0$ for any $F \in \mathcal{F}_h^\partial$.

We easily have, for any $\psi_h, \varphi_h \in V_{h0}^{CR}$,

$$|a_h^{CR}(\psi_h, \varphi_h)| \leq |\psi_h|_{H^1(\mathbb{T}_h)}|\varphi_h|_{H^1(\mathbb{T}_h)},$$
$$a_h^{CR}(\psi_h, \psi_h) = |\psi_h|^2_{H^1(\mathbb{T}_h)}.$$

The discrete Poincaré inequality (11.2.2) yields

$$\|\ell_h\|_{(V_{h0}^{CR})'} = \sup_{\varphi_h \in V_{h0}^{CR}} \frac{|\ell_h(\varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}} \leq C(\Omega)\|f\|_{L^2(\Omega)} < \infty,$$

and thus, $\ell_h \in (V_{h0}^{CR})'$. By the Lax–Milgram lemma, there exists a unique solution $u_h^{CR} \in V_{h0}^{CR}$ for any $f \in L^2(\Omega)$. For the Crouzeix–Raviart approximate solution $u_h^{CR} \in V_{h0}^{CR}$ of (11.2.1), we have a stability estimate, using (11.2.2),

$$|u_h^{CR}|^2_{H^1(\mathbb{T}_h)} \leq \|f\|_{L^2(\Omega)}\|u_h^{CR}\|_{L^2(\Omega)} \leq C(\Omega)\|f\|_{L^2(\Omega)}|u_h^{CR}|_{H^1(\mathbb{T}_h)},$$

which leads to (11.2.4). □

11.3 Discrete Trace Inequality

We introduce a discrete trace inequality. Let $T \in \mathbb{T}_h$ be an element. The faces are defined to be the image by Φ of the faces of the reference element $\widehat{T}$, and these geometric entries are collected in the set $\mathcal{F}_T$.

Lemma 11.3.1 (Discrete trace inequality)**.** *There exists a positive constant* c *such that, for any* $p \in [1, \infty]$, *any* $\varphi \in W^{1,p}(T)$, *any* $T \in \mathbb{T}_h$, *any* $F \in \mathcal{F}_T$, *and any* h,

$$\|\varphi\|_{L^p(F)} \leq c\ell_{T,F}^{-\frac{1}{p}}\|\varphi\|_{L^p(T)}^{1-\frac{1}{p}}\left(\|\varphi\|_{L^p(T)}^{\frac{1}{p}} + h_T\|\nabla\varphi\|_{L^p(T)}^{\frac{1}{p}}\right). \tag{11.3.1}$$

where $\ell_{T,F}$ *denotes the distance of the vertex of* T *opposite to* F *to the face.*

Proof. We follow [31, Lemma 12.15]. Let $T \in \mathbb{T}_h$ and $\varphi \in W^{1,p}(T)$. Assume that $p \in [1, \infty)$. Let F be a face of T and let P_F the vertex of T opposite to F. Consider the Raviart–Thomas function

$$\theta_F(x) := \frac{|F|}{d|T|}(x - P_F).$$

One can verify that the normal component of θ_F is equal to 1 on F and 0 on the other faces of T. Because $\operatorname{div}\theta_F = \frac{|F|}{|T|}$, we infer using the divergence theorem that

$$\begin{aligned}
\|\varphi\|^p_{L^p(F)} &= \int_{\partial T} |\varphi|^p(\theta_F \cdot n)ds = \int_T \operatorname{div}(|\varphi|^p \theta_F)dx \\
&= \int_T (|\varphi|^p \operatorname{div}\theta_F + p\varphi|\varphi|^{p-2}(\theta_F \cdot \nabla)\varphi)dx \\
&= \frac{|F|}{|T|}\|\varphi\|^p_{L^p(T)} + \frac{p|F|}{d|T|}\int_T \varphi|\varphi|^{p-2}(x - P_F)\cdot\nabla\varphi dx.
\end{aligned}$$

Using Hölder's inequality and introducing the length $\ell_F^\perp$ defined as the largest length of an edge of T having P_F as an endpoint, we infer that

$$\|\varphi\|^p_{L^p(F)} \le \frac{|F|}{|T|}\|\varphi\|^p_{L^p(T)} + \frac{p|F|\ell_F^\perp}{d|T|}\|\varphi\|^{p-1}_{L^p(T)}\|\nabla\varphi\|_{L^p(T)}.$$

Together with the fact that $p^{\frac{1}{p}} \le e^{\frac{1}{e}} \le \frac{3}{2}$, $\ell_F^\perp \le h_T$ and $\frac{|F|}{|T|} = \frac{d!}{\ell_{T,F}}$, we conclude that

$$\|\varphi\|_{L^p(F)} \le c\ell_{T,F}^{-\frac{1}{p}}\|\varphi\|^{1-\frac{1}{p}}_{L^p(T)}\left(\|\varphi\|^{\frac{1}{p}}_{L^p(T)} + h_T\|\nabla\varphi\|^{\frac{1}{p}}_{L^p(T)}\right),$$

which is the desired result for $p < \infty$. The bound for $p = \infty$ is obtained by passing to the limit $p \to \infty$ in (11.3.1) because c is uniform w.r.t. p and because $\lim_{p\to\infty}\|\cdot\|_{L^p(T)} = \|\cdot\|_{L^\infty(T)}$. □

Lemma 11.3.2 (Discrete Poincaré inequality on faces)**.** *There exists a positive constant c such that, for any $T \in \mathbb{T}_h$, any $F \in \mathcal{F}_T$, and any h,*

$$\|\varphi - \varphi_F\|_{L^2(F)} \le c\ell_{T,F}^{-\frac{1}{2}}h_T|\varphi|_{H^1(T)}, \tag{11.3.2}$$

for any $\varphi \in H^1(T)$ with $\varphi_F := \frac{1}{|F|}\int_F \varphi ds$.

Proof. We use the idea of the proof in [32, Lemma 36.8]. Let $\overline{\varphi} := \varphi - \frac{1}{|T|}\int_T \varphi dx$. We then have

$$\varphi - \varphi_F = \overline{\varphi} + \frac{1}{|T|}\int_T \varphi dx - \varphi_F = \overline{\varphi} - \frac{1}{|F|}\int_F \overline{\varphi} ds.$$

The Cauchy–Schwarz inequality yields

$$\begin{aligned}\left\| \frac{1}{|F|} \int_F \overline{\varphi} ds \right\|_{L^2(F)} &= |F|^{-\frac{1}{2}} \int_F \overline{\varphi} ds \le |F|^{-\frac{1}{2}} \left(\int_F 1^2 ds \right)^{1/2} \left(\int_F \overline{\varphi}^2 ds \right)^{1/2} \\ &= \|\overline{\varphi}\|_{L^2(F)}.\end{aligned}$$

The triangle inequality implies that

$$\|\varphi - \varphi_F\|_{L^2(F)} \le 2 \|\overline{\varphi}\|_{L^2(F)}.$$

Using the trace inequality (11.3.1) together with (1.6.12) and $\nabla \overline{\varphi} = \nabla \varphi$ yields

$$\begin{aligned}\|\overline{\varphi}\|_{L^2(F)} &\le c \ell_{T,F}^{-\frac{1}{2}} \|\overline{\varphi}\|_{L^2(T)}^{\frac{1}{2}} \left(\|\overline{\varphi}\|_{L^2(T)}^{\frac{1}{2}} + h_T \|\nabla \overline{\varphi}\|_{L^2(T)}^{\frac{1}{2}} \right) \\ &\le c \ell_{T,F}^{-\frac{1}{2}} h_T |\varphi|_{H^1(T)},\end{aligned}$$

which leads to the desire result. □

11.4 Second Strang Lemma

The starting point for error analysis is the Second Strang Lemma, e.g. see [30, Lemma 2.25].

We introduce the space $V_\# := H^2(\Omega) + V_{h0}^{CR}$ equipped with the norm $|\cdot|_{V_\#}$ defined as

$$|\varphi|_{V_\#} := \left(|\varphi|_{H^1(\mathbb{T}_h)}^2 + \sum_{T \in \mathbb{T}_h} \sum_{F \subset \partial T} \ell_{T,F} \|(n_T \cdot \nabla)\varphi|_T\|_{L^2(F)}^2 \right)^{1/2}. \qquad (11.4.1)$$

We assume that Ω is convex, and we impose Condition 4.3.1 with $h \le 1$. For any $\varphi_h \in V_{h0}^{CR}$, using the discrete Poincaré and trace inequalities,

$$|\varphi_h|_{V_\#} \le c_\# |\varphi_h|_{H^1(\mathbb{T}_h)}. \qquad (11.4.2)$$

Lemma 11.4.1. *We assume that Ω is convex, and we impose Condition 4.3.1 with $h \le 1$. For any $f \in L^2(\Omega)$, let u be the solution of* (11.1.2) *and u_h^{CR} the solution of* (11.2.1)*. It then holds that*

$$|u - u_h^{CR}|_{V_\#} \le (1 + c_\#) \inf_{v_h \in V_{h0}^{CR}} |u - v_h|_{V_\#} + c_\# \sup_{\varphi_h \in V_{h0}^{CR}} \frac{|a_h^{CR}(u, \varphi_h) - (f, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}}. \qquad (11.4.3)$$

Proof. For any $v_h \in V_{h0}^{CR}$, using (11.4.2) yields

$$\begin{aligned} |u - u_h^{CR}|_{V_\#} &\le |u - v_h|_{V_\#} + |v_h - u_h^{CR}|_{V_\#} \\ &\le |u - v_h|_{V_\#} + c_\# |v_h - u_h^{CR}|_{H^1(\mathbb{T}_h)} \\ &\le |u - v_h|_{V_\#} + c_\# \sup_{\varphi_h \in V_{h0}^{CR}} \frac{|a_h^{CR}(v_h - u_h^{CR}, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}} \\ &\le (1 + c_\#)|u - v_h|_{V_\#} + c_\# \sup_{\varphi_h \in V_{h0}^{CR}} \frac{|a_h^{CR}(u - u_h^{CR}, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}}, \end{aligned}$$

which leads to (11.4.3). □

We define the global interpolation $I_{h0}^{CR} : H_0^1(\Omega) \to V_{h0}^{CR}$ as follows.

$$(I_{h0}^{CR}\varphi)|_T := I_T^{CR}(\varphi|_T) \quad \forall T \in \mathbb{T}_h, \ \forall \varphi \in H_0^1(\Omega).$$

Lemma 11.4.2 (Best approximation)**.** *We assume that Ω is convex. For any $f \in L^2(\Omega)$, let u be the solution of* (11.1.2)*. It then holds that*

(I) *if Condition 3.3.1 is not imposed,*

$$\inf_{v_h \in V_{h0}^{CR}} |u - v_h|_{V_\#} \le c \left(\sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} h_i \left\| \frac{\partial^2 u}{\partial r_i \partial x_j} \right\|_{L^2(T)} + h|u|_{H^2(\Omega)} \right). \tag{11.4.4}$$

(II) *if Condition 3.3.1 is imposed,*

$$\inf_{v_h \in V_{h0}^{CR}} |u - v_h|_{V_\#} \le c \left(\sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} \mathscr{H}_i \left\| \frac{\partial^2 (u \circ \Phi_{T^s})}{\partial x_i^s \partial x_j^s} \right\|_{L^2(\Phi_{T^s}^{-1}(T))} + h|u|_{H^2(\Omega)} \right). \tag{11.4.5}$$

Proof. Because Ω is convex, $u \in H^2(\Omega)$. From the trace inequality (11.3.1), we have, for any $F \subset \partial T$,

$$\begin{aligned} &\|(n_T \cdot \nabla)(u - I_{h0}^{CR} u)|_T\|_{L^2(F)}^2 \\ &\quad \le c\ell_{T,F}^{-1} \left(|u - I_{h0}^{CR} u|_{H^1(T)}^2 + h_T |u - I_{h0}^{CR} u|_{H^1(T)} |u|_{H^2(T)} \right) \\ &\quad \le c\ell_{T,F}^{-1} \left(\frac{3}{2} |u - I_{h0}^{CR} u|_{H^1(T)}^2 + \frac{1}{2} h_T^2 |u|_{H^2(T)}^2 \right). \end{aligned} \tag{11.4.6}$$

If Condition 3.3.1 is not imposed, from (1.6.1), (7.3.1) and (11.4.6), we have

$$\inf_{v_h \in V_{h0}^{CR}} |u - v_h|_{V_\#} \leq |u - I_{h0}^{CR} u|_{V_\#}$$
$$= \left(|u - I_{h0}^{CR} u|^2_{H^1(\mathbb{T}_h)} + \sum_{T \in \mathbb{T}_h} \sum_{F \subset \partial T} \ell_{T,F} \|(n_T \cdot \nabla)(u - I_{h0}^{CR} u)|_T\|^2_{L^2(F)} \right)^{1/2}$$
$$\leq c \left(\sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} h_i \left\| \frac{\partial^2 u}{\partial r_i \partial x_j} \right\|_{L^2(T)} + h|u|_{H^2(\Omega)} \right).$$

By analogous argument, if Condition 3.3.1 is imposed, from (1.6.1), (7.3.3) and (11.4.6), we have

$$\inf_{v_h \in V_{h0}^{CR}} |u - v_h|_{V_\#} \leq c \left(\sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} \mathscr{H}_i \left\| \frac{\partial^2 u^s}{\partial x_i^s \partial x_j^s} \right\|_{L^2(T^s)} + h|u|_{H^2(\Omega)} \right).$$

□

11.5 Classical Consistency Error Analysis

We first investigate the jumps of functions in V_{h0}^{CR} on meshes violating the shape-regularity condition. We use the idea of the proof of [32, Lemma 36.9].

Lemma 11.5.1. *Let $F \in \mathcal{F}_h^i$ with $F = T_1 \cap T_2$, $T_1, T_2 \in \mathbb{T}_h$ and $F \in \mathcal{F}_h^\partial$ with $F = T \cap \partial\Omega$, $T \in \mathbb{T}_h$. There exists a positive constant c such that, for any $\varphi_h \in V_{h0}^{CR}$,*

$$\sum_{F \in \mathcal{F}_h} \min\{\ell_{T_1,F} h_{T_1}^{-2}, \ell_{T_2,F} h_{T_2}^{-2}\} \|[\![\varphi_h]\!]\|^2_{L^2(F)} \leq c \inf_{v \in H_0^1(\Omega)} \|\nabla_h(\varphi_h - v)\|^2_{L^2(\Omega)}, \tag{11.5.1}$$

where if $F \in \mathcal{F}_h^\partial$, the coefficient $\min\{\ell_{T_1,F} h_{T_1}^{-2}, \ell_{T_2,F} h_{T_2}^{-2}\}$ means that $\ell_{T,F} h_T^{-2}$.

Proof. Let $\varphi_h \in V_{h0}^{CR}$. For any $T \in \mathbb{T}_h$, we set $H_*^1(T) := \{\varphi \in H^1(T) : \int_T \varphi dx = 0\}$ and let $\mathcal{F}_T$ be the collection of the faces of T. For any $F \in \mathcal{F}_T$, we consider the Neumann problem as follows. Find $\psi_{T,F} \in H_*^1(T)$ such that

$$-\Delta \psi_{T,F} = 0 \text{ in } T, \quad \frac{\partial \psi_{T,F}}{\partial n} = \varepsilon_{T,F} \ell_{T,F}^{\frac{1}{2}} h_T^{-1} [\![\varphi_h]\!] \text{ on } F, \quad \frac{\partial \psi_{T,F}}{\partial n} = 0 \text{ otherwise},$$

which leads to the weak form:

$$\int_T \nabla\psi_{T,F}\cdot\nabla\phi dx = \varepsilon_{T,F}\ell_{T,F}^{\frac{1}{2}}h_T^{-1}\int_F [\![\varphi_h]\!]\phi ds \quad \forall\phi\in H_*^1(T), \tag{11.5.2}$$

where $\varepsilon_{T,F} := n_T\cdot n_F = \pm 1$. This problem is well-posed because $\varphi_h \in V_{h0}^{CR}$, and the compatibility condition: for any $F\in\mathcal{F}_h$,

$$\ell_{T,F}^{\frac{1}{2}}h_T^{-1}\int_F [\![\varphi_h]\!]ds = 0$$

holds. Because $\psi_{T,F}\in H_*^1(T)$, the trace inequality (11.3.1) together with the Poincaré inequality (1.6.12) implies that

$$\|\psi_{T,F}\|_{L^2(F)} \le c\ell_{T,F}^{-\frac{1}{2}}h_T|\psi_{T,F}|_{H^1(T)}.$$

Setting $\phi := \psi_{T,F}$ as a test function in (11.5.2), we have

$$\begin{aligned}|\psi_{T,F}|_{H^1(T)}^2 &= \varepsilon_{T,F}\ell_{T,F}^{\frac{1}{2}}h_T^{-1}\int_F [\![\varphi_h]\!]\psi_{T,F}ds\\ &\le \ell_{T,F}^{\frac{1}{2}}h_T^{-1}\|[\![\varphi_h]\!]\|_{L^2(F)}\|\psi_{T,F}\|_{L^2(F)}\\ &\le c\|[\![\varphi_h]\!]\|_{L^2(F)}|\psi_{T,F}|_{H^1(T)},\end{aligned}$$

which leads to

$$|\psi_{T,F}|_{H^1(T)} \le c\|[\![\varphi_h]\!]\|_{L^2(F)}. \tag{11.5.3}$$

Let $v\in H_0^1(\Omega)$. Let c_T be the mean value of the function $\varphi_h - v$ over T. The restriction of $(\varphi_h - v - c_T)$ to T is in $H_*^1(T)$. Let $F\in\mathcal{F}_h^i$ with $F = T_1\cap T_2$, $T_1, T_2\in\mathbb{T}_h$ or $F\in\mathcal{F}_h^\partial$. Setting $\phi_T := (\varphi_h - v)|_T - c_T$ as a test function in (11.5.2) and summing over $T\in\mathbb{T}_F$, we have, if $F\in\mathcal{F}_h^i$,

$$\begin{aligned}&\sum_{T\in\mathbb{T}_F}\int_T \nabla\psi_{T,F}\cdot\nabla(\varphi_h - v)|_T dx\\ &\quad= \sum_{T\in\mathbb{T}_F}\int_T \nabla\psi_{T,F}\cdot\nabla\phi_T dx\\ &\quad= \sum_{T\in\mathbb{T}_F}\varepsilon_{T,F}\ell_{T,F}^{\frac{1}{2}}h_T^{-1}\int_F [\![\varphi_h]\!]\phi_T ds\\ &\quad= \sum_{T\in\mathbb{T}_F}\varepsilon_{T,F}\ell_{T,F}^{\frac{1}{2}}h_T^{-1}\int_F [\![\varphi_h]\!](\varphi_h|_T - v - c_T)ds\end{aligned}$$

$$\geq \min\{\ell_{T_1,F}^{\frac{1}{2}} h_{T_1}^{-1}, \ell_{T_2,F}^{\frac{1}{2}} h_{T_2}^{-1}\} \sum_{T \in \mathbb{T}_F} \varepsilon_{T,F} \int_F [\![\varphi_h]\!] (\varphi_h|_T - v - c_T) ds$$
$$\geq \min\{\ell_{T_1,F}^{\frac{1}{2}} h_{T_1}^{-1}, \ell_{T_2,F}^{\frac{1}{2}} h_{T_2}^{-1}\} \int_F [\![\varphi_h]\!] [\![\varphi_h - v - c_T]\!] ds$$
$$\geq \min\{\ell_{T_1,F}^{\frac{1}{2}} h_{T_1}^{-1}, \ell_{T_2,F}^{\frac{1}{2}} h_{T_2}^{-1}\} \int_F [\![\varphi_h]\!] [\![\varphi_h - v]\!] ds$$
$$\geq \min\{\ell_{T_1,F}^{\frac{1}{2}} h_{T_1}^{-1}, \ell_{T_2,F}^{\frac{1}{2}} h_{T_2}^{-1}\} \int_F [\![\varphi_h]\!]^2 ds,$$

and $F \in \mathcal{F}_h^{\partial}$,

$$\int_T \nabla \psi_{T,F} \cdot \nabla(\varphi_h - v)|_T dx = \ell_{T,F}^{\frac{1}{2}} h_T^{-1} \int_F [\![\varphi_h]\!]^2 ds,$$

where we used that $\int_F [\![\varphi_h]\!] ds = 0$ to eliminate c_T and the fact that $v \in H_0^1(\Omega)$ to eliminate $[\![v]\!] = 0$. Using the Hölder inequality and (11.5.3) yields

$$\begin{aligned} \int_T \nabla \psi_{T,F} \cdot \nabla(\varphi_h - v)|_T dx &\leq \|\nabla \psi_{T,F}\|_{L^2(T)} \|\nabla(\varphi_h - v)|_T\|_{L^2(T)} \\ &\leq c \|[\![\varphi_h]\!]\|_{L^2(F)} \|\nabla(\varphi_h - v)|_T\|_{L^2(T)}, \end{aligned}$$

which leads to

$$\min\{\ell_{T_1,F}^{\frac{1}{2}} h_{T_1}^{-1}, \ell_{T_2,F}^{\frac{1}{2}} h_{T_2}^{-1}\} \|[\![\varphi_h]\!]\|_{L^2(F)} \leq c \sum_{T \in \mathbb{T}_F} \|\nabla(\varphi_h - v)|_T\|_{L^2(T)}.$$

Therefore, we have

$$\min\{\ell_{T_1,F} h_{T_1}^{-2}, \ell_{T_2,F} h_{T_2}^{-2}\} \|[\![\varphi_h]\!]\|_{L^2(F)}^2 \leq c \sum_{T \in \mathbb{T}_F} \|\nabla(\varphi_h - v)|_T\|_{L^2(T)}^2,$$

which leads to (11.5.1). □

From the discrete Poincaré inequality on faces (11.3.2), we have a consistency error inequality.

Lemma 11.5.2 (Asymptotic Consistency)**.** *We assume that Ω is convex. For any $f \in L^2(\Omega)$, let u be the solution of* (11.1.2)*. It then holds that, for any h and any $\varphi_h \in V_{h0}^{CR}$,*

$$\frac{|a_h^{CR}(u, \varphi_h) - (f, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}} \leq c \left(\sum_{T \in \mathbb{T}_h} \frac{h_T^4}{(\min_{F \in \partial \mathbb{T}_h} \ell_{T,F})^2} |u|_{H^2(T)}^2 \right)^{1/2}, \quad (11.5.4)$$

where $\partial \mathbb{T}_h$ denotes the set of all faces F of $T \in \mathbb{T}_h$

Proof. Because Ω is convex, $u \in H^2(\Omega)$. The normal derivative $\int_F (n_T \cdot \nabla)u$ is then meaningful in $L^2(\partial T)$. Let $\varphi_h \in V_{h0}^{CR}$. Because $-\Delta u = f$, we have

$$\begin{aligned} a_h^{CR}(u, \varphi_h) - (f, \varphi_h) &= \sum_{T \in \mathbb{T}_h} \int_T (\nabla u \cdot \nabla \varphi_h - f\varphi_h) dx \\ &= \sum_{T \in \mathbb{T}_h} \sum_{F \in \partial \mathbb{T}_h} \int_F (n_T \cdot \nabla) u \varphi_h ds. \end{aligned}$$

Because each face F of an element T located inside Ω appears twice in the above sum, we have

$$a_h^{CR}(u, \varphi_h) - (f, \varphi_h) = \sum_{T \in \mathbb{T}_h} \sum_{F \in \partial \mathbb{T}_h} \int_F (n_T \cdot \nabla) u \left(\varphi_h - \overline{\varphi_h}\right) ds$$

with the mean value

$$\overline{\varphi_h} := \frac{1}{|F|} \int_F \varphi_h ds.$$

Furthermore, we get

$$a_h^{CR}(u, \varphi_h) - (f, \varphi_h) = \sum_{T \in \mathbb{T}_h} \sum_{F \in \partial \mathbb{T}_h} \int_F n_T \cdot \left(\nabla u - \overline{\nabla u}\right) \left(\varphi_h - \overline{\varphi_h}\right) ds$$

with the mean value

$$\overline{\nabla u} := \frac{1}{|F|} \int_F \nabla u ds.$$

The Cauchy–Schwarz inequality yields

$$|a_h^{CR}(u, \varphi_h) - (f, \varphi_h)| \leq \sum_{T \in \mathbb{T}_h} \sum_{F \in \partial \mathbb{T}_h} \|\nabla u - \overline{\nabla u}\|_{L^2(F)^d} \|\varphi_h - \overline{\varphi_h}\|_{L^2(F)}.$$

From the discrete Poincaré inequality on faces (11.3.2), we have

$$\begin{aligned} \|\nabla u - \overline{\nabla u}\|_{L^2(F)^d} &\leq c \ell_{T,F}^{-\frac{1}{2}} h_T |u|_{H^2(T)}, \\ \|\varphi_h - \overline{\varphi_h}\|_{L^2(F)} &\leq c \ell_{T,F}^{-\frac{1}{2}} h_T |\varphi_h|_{H^1(T)}. \end{aligned}$$

We consequently get

$$\begin{aligned} |a_h^{CR}(u, \varphi_h) - (f, \varphi_h)| &\leq c \sum_{T \in \mathbb{T}_h} \sum_{F \in \partial \mathbb{T}_h} \frac{h_T^2}{\ell_{T,F}} |u|_{H^2(T)} |\varphi_h|_{H^1(T)} \\ &\leq c \sum_{T \in \mathbb{T}_h} \frac{h_T^2}{\min_{F \in \partial \mathbb{T}_h} \ell_{T,F}} |u|_{H^2(T)} |\varphi_h|_{H^1(T)} \\ &\leq c \left(\sum_{T \in \mathbb{T}_h} \frac{h_T^4}{(\min_{F \in \partial \mathbb{T}_h} \ell_{T,F})^2} |u|_{H^2(T)}^2 \sum_{T \in \mathbb{T}_h} |\varphi_h|_{H^1(T)}^2 \right)^{1/2}, \end{aligned}$$

which leads to (11.5.4). □

Because the order of the nonconforming term does not necessarily become the order h, this inequality may be overestimated.

Example 11.5.3. Let $s \in \mathbb{R}$ with $0 < s \ll 1$. When we use meshes including the tetrahedra T with vertices $(0,0,0)^T$, $(s,0,0)^T$, $(0,s,0)^T$, and $(0,0,s^{\varepsilon})^T$, we have, for $c_1 h_T \leq s \leq c_2 h_T$,

$$\frac{h_T^4}{(\min_{F \in \partial \mathbb{T}_h} \ell_{T,F})^2} |u|^2_{H^2(T)} \leq c h_T^{2(2-\varepsilon)} |u|^2_{H^2(T)}.$$

To overcome the difficulty, we use the relation (11.6.1) in Lemma 11.6.1, e.g., see also [2, 67].

11.6 Error Analysis on Anisotropic Meshes

The following relation plays a vital role in the Crouzeix–Raviart finite element analysis on anisotropic meshes.

Lemma 11.6.1. *It holds that*

$$\int_\Omega (v_h \cdot \nabla_h)\psi_h dx + \int_\Omega \operatorname{div} v_h \psi_h dx = 0 \quad \forall v_h \in V_h^{RT^0}, \quad \forall \psi_h \in H_0^1(\Omega) + V_{h0}^{CR}. \tag{11.6.1}$$

Proof. For any $v_h \in V_h^{RT^0}$ and $\psi_h \in H_0^1(\Omega) + V_{h0}^{CR}$, using Green formula and the fact $v_h \cdot n_F \in \mathcal{P}^0(F)$ for any $F \in \mathcal{F}_h$, we can derive

$$\begin{aligned}
&\int_\Omega (v_h \cdot \nabla_h)\psi_h dx + \int_\Omega \operatorname{div} v_h \psi_h dx \\
&= \sum_{T \in \mathbb{T}_h} \int_{\partial T} (v_h \cdot n_T)\psi_h ds \\
&= \sum_{F \in \mathcal{F}_h} \int_F [\![(v_h \psi_h) \cdot n_F]\!] ds \\
&= \sum_{F \in \mathcal{F}_h} \int_F \left([\![v_h \cdot n_F]\!] \{\{\psi_h\}\} + \{\{v_h\}\} \cdot n_F [\![\psi_h]\!] \right) ds \\
&= 0.
\end{aligned}$$

□

Lemma 11.6.2 (Asymptotic Consistency)**.** *We assume that Ω is convex, and we impose Condition 4.3.1 with $h \leq 1$. For any $f \in L^2(\Omega)$, let u be the solution of* (11.1.2)*. It then holds that*

(I) *if all elements are composed of the type $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is not imposed,*

$$\sup_{\varphi_h \in V_{h0}^{CR}} \frac{|a_h^{CR}(u, \varphi_h) - (f, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}} \leq c \left(\sum_{T \in \mathbb{T}_h} \sum_{i=1}^{d} h_i \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^2(T)} + h \|f\|_{L^2(\Omega)} \right); \tag{11.6.2}$$

(II) *if all elements are composed of the type $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is imposed,*

$$\begin{aligned} &\sup_{\varphi_h \in V_{h0}^{CR}} \frac{|a_h^{CR}(u, \varphi_h) - (f, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}} \\ &\quad \leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{i=1}^{d} \mathscr{H}_i \left\| \frac{\partial}{\partial x_i^s} (\Psi_{T_1}^{-1} \nabla u) \right\|_{L^2(\Phi_{T_1}^{-1}(T))^d} + h_T \|\nabla_{x^s} \cdot (\Psi_{T_1}^{-1} \nabla u)\|_{L^2(\Phi_{T_1}^{-1}(T))} \right) \\ &\qquad + ch\|f\|_{L^2(\Omega)}; \end{aligned} \tag{11.6.3}$$

(III) *if all elements are composed of the type $T_2 \in \mathfrak{T}_2^{(3)}$ and Condition 3.3.1 is not imposed,*

$$\sup_{\varphi_h \in V_{h0}^{CR}} \frac{|a_h^{CR}(u, \varphi_h) - (f, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}} \leq c \left(\sum_{T \in \mathbb{T}_h} \sum_{i=1}^{3} h_i \left\| \frac{\partial}{\partial r_i} \nabla u \right\|_{L^p(T)^3} + h \|f\|_{L^2(\Omega)} \right); \tag{11.6.4}$$

(IV) *if all elements are composed of the type $T_2 \in \mathfrak{T}_2^{(3)}$ and Condition 3.3.1 is imposed,*

$$\begin{aligned} &\sup_{\varphi_h \in V_{h0}^{CR}} \frac{|a_h^{CR}(u, \varphi_h) - (f, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}} \\ &\quad \leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{i=1}^{3} \mathscr{H}_i \left\| \frac{\partial}{\partial x_i^s} (\Psi_{T_2}^{-1} \nabla u) \right\|_{L^2(\Phi_{T_2}^{-1}(T))^3} + h_T \sum_{k=1}^{3} \left\| \frac{\partial (\Psi_{T_2}^{-1} \nabla u)}{\partial r_k^s} \right\|_{L^2(\Phi_{T_2}^{-1}(T))^3} \right) \\ &\qquad + ch\|f\|_{L^2(\Omega)}; \end{aligned} \tag{11.6.5}$$

Proof. Because Ω is convex, $u \in H^2(\Omega)$. Using (11.6.1), we have, for any $w_h \in V_h^{RT^0}$,

$$\sup_{\varphi_h \in V_{h0}^C R} \frac{|a_h^{CR}(u, \varphi_h) - (f, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}} = \sup_{\varphi_h \in V_{h0}^{CR}} \frac{|(\nabla u - w_h, \nabla_h \varphi_h) - (\operatorname{div} w_h + f, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}}. \tag{11.6.6}$$

We set $w_h := I_h^{RT^0} \nabla u$. From (9.7.1), we get

$$\operatorname{div}(I_h^{RT^0} \nabla u) = \Pi_h^0 \operatorname{div}(\nabla u) = \Pi_h^0 (\Delta u).$$

Furthermore, we have, for any $\varphi_h \in V_{h0}^{CR}$,

$$(-\Pi_h^0 f + f, \Pi_h^0 \varphi_h) = 0.$$

We thus obtain

$$\begin{aligned} &|(\nabla u - I_h^{RT^0} \nabla u, \nabla_h \varphi_h) - (-\Pi_h^0 f + f, \varphi_h)| \\ &= |(\nabla u - I_h^{RT^0} \nabla u, \nabla_h \varphi_h) - (-\Pi_h^0 f + f, \varphi_h - \Pi_h^0 \varphi_h)| \\ &\le \|\nabla u - I_h^{RT^0} \nabla u\|_{L^2(\Omega)^d} |\varphi_h|_{H^1(\mathbb{T}_h)} + \|f - \Pi_h^0 f\|_{L^2(\Omega)} \|\varphi_h - \Pi_h^0 \varphi_h\|_{L^2(\Omega)}. \end{aligned} \tag{11.6.7}$$

From the definition of Π_h^0, we have

$$\begin{aligned} \|f - \Pi_h^0 f\|_{L^2(\Omega)}^2 &\le (f - \Pi_h^0 f, f - \Pi_h^0 f) = (f - \Pi_h^0 f, f) \\ &\le \|f - \Pi_h^0 f\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)}, \end{aligned}$$

which leads to

$$\|f - \Pi_h^0 f\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}. \tag{11.6.8}$$

From Theorem 6.3.1,

$$\|\varphi_h - \Pi_h^0 \varphi_h\|_{L^2(\Omega)} \le ch |\varphi_h|_{H^1(\mathbb{T}_h)}. \tag{11.6.9}$$

If all elements are composed of the type $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is not imposed, from (9.7.2),

$$\|I_h^{RT^0} \nabla u - \nabla u\|_{L^2(\Omega)^d} \le c \left(\sum_{T \in \mathbb{T}_h} \sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{\varepsilon} \nabla u\|_{L^2(T)^d} + h \|\Delta u\|_{L^2(\Omega)} \right). \tag{11.6.10}$$

Therefore, (11.6.2) follows from (11.6.6), (11.6.7), (11.6.8), (11.6.9), and (11.6.10).

If all elements are composed of the type $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is imposed, from (9.7.3),

$$\begin{aligned}
&\|I_h^{RT^0} \nabla u - \nabla u\|_{L^2(\Omega)^d} \\
&\quad \leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon|=1} \mathscr{H}^{\varepsilon} \|\partial_{x^s}^{\varepsilon} (\Psi_{T_1}^{-1} \nabla u)\|_{L^2(\Phi_{T_1}^{-1}(T))^d} + h_T \|\nabla_{x^s} \cdot (\Psi_{T_1}^{-1} \nabla u)\|_{L^2(\Phi_{T_1}^{-1}(T))} \right);
\end{aligned} \tag{11.6.11}$$

Therefore, (11.6.3) follows from (11.6.6), (11.6.7), (11.6.8), (11.6.9), and (11.6.11).

If all elements are composed of the type $T_2 \in \mathfrak{T}_2^{(3)}$ and Condition 3.3.1 is not imposed, from (12.4.6),

$$\begin{aligned}
&\|I_h^{RT^0} \nabla u - \nabla u\|_{L^2(\Omega)^3} \\
&\quad \leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{\varepsilon} \nabla u\|_{L^p(T)^3} + h_T \sum_{k=1}^{3} \left\| \frac{\partial}{\partial r_k} \nabla u \right\|_{L^2(T)^3} \right) \\
&\quad \leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{\varepsilon} \nabla u\|_{L^p(T)^3} + h_T |u|_{H^2(T)} \right).
\end{aligned} \tag{11.6.12}$$

Therefore, (11.6.4) follows from (11.6.6), (11.6.7), (11.6.8), (11.6.9), and (11.6.12).

If all elements are composed of the type $T_2 \in \mathfrak{T}_2^{(3)}$ and Condition 3.3.1 is imposed, from (9.7.5),

$$\begin{aligned}
&\|I_h^{RT^0} \nabla u - \nabla u\|_{L^2(\Omega)^3} \\
&\quad \leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon|=1} \mathscr{H}^{\varepsilon} \|\partial^{\varepsilon} (\Psi_{T_2}^{-1} \nabla u)\|_{L^2(\Phi_{T_2}^{-1}(T))^3} + h_T \sum_{k=1}^{3} \left\| \frac{\partial (\Psi_{T_2}^{-1} \nabla u)}{\partial r_k^s} \right\|_{L^2(\Phi_{T_2}^{-1}(T))^3} \right).
\end{aligned} \tag{11.6.13}$$

Therefore, (11.6.5) follows from (11.6.6), (11.6.7), (11.6.8), (11.6.9), and (11.6.13). □

We consequently obtain the error estimate of the Crouzeix–Raviart finite element method on anisotropic meshes.

Theorem 11.6.3. *We assume that* Ω *is convex, and we impose Condition 4.3.1 with* $h \leq 1$. *For any* $f \in L^2(\Omega)$, *let* u *be the solution of* (11.1.2) *and* u_h^{CR} *the solution of* (11.2.1). *It then holds that*

(I) *if all elements are composed of the type* $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ *and Condition 3.3.1 is not imposed,*

$$|u - u_h^{CR}|_{V_\#} \leq c \left(\sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} h_i \left\| \frac{\partial^2 u}{\partial r_i \partial x_j} \right\|_{L^2(T)} + h \|f\|_{L^2(\Omega)} \right); \quad (11.6.14)$$

(II) *if all elements are composed of the type* $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ *and Condition 3.3.1 is imposed,*

$$\begin{aligned} &|u - u_h^{CR}|_{V_\#} \\ &\quad \leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{i,j=1}^{d} \mathscr{H}_i \left\| \frac{\partial^2 (u \circ \Phi_{T_1})}{\partial x_i^s \partial x_j^s} \right\|_{L^2(\Phi_{T_1}^{-1}(T))} + h_T \|\nabla_{x^s} \cdot (\Psi_{T_1}^{-1} \nabla u)\|_{L^2(\Phi_{T_1}^{-1}(T))} \right) \\ &\qquad + ch\|f\|_{L^2(\Omega)}; \end{aligned} \quad (11.6.15)$$

(III) *if all elements are composed of the type* $T_2 \in \mathfrak{T}_2^{(3)}$ *and Condition 3.3.1 is not imposed,*

$$|u - u_h^{CR}|_{V_\#} \leq c \left(\sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{3} h_i \left\| \frac{\partial^2 u}{\partial r_i \partial x_j} \right\|_{L^2(T)} + h \|f\|_{L^2(\Omega)} \right); \quad (11.6.16)$$

(IV) *if all elements are composed of the type* $T_2 \in \mathfrak{T}_2^{(3)}$ *and Condition 3.3.1 is imposed,*

$$\begin{aligned} &|u - u_h^{CR}|_{V_\#} \\ &\quad \leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{i,j=1}^{3} \mathscr{H}_i \left\| \frac{\partial^2 (u \circ \Phi_{T_2})}{\partial x_i^s \partial x_j^s} \right\|_{L^2(\Phi_{T_2}^{-1}(T))} + h_T \sum_{k=1}^{3} \left\| \frac{\partial (\Psi_{T_2}^{-1} \nabla u)}{\partial r_k^s} \right\|_{L^2(\Phi_{T_2}^{-1}(T))^3} \right) \\ &\qquad + ch\|f\|_{L^2(\Omega)}. \end{aligned} \quad (11.6.17)$$

Proof. From (11.4.3), Lemma 11.4.2, and Lemma 11.6.2,

$$\begin{aligned} |u - u_h^{CR}|_{V_\#} &\leq (1 + c_\#) \inf_{v_h \in V_{h0}^{CR}} |u - v_h|_{V_\#} + c_\# \sup_{\varphi_h \in V_{h0}^{CR}} \frac{|a_h^{CR}(u, \varphi_h) - (f, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}} \\ &\leq (1 + c_\#)|u - I_{h0}^{CR} u|_{V_\#} + c_\# \sup_{\varphi_h \in V_{h0}^{CR}} \frac{|a_h^{CR}(u, \varphi_h) - (f, \varphi_h)|}{|\varphi_h|_{H^1(\mathbb{T}_h)}}, \end{aligned}$$

which leads to the desired results. □

11.7 L^2 Error Estimate

We next give the L^2 error estimate of the Crouzeix–Raviart finite element method on anisotropic meshes, also see [67, 22].

Theorem 11.7.1. *We assume that Ω is convex, and we impose Condition 4.3.1 with $h \leq 1$. For any $f \in L^2(\Omega)$, let u be the solution of* (11.1.2) *and u_h^{CR} the solution of* (11.2.1)*. If Condition 3.3.1 is not imposed, it then holds that*

$$\|u - u_h^{CR}\|_{L^2(\Omega)} \leq ch \left(\sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} h_i \left\| \frac{\partial^2 u}{\partial r_i \partial x_j} \right\|_{L^2(T)} + h\|f\|_{L^2(\Omega)} \right). \quad (11.7.1)$$

Proof. We set $e_h := u - u_h^{CR}$. Let $z \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy

$$a_0(\varphi, z) = (\varphi, e_h) \quad \forall \varphi \in H_0^1(\Omega) \quad (11.7.2)$$

and $z_h^{CR} \in V_{h0}^{CR}$ satisfy

$$a_h^{CR}(\varphi_h, z_h^{CR}) = (\varphi_h, e_h) \quad \forall \varphi_h \in V_{h0}^{CR}. \quad (11.7.3)$$

We then have

$$\begin{aligned}
\|e_h\|_{L^2(\Omega)}^2 &= (e_h, e_h) = a_h^{CR}(u, z) - a_h^{CR}(u_h^{CR}, z_h^{CR}) \\
&= a_h^{CR}(u - u_h^{CR}, z - z_h^{CR}) + a_h^{CR}(u - u_h^{CR}, z_h^{CR}) + a_h^{CR}(u_h^{CR}, z - z_h^{CR}) \\
&= a_h^{CR}(u - u_h^{CR}, z - z_h^{CR}) \\
&\quad + a_h^{CR}(u - u_h^{CR}, z_h^{CR} - I_{h0}^{CR} z) + a_h^{CR}(u - u_h^{CR}, I_{h0}^{CR} z) \\
&\quad + a_h^{CR}(u_h^{CR} - I_{h0}^{CR} u, z - z_h^{CR}) + a_h^{CR}(I_{h0}^{CR} u, z - z_h^{CR}). \quad (11.7.4)
\end{aligned}$$

Theorem 11.6.3 and Theorem 7.3.1 yield

$$|z - z_h^{CR}|_{H^1(\mathbb{T}_h)} \leq ch\|e_h\|_{L^2(\Omega)}, \quad (11.7.5)$$

$$|z - I_{h0}^{CR} z|_{H^1(\mathbb{T}_h)} \leq ch\|e_h\|_{L^2(\Omega)}, \quad (11.7.6)$$

$$\|z - I_{h0}^{CR} z\|_{L^2(\Omega)} \leq ch^2\|e_h\|_{L^2(\Omega)}. \quad (11.7.7)$$

Using Theorem 11.6.3 and (11.7.5), the first term on the right hand side of (11.7.4) can be estimated as

$$\begin{aligned}
a_{0h}(u - u_h^{CR}, z - z_h^{CR}) &\leq |u - u_h^{CR}|_{H^1(\mathbb{T}_h)} |z - z_h^{CR}|_{H^1(\mathbb{T}_h)} \\
&\leq ch\|e_h\|_{L^2(\Omega)} \left(\sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} h_i \left\| \frac{\partial^2 u}{\partial r_i \partial x_j} \right\|_{L^2(T)} + h\|f\|_{L^2(\Omega)} \right). \quad (11.7.8)
\end{aligned}$$

For the second and fourth terms on the right hand side of (11.7.5), using Theorem 11.6.3, (11.7.5) and (11.7.6), we have

$$\begin{aligned}
&a_h^{CR}(u-u_h^{CR}, z_h^{CR}-I_{h0}^{CR}z)\\
&\quad = a_{0h}(u-u_h^{CR}, z_h^{CR}-z) + a_{0h}(u-u_h^{CR}, z-I_{h0}^{CR}z)\\
&\quad \le |u-u_h^{CR}|_{H^1(\mathbb{T}_h)}\left(|z_h^{CR}-z|_{H^1(\mathbb{T}_h)} + |z-I_{h0}^{CR}z|_{H^1(\mathbb{T}_h)}\right)\\
&\quad \le ch\|e_h\|_{L^2(\Omega)}\left(\sum_{T\in\mathbb{T}_h}\sum_{i,j=1}^{d} h_i \left\|\frac{\partial^2 u}{\partial r_i \partial x_j}\right\|_{L^2(T)} + h\|f\|_{L^2(\Omega)}\right), \qquad (11.7.9)
\end{aligned}$$

and, analogously,

$$\begin{aligned}
&a_h^{CR}(u_h^{CR}-I_{h0}^{CR}u, z-z_h^{CR})\\
&\quad \le ch\|e_h\|_{L^2(\Omega)}\left(\sum_{T\in\mathbb{T}_h}\sum_{i,j=1}^{d} h_i \left\|\frac{\partial^2 u}{\partial r_i \partial x_j}\right\|_{L^2(T)} + h\|f\|_{L^2(\Omega)}\right), \qquad (11.7.10)
\end{aligned}$$

From (11.7.2), (11.7.3) and (11.6.1), we have

$$\begin{aligned}
&a_h^{CR}(u-u_h^{CR}, I_{h0}^{CR}z)\\
&\quad = a_h^{CR}(u, I_{h0}^{CR}z) - a_h^{CR}(u_h^{CR}, I_{h0}^{CR}z) = (\nabla u, \nabla_h I_{h0}^{CR}z) - (f, I_{h0}^{CR}z)\\
&\quad = (\nabla u, \nabla_h I_{h0}^{CR}z - \nabla z) - (f, I_{h0}^{CR}z - z) + (\nabla u, \nabla z) - (f, z)\\
&\quad = (\nabla u - I_h^{RT^0}\nabla u, \nabla_h I_{h0}^{CR}z - \nabla z) - (f + \operatorname{div}(I_h^{RT^0}\nabla u), I_{h0}^{CR}z - z).
\end{aligned}$$

From $\operatorname{div}(I_h^{RT^0}\nabla u) = \Pi_h^0(\operatorname{div}\nabla u) = -\Pi_h^0 f$, using (9.7.2), (11.6.8), (11.7.6), and (11.7.7), we have

$$\begin{aligned}
&a_{0h}(u-u_h^{CR}, I_{h0}^{CR}z)\\
&\quad = (\nabla u - I_h^{RT^0}\nabla u, \nabla_h I_{h0}^{CR}z - \nabla z) - (f - \Pi_h^0 f, I_{h0}^{CR}z - z)\\
&\quad \le \|\nabla u - I_h^{RT^0}\nabla u\|_{L^2(\Omega)^d}|I_{h0}^{CR}z - z|_{H^1(\mathbb{T}_h)} + \|f-\Pi_h^0 f\|_{L^2(\Omega)}\|I_{h0}^{CR}z - z\|_{L^2(\Omega)}\\
&\quad \le ch\|e_h\|_{L^2(\Omega)}\left(\sum_{T\in\mathbb{T}_h}\sum_{i,j=1}^{d} h_i \left\|\frac{\partial^2 u}{\partial r_i \partial x_j}\right\|_{L^2(T)} + h\|f\|_{L^2(\Omega)}\right). \qquad (11.7.11)
\end{aligned}$$

Analogously, from $\operatorname{div}(I_h^{RT^0}\nabla z) = -\Pi_h^0 e_h$, we have

$$\begin{aligned}
&a_h^{CR}(I_{h0}^{CR}u, z-z_h^{CR})\\
&\quad = (\nabla_h I_{h0}^{CR}u - \nabla u, \nabla z - I_h^{RT^0}\nabla z) - (I_{h0}^{CR}u - u, e_h + \operatorname{div}(I_h^{RT^0}\nabla z))\\
&\quad \le |I_{h0}^{CR}u - u|_{H^1(\mathbb{T}_h)}\|\nabla z - I_h^{RT^0}\nabla z\|_{L^2(\Omega)^d} + \|I_{h0}^{CR}u - u\|_{L^2(\Omega)}\|e_h - \Pi_h^0 e_h\|_{L^2(\Omega)}\\
&\quad \le ch\|e_h\|_{L^2(\Omega)}\left(\sum_{T\in\mathbb{T}_h}\sum_{i,j=1}^{d} h_i \left\|\frac{\partial^2 u}{\partial r_i \partial x_j}\right\|_{L^2(T)} + h\|f\|_{L^2(\Omega)}\right), \qquad (11.7.12)
\end{aligned}$$

where we used

$$|I_{h0}^{CR}u - u|_{H^1(\mathbb{T}_h)} \leq c \sum_{T\in\mathbb{T}_h} \sum_{i,j=1}^{d} h_i \left\| \frac{\partial^2 u}{\partial r_i \partial x_j} \right\|_{L^2(T)},$$

$$\|\nabla z - I_h^{RT^0} \nabla z\|_{L^2(\Omega)^d} \leq ch|z|_{H^2(\Omega)}.$$

Combining (11.7.4), (11.7.8), (11.7.9), (11.7.10), (11.7.11), and (11.7.12), we finally get

$$\|e_h\|_{L^2(\Omega)} \leq ch \left(\sum_{T\in\mathbb{T}_h} \sum_{i,j=1}^{d} h_i \left\| \frac{\partial^2 u}{\partial r_i \partial x_j} \right\|_{L^2(T)} + h\|f\|_{L^2(\Omega)} \right),$$

which leads to the target estimate. □

11.8 What happens if the Synge's condition is violated? - Numerical Results

This section presents results of numerical examples. Let $\Omega := (0,1)^3$. Let u_h^L and u_h^{CR} be the $\mathcal{P}^1$-Lagrange and $\mathcal{P}^1$-Crouzeix–Raviart finite element solutions, respectively, for the model problem

$$\begin{aligned} -\Delta u &= 2y(1-y)z(1-z) + 2x(1-x)z(1-z) + 2x(1-x)y(1-y) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

which is the exact solution $u = x(1-x)y(1-y)z(1-z)$. Then, $u \in H_0^1(\Omega) \cap W^{2,\infty}(\Omega)$.

Let M be the division number of each side of the bottom face and N the division number of the height of Ω with $N \sim M^\gamma$ (see Fig. 11.1). There are two elements as shown in Fig. 11.2. We set $h := \frac{1}{M}$ with $M = 4, 8, 16, 32$. In this mesh sequence, the Synge's condition is not satisfied for $\gamma > 1$ because

$$\frac{H_T}{h_T} = h_T^{1-\gamma} \to \infty \quad \text{as } h_T \to 0.$$

We set $H := H(h) := \max_{T\in\mathbb{T}_h} H_T. = \mathcal{O}(h^{2-\gamma})$, and it holds that

$$\frac{h_{\max}}{h_{\min}} \leq c.$$

For $T \in \mathbb{T}_h$, let $I_T^L : \mathcal{C}(T) \to \mathcal{P}^1$ be the local Lagrange interpolation operator and let $I_h^L : \mathcal{C}(\overline{\Omega}) \to V_h$ the (global) Lagrange interpolation operator such that $I_T^L(u|_T) = (I_h^L u)|_T$, where $V_h \subset H_0^1(\Omega)$ is the piecewise linear Lagrange finite element space.

From (5.8.11) in Corollary 5.8.3 with (5.3.11), we have

$$|u - I_T^L u|_{H^1(T)} \le c|T|^{\frac{1}{2}} \frac{H_T}{h_T} \sum_{|\varepsilon|=1} h^\varepsilon |\partial_r^\varepsilon \varphi^s|_{W^{1,\infty}(T^s)} \le c h_T^{3-\frac{\gamma}{2}} |u|_{W^{2,\infty}(T)}.$$

The theoretical error estimate in the Lagrange element finite element method is then as follows:

$$\begin{aligned} |u - u_h^L|_{H^1(\Omega)} &\le \inf_{v_h \in V_h} |u - v_h|_{H^1(\Omega)} \\ &\le |u - I_h^L u|_{H^1(\Omega)} \le c h^{3-\frac{\gamma}{2}} |u|_{W^{2,\infty}(\Omega)} \quad \text{if } \varepsilon < 6. \end{aligned} \tag{11.8.1}$$

However, expected theoretical error estimates are then as follows:

$$\begin{aligned} |u - u_h^L|_{H^1(\Omega)} &= \mathcal{O}(H), \quad \|u - u_h^L\| = \mathcal{O}(H^2), \\ |u - u_h^{CR}|_{H^1(\mathbb{T}_h)} &= \mathcal{O}(H), \quad \|u - u_h^{CR}\| = \mathcal{O}(H^2). \end{aligned}$$

If an exact solution u is known, the error $e_h := u - u_h$ and $e_{h/2} := u - u_{h/2}$ are computed numerically for two mesh sizes h and $h/2$. The convergence indicator r is defined by

$$r = \frac{1}{\log(2)} \log\left(\frac{\|e_h\|_X}{\|e_{h/2}\|_X}\right).$$

We compute the convergence order with respect to H_0^1 and L^2 norms defined by

$$\begin{aligned} Err_h^L(H^1) &:= \frac{|u - u_h^L|_{H^1(\Omega)}}{\|\Delta u\|}, \quad Err_h^L(L^2) := \frac{\|u - u_h^L\|}{\|\Delta u\|}, \\ Err_h^{CR}(H^1) &:= \frac{|u - u_h^{CR}|_{H^1(\mathbb{T}_h)}}{\|\Delta u\|}, \quad Err_h^{CR}(L^2) := \frac{\|u - u_h^{CR}\|}{\|\Delta u\|}, \end{aligned}$$

for three cases: $\gamma = 1.5$, $\gamma = 1.9$ and $\gamma = 2.0$. In order to compute the above norms, we use the five-order fifteen-point numerical integration introduced in [54]. The results are give in Table 11.1, Table 11.2 when $\gamma = 1.5$, Table 11.3, Table 11.4 when $\gamma = 1.9$, and Table 11.5, Table 11.6 when $\gamma = 2.0$. Further, N_p^L and N_p^{CR} denote respectively the degrees of freedom for the $\mathcal{P}^1$-Lagrange finite element and the $\mathcal{P}^1$-Crouzeix–Raviart finite element.

Observing the numerical results, the convergence indicators r in each norms are respectively

$$\begin{aligned} |u - u_h^L|_{H^1(\Omega)} &= \mathcal{O}(H), \quad \|u - u_h^L\| = \mathcal{O}(H^2), \\ |u - u_h^{CR}|_{H^1(\mathbb{T}_h)} &= \mathcal{O}(h), \quad \|u - u_h^{CR}\| = \mathcal{O}(h^2). \end{aligned}$$

Table 11.1: Error of the $\mathcal{P}^1$-Lagrange finite element solution ($\gamma = 1.5$)

M	N	h	H	N_p^L	$Err_h^L(H^1)$	r	$Err_h^L(L^2)$	r
4	8	2.50e-01	5.00e-01	225	1.2043e-01		9.5321e-03	
8	22	1.25e-01	3.54e-01	1,863	7.0318e-02	0.78	3.1646e-03	1.59
16	64	6.25e-02	2.50e-01	18,785	4.4662e-02	0.65	1.2570e-03	1.33
32	182	3.13e-02	1.77e-01	199,287	2.9479e-02	0.60	5.4477e-04	1.21

Table 11.2: Error of the $\mathcal{P}^1$-CR finite element solution ($\gamma = 1.5$)

M	N	h	H	N_p^{CR}	$Err_h^{CR}(H^1)$	r	$Err_h^{CR}(L^2)$	r
4	8	2.50e-01	5.00e-01	1,440	8.2569e-02		3.8242e-03	
8	22	1.25e-01	3.54e-01	14,912	4.0629e-02	1.02	8.8356e-04	2.11
16	64	6.25e-02	2.50e-01	168,448	2.0042e-02	1.02	2.0485e-04	2.11
32	182	3.13e-02	1.77e-01	1,889,024	9.9579e-03	1.01	4.8960e-05	2.07

Table 11.3: Error of the $\mathcal{P}^1$-Lagrange finite element solution ($\gamma = 1.9$)

M	N	h	H	N_p^L	$Err_h^L(H^1)$	r	$Err_h^L(L^2)$	r
4	14	2.50e-01	8.71e-01	345	1.4873e-01		1.4032e-02	
8	52	1.25e-01	8.12e-01	4,293	1.2167e-01	0.29	9.3061e-03	0.59
16	194	6.25e-02	7.58e-01	56,355	1.0919e-01	0.16	7.4989e-03	0.31
32	724	3.13e-02	7.07e-01	789,525	1.0128e-01	0.11	6.4558e-03	0.22

Table 11.4: Error of the $\mathcal{P}^1$-CR finite element solution ($\gamma = 1.9$)

M	N	h	H	N_p^{CR}	$Err_h^{CR}(H^1)$	r	$Err_h^{CR}(L^2)$	r
4	14	2.50e-01	8.71e-01	2,496	7.9756e-02		3.2993e-03	
8	52	1.25e-01	8.12e-01	35,072	3.9708e-02	1.01	7.7177e-04	2.10
16	194	6.25e-02	7.58e-01	509,568	1.9814e-02	1.00	1.8781e-04	2.04
32	724	3.13e-02	7.07e-01	7,508,480	9.9003e-03	1.00	4.6546e-05	2.01

Table 11.5: Error of the $\mathcal{P}^1$-Lagrange finite element solution ($\gamma = 2.0$)

M	N	h	H	N_p^L	$Err_h^L(H^1)$	r	$Err_h^L(L^2)$	r
4	16	2.50e-01	1.00	425	1.5862e-01		1.5909e-02	
8	64	1.25e-01	1.00	5,265	1.4079e-01	0.17	1.2472e-02	0.35
16	256	6.25e-02	1.00	74,273	1.3597e-01	0.05	1.1646e-02	0.10
32	1,024	3.13e-02	1.00	1,116,225	1.3474e-01	0.01	1.1442e-02	0.03

Table 11.6: Error of the $\mathcal{P}^1$-CR finite element solution ($\gamma = 2.0$)

M	N	h	H	N_p^{CR}	$Err_h^{CR}(H^1)$	r	$Err_h^{CR}(L^2)$	r
4	16	2.50e-01	1.00	2,848	7.9473e-02		3.2264e-03	
8	64	1.25e-01	1.00	43,136	3.9647e-02	1.00	7.6153e-04	2.08
16	256	6.25e-02	1.00	672,256	1.9803e-02	1.00	1.8680e-04	2.03
32	1,024	3.13e-02	1.00	10,618,880	9.8984e-03	1.00	4.6458e-05	2.01

If Ω is convex, $u \in H^2(\Omega) \cap H_0^1(\Omega)$. In these numerical examples, the Crouzeix–Raviart finite element approximation is superior to the Lagrange finite element approximation on these anisotropic meshes. The theoretical explanation of this point is still open.

Remark 11.8.1. As described in Remark 4.1.3, imposing the Synge's condition for mesh partitions guarantees the convergence of finite element methods.

As this numerical example, when $u \in H_0^1(\Omega) \cap W^{2,\infty}(\Omega)$, the theoretical error estimate was such as (11.8.1). That is, we impose the assumption $\gamma < 6$. However, the numerical result implies that the numerical solution diverges when $\gamma \geq 2$. There is a gap in the parameter γ range. We can see from this argument that Synge's condition is sufficient for safe numerical calculation.

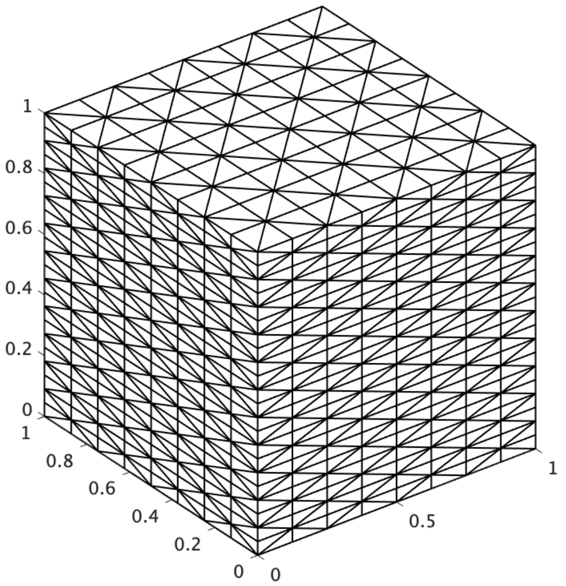

Fig. 11.1: Mesh: $M = 8$, $N = 22$

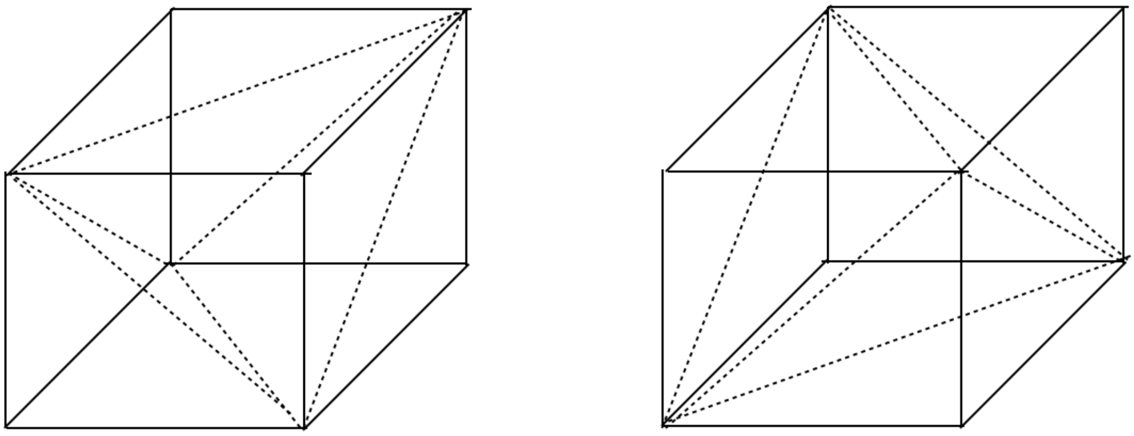

Fig. 11.2: Elements

Chapter 12

Dual Mixed Formulation of Elliptic Problem

12.1 Babuška–Brezzi Theorem

In this section, we consider an abstract problem, e.g., see [32, Chapter 49.4 and 50.1].

Let V and M be two real Banach spaces. Let $A : V \to V'$ and $B : V \to M$ be two bounded linear operator. We consider the model problem as follows. Find $(u, p) \in V \times M'$ such that

$$A(u) + B^*(p) = f, \tag{12.1.1a}$$
$$B(u) = g, \tag{12.1.1b}$$

where $B^* : M' \to V'$ is the adjoint of B, $f \in V'$, and $g \in M$.

Let us assume that V and M be reflexive Banach spaces and let us set $Q := M'$. We thus have $B \in \mathcal{L}(V; Q')$ and $B^* \in \mathcal{L}(Q; V')$. Note that $Q' = M'' = M$. Consider the two bounded bilinear forms a and b defined on $V \times V$ and on $V \times Q$ such that

$$a(v, w) := \langle A(v), w\rangle_{V',V},$$
$$b(v, q) := \langle B(v), q\rangle_{Q',Q}.$$

We set

$$\|a\| := \sup_{v\in V}\sup_{w\in V} \frac{|a(v, w)|}{\|v\|_V \|w\|_V}, \quad \|b\| := \sup_{v\in V}\sup_{q\in Q} \frac{|b(v, q)|}{\|v\|_V \|q\|_Q}. \tag{12.1.2}$$

For any $f \in V'$ and $g \in Q'$, the abstract problem (12.1.1) is reformulated as follows. Find $(u, p) \in V \times Q$ such that

$$a(u, w) + b(w, p) = f(w) \quad \forall w \in V, \tag{12.1.3a}$$

$$b(u, q) = g(q) \quad \forall q \in Q, \tag{12.1.3b}$$

where $f(w) := \langle f, w \rangle_{V',V}$ and $g(q) := \langle g, q \rangle_{Q',Q}$.

Theorem 12.1.1 (Babuška–Brezzi)**.** *The problem* (12.1.3) *is well-posed if and only if*

$$\inf_{v \in \ker(B)} \sup_{w \in \ker(B)} \frac{|a(v, w)|}{\|v\|_V \|w\|_V} =: \alpha > 0, \tag{12.1.4a}$$

$$\forall w \in \ker(B), \quad [\forall v \in \ker(B),\ a(v, w) = 0] \Rightarrow [w = 0], \tag{12.1.4b}$$

and the Babuška–Brezzi condition holds:

$$\inf_{q \in Q} \sup_{v \in V} \frac{|b(v, q)|}{\|v\|_V \|q\|_Q} =: \beta > 0. \tag{12.1.5}$$

Furthermore, we have the following a priori estimates:

$$\|u\|_V \leq c_1 \|f\|_{V'} + c_2 \|q\|_{Q'}, \tag{12.1.6a}$$

$$\|p\|_Q \leq c_3 \|f\|_{V'} + c_4 \|q\|_{Q'}, \tag{12.1.6b}$$

where $c_1 := \frac{1}{\alpha}$, $c_2 := \frac{1}{\beta}(1 + \frac{\|a\|}{\alpha})$, $c_3 := \frac{1}{\beta}(1 + \frac{\|a\|}{\alpha})$, *and* $c_4 := \frac{\|a\|}{\beta^2}(a + \frac{\|a\|}{\alpha})$.

Proof. The proof is found in [32, p. 358]. □

Remark 12.1.2 (Coercivity)**.** The conditions in (12.1.4) are automatically fulfilled if the bilinear form a is coercive on $\ker(B)$. Let $v \in \ker(B)$. Assume that

$$\exists \alpha_0 > 0, \quad a(v, v) \geq \alpha_0 \|v\|_V^2.$$

Condition (12.1.4a) is readily deduced from

$$\alpha_0 \|v\|_V \leq \frac{a(v, v)}{\|v\|_V} \leq \sup_{w \in \ker(B)} \frac{a(v, w)}{\|w\|_V}.$$

Let $w \in \ker(B)$. Setting $v := w$ yields

$$\sup_{v \in \ker(B)} a(v, w) \geq a(w, w) \geq \alpha_0 \|w\|_V^2.$$

Therefore, $\sup_{v \in \ker(B)} a(v, w) = 0$ implies $w = 0$.

A conforming Galerkin approximation of (12.1.3) is obtained by considering finite-dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$. The discrete problem is as follows. Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$a(u_h, w_h) + b(w_h, p_h) = f(w_h) \quad \forall w_h \in V_h, \tag{12.1.7a}$$

$$b(u_h, q_h) = g(q_h) \quad \forall q_h \in Q_h. \tag{12.1.7b}$$

Let $B_h : V_h \to Q'_h$ be the discrete operator of the operator $B : V \to Q'$, that is,

$$\langle B_h(v_h), q_h\rangle_{Q'_h,Q_h} := \langle B(v_h), q_h\rangle_{Q'_h,Q_h} = b(v_h, q_h) \quad \forall (v_h, q_h) \in V_h \times Q_h.$$

The null space of B_h is such that

$$\ker(B_h) := \{v_h \in V_h : \; b(v_h, q_h) = 0 \; \forall q_h \in Q_h\}.$$

Note 12.1.3. One important aspect of the discretisation is that the surjectivity of B does not imply that of B_h. In general, $\ker(B_h)$ is not necessarily a subspace of $\ker(B)$.

Proposition 12.1.4. *The problem* (12.1.7) *is well-posed if and only if*

$$\inf_{v_h\in\ker(B_h)} \sup_{w_h\in\ker(B_h)} \frac{|a(v_h, w_h)|}{\|v_h\|_V \|w_h\|_V} =: \alpha_h > 0, \tag{12.1.8a}$$

$$\inf_{q_h\in Q_h} \sup_{v_h\in V_h} \frac{|b(v_h, q_h)|}{\|v_h\|_V \|q_h\|_Q} =: \beta_h > 0. \tag{12.1.8b}$$

Proof. The proof is found in [32, p. 364]. □

12.2 Dual Mixed Formulation

The Poisson equation (11.1.1) $-\Delta u = -\operatorname{div} \nabla u = f$ can be written as the following system. Find $(\sigma, u) : \Omega \to \mathbb{R}^d \times \mathbb{R}$ such that

$$\sigma - \nabla u = 0 \quad \text{in } \Omega, \tag{12.2.1a}$$

$$\operatorname{div} \sigma = -f \quad \text{in } \Omega, \tag{12.2.1b}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{12.2.1c}$$

We consider the following dual mixed formulation: Find $(\sigma, u) \in V := H(\text{div}; \Omega) \times Q := L^2(\Omega)$ such that

$$a(\sigma, v) + b(v, u) = 0 \quad \forall v \in V, \tag{12.2.2a}$$

$$b(\sigma, q) = -(f, q) \quad \forall q \in Q, \tag{12.2.2b}$$

where bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ are defined by

$$a(\sigma, v) := (\sigma, v), \quad b(v, q) := (\operatorname{div} v, q).$$

To show that the problem is well-posed, we introduce the inf-sup condition, e.g., see [32, Lemma 51.2].

Lemma 12.2.1. *Let $D \subset \mathbb{R}^d$ be a Lipschitz domain. The operator* div : $H(\operatorname{div}; D) \to L^2(D)$ *is surjective, and it holds that*

$$\inf_{q \in L^2(D)} \sup_{v \in H(\operatorname{div};D)} \frac{|\int_D \operatorname{div} v q dx|}{\|v\|_{H(\operatorname{div};D)} \|q\|_{L^2(D)}} \geq \beta, \tag{12.2.3}$$

where $\beta := (C_p(D)^2 + 1)^{-\frac{1}{2}}$ with $C_P(D)$ is the Poincaré constant.

Proof. Let $q \in L^2(D)$. Let $\varphi \in H_0^1(D)$ be such that

$$(\nabla \varphi, \nabla \psi) = (q, \psi) \quad \forall \psi \in H_0^1(D).$$

We then have

$$|\varphi|_{H^1(D)} \leq C_p(D) \|q\|_{L^2(D)},$$

where $C_P(D)$ is the Poincaré constant.

Setting $v_0 := -\nabla \varphi$, we have $v_0 \in H(\operatorname{div}; D)$, $\operatorname{div} v_0 = q$, and

$$\|v_0\|^2_{H(\operatorname{div};D)} = \|\nabla v_0\|^2_{L^2(D)} + \|q\|^2_{L^2(D)} \leq (C_p(D)^2 + 1) \|q\|^2_{L^2(D)}.$$

We hence obtain

$$\begin{aligned} \sup_{v \in H(\operatorname{div};D)} \frac{\int_D \operatorname{div} v q dx}{\|v\|_{H(\operatorname{div};D)}} \geq \frac{\int_D \operatorname{div} v_0 q dx}{\|v_0\|_{H(\operatorname{div};D)}} &= \frac{\|q\|^2_{L^2(D)}}{\|q\|_{L^2(D)}} \frac{\|q\|_{L^2(D)}}{\|v_0\|_{H(\operatorname{div};D)}} \\ &\geq (C_p(D)^2 + 1)^{-\frac{1}{2}} \|q\|_{L^2(D)}. \end{aligned}$$

□

Proposition 12.2.2 (Well-posedness)**.** *For any $f \in L^2(\Omega)$, the problem* (12.2.2) *is well-posed.*

Proof. We apply Theorem 12.1.1. We set $X_0 := \{v \in V;\ b(v, q) = 0\ \forall q \in Q\}$. Because

$$0 = b(v, \operatorname{div} v) = \|\operatorname{div} v\|^2_{L^2(\Omega)} \quad \forall v \in X_0,$$

we have $\|v\|_V = \|v\|_{L^2(\Omega)^d}$ for any $v \in X_0$. We observe that

$$a(v,v) = \|v\|_V^2 \quad \forall v \in X_0,$$

and the bilinear form $b(.,.)$ satisfies the inf-sup condition

$$\inf_{0\neq q\in Q} \sup_{0\neq v\in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta > 0, \tag{12.2.4}$$

where $\|\cdot\|_V := \|\cdot\|_{H(\text{div};\Omega)}$ and $\|\cdot\|_{Q_h} := \|\cdot\|_{L^2(\Omega)}$. The problem (12.2.2) is then well-posed. □

12.3 Raviart–Thomas Finite Element Approximation

Let $k \in \mathbb{N}_0$. We consider the following Raviart–Thomas finite element approximate problem. Find $(\sigma_h^{RT^k}, u_h^{RT^k}) \in V_h := V_h^{RT^k} \times Q_h := M_h^k$ such that

$$a(\sigma_h^{RT^k}, v_h) + b(v_h, u_h^{RT^k}) = 0, \quad \forall v_h \in V_h, \tag{12.3.1a}$$

$$b(\sigma_h^{RT^k}, q_h) = -(f, q_h), \quad \forall q_h \in Q_h. \tag{12.3.1b}$$

This setting is conforming because $V_h \times Q_h \subset H(\text{div};\Omega) \times L^2(\Omega)$.

The following lemma is fundamental in the analysis of mixed finite element approximations.

Lemma 12.3.1. *Let $D \subset \mathbb{R}^d$ be a Lipschitz domain. For any $g \in L^2(D)$, there exists $v \in H^1(D)^d$ such that*

$$\operatorname{div} v = g \quad \text{in } D \tag{12.3.2}$$

and

$$|v|_{H^1(D)^d} \leq \|g\|_{L^2(D)}, \quad \|v\|_{L^2(\Omega)^d} \leq C_P(D)\|g\|_{L^2(D)}, \tag{12.3.3}$$

where $C_P(D)$ is the Poincaré constant.

Proof. We follow [17, Lemma 2.2].

Let $B \subset \mathbb{R}^d$ be a ball containing D. We set

$$\tilde{g} := \begin{cases} g & \text{in } D, \\ 0 & \text{in } B\backslash D. \end{cases}$$

Then, there exists a unique solution $p \in H_0^1(B) \cap H^2(B)$ such that

$$\mathrm{div}(\nabla p) = \Delta p = \tilde{g} \quad \text{in } B, \quad p = 0 \quad \text{on } \partial B.$$

It is known that p satisfies the a priori estimate (e.g., see [40, Theorem, 2.4.2.5, Theorem 3.1.1.2])

$$|p|_{H^2(D)} \le |p|_{H^2(B)} \le \|\Delta p\|_{L^2(B)} \le \|\tilde{g}\|_{L^2(B)} = \|g\|_{L^2(D)}.$$

We also get the a priori estimate

$$|p|_{H^1(D)} \le |p|_{H^1(B)} \le C_P(D)\|\tilde{g}\|_{L^2(B)} = C_P(D)\|g\|_{L^2(D)}.$$

Therefore, setting $v := \nabla p \in H^1(D)^d$, we have (12.3.2) and (12.3.3). □

To show that the problem is well-posed, we introduce the discrete inf-sup condition,

Lemma 12.3.2 (Discrete inf–sup condition)**.** *We impose Condition 4.3.1 with $h \le 1$. Then, there exists a constant β_*, depending only on the Poincaré constant, such that*

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{|b(v_h, q_h)|}{\|v_h\|_{V_h}\|q_h\|_{Q_h}} \ge \beta_* > 0, \tag{12.3.4}$$

where $\|\cdot\|_{V_h} := \|\cdot\|_V$ and $\|\cdot\|_{Q_h} := \|\cdot\|_Q$.ß

Proof. Let $q_h \in Q_h$. From Lemma 12.3.1, there exists $v \in H^1(\Omega)^d$ such that $\mathrm{div}\, v = q_h$ in Ω, $|v|_{H^1(\Omega)^d} \le \|q_h\|_{Q_h}$, and $\|v\|_{L^2(\Omega)^d} \le C_P(\Omega)\|q_h\|_{Q_h}$. From the Gauss–Gauss formula and the definition of the Raviart–Thomas interpolation, we conclude that, for any $p_h \in M_h^k$,

$$\begin{aligned}
\int_\Omega \mathrm{div}(I_h^{RT^k} v) p_h dx &= \sum_{T \in \mathbb{T}_h} \int_T \mathrm{div}(I_T^{RT^k} v) p_h dx \\
&= \sum_{T \in \mathbb{T}_h} \int_{\partial T} (I_T^{RT^k} v) \cdot n_T p_h ds - \sum_{T \in \mathbb{T}_h} \int_T [(I_T^{RT^k} v) \cdot \nabla] p_h dx \\
&= \sum_{T \in \mathbb{T}_h} \int_{\partial T} (v \cdot n_T) p_h ds - \sum_{T \in \mathbb{T}_h} \int_T (v \cdot \nabla) p_h dx \\
&= \sum_{T \in \mathbb{T}_h} \int_T \mathrm{div}\, v p_h dx = \int_\Omega \mathrm{div}\, v p_h dx = \int_\Omega q_h p_h dx,
\end{aligned}$$

which leads to $\mathrm{div}(I_h^{RT^k} v) = q_h$. Furthermore, setting $p_h := q_h$ yields

$$\int_\Omega \mathrm{div}(I_h^{RT^k} v) q_h dx = \|q_h\|^2_{L^2(\Omega)}.$$

From the stability of the Raviart–Thomas interpolation,

$$\begin{aligned}
\|I_h^{RT^k} v\|^2_{V_h} &= \|I_h^{RT^k} v\|^2_{L^2(\Omega)^d} + \|\,\mathrm{div}(I_h^{RT^k} v)\|^2_{L^2(\Omega)} \\
&\leq c \left(\|v\|^2_{L^2(\Omega)^d} + |v|^2_{H^1(\Omega)^d} \right) + \|q_h\|^2_{L^2(\Omega)} \\
&\leq \left(c \left(C_P(\Omega)^2 + 1 \right) + 1 \right) \|q_h\|^2_{L^2(\Omega)}.
\end{aligned}$$

We thus have

$$\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_{V_h}} \geq \frac{b(I_h^{RT^k} v, q_h)}{\|I_h^{RT^k} v\|_{V_h}} \geq \frac{1}{\left(c\left(C_P(\Omega)^2 + 1 \right) + 1 \right)^{1/2}} \|q_h\|_{L^2(\Omega)}.$$

□

Proposition 12.3.3 (Well-posedness)**.** *For any* $f \in L^2(\Omega)$*, the problem* (12.3.1) *is well-posed.*

Proof. We apply Proposition 12.1.4. We set $X_{h0} := \{v_h \in V_h;\ b(v_h, q_h) = 0\ \forall q_h \in Q_h\}$. Because $\mathrm{div}\, v_h \in \mathcal{P}^k$ for any $v_h \in X_{h0}$, and

$$0 = b(v_h, \mathrm{div}\, v_h) = \|\,\mathrm{div}\, v_h\|^2_{L^2(\Omega)} \quad \forall v_h \in X_{h0},$$

we have $\|v_h\|_{V_h} = \|v_h\|_{L^2(\Omega)^d}$ for any $v_h \in X_{h0}$. We observe that

$$a(v_h, v_h) = \|v_h\|^2_V \quad \forall v_h \in X_{h0},$$

and the bilinear form $b(.,.)$ satisfies the inf-sup condition

$$\inf_{0 \neq q_h \in Q_h} \sup_{0 \neq v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_{V_h} \|q_h\|_{Q_h}} \geq \beta_* > 0. \qquad (12.3.5)$$

The problem (12.3.1) is then well-posed. □

Remark 12.3.4. If the extension of $I_h^{RT^k}$ to $H(\mathrm{div};\Omega)$ is done, we can adopt the operator $I_h^{RT^k}$ as the Fortin operator. Unfortunately, this is not possible (Chapter B). However, the operator $I_h^{RT^k}$ is well-defined on the space $\widetilde{V} := H^1(\Omega)^d$. Thus, if the domain allows for an inf-sup condition of the form

$$\inf_{q \in Q} \sup_{\tilde{v} \in \widetilde{V}} \frac{\int_\Omega \mathrm{div}\, \tilde{v} q dx}{\|v\|_{\widetilde{V}} \|q\|_Q} \geq \beta > 0.$$

where $\|\cdot\|_{\widetilde{V}} := \|\cdot\|_{H^1(\Omega)^d}$, we can construct the Fortin operator $I_h^{RT^k}$. The case of minimum regularity $H(\mathrm{div};\Omega)$ require us to extend domain of the Fortin operator, e.g., see [32, Lemma 51.10].

12.4 Error Analysis

From the discrete equations (12.3.1) and their continuous counterpart (12.2.2), we obtain the Galerkin orthogonality

$$a(\sigma - \sigma_h^{RT^k}, v_h) + b(v_h, u - u_h^{RT^k}) = 0 \quad \forall v_h \in V_h, \tag{12.4.1a}$$

$$b(\sigma - \sigma_h^{RT^k}, q_h) = 0 \quad \forall q_h \in Q_h. \tag{12.4.1b}$$

We then get the following Céa-lemma-type estimates with the help of (12.4.1) and the inf–sup condition (12.3.4).

Theorem 12.4.1. *Let* $\sigma \in H^1(\Omega)^d$ *and* $\sigma_h^{RT^k} \in V_h$ *be the solutions of* (12.2.2) *and* (12.3.1)*, respectively. We then have*

$$\|\sigma - \sigma_h^{RT^k}\|_{L^2(\Omega)^d} \le \|\sigma - I_h^{RT^k}\sigma\|_{L^2(\Omega)^d}. \tag{12.4.2}$$

Furthermore, let $(\sigma, u) \in H^1(\Omega)^d \times L^2(\Omega)$ *and* $(\sigma_h^{RT^k}, u_h^{RT^k}) \in V_h \times Q_h$ *be the solutions of* (12.2.2) *and* (12.3.1)*, respectively. We impose Condition 4.3.1 with* $h \le 1$. *It then holds that*

$$\|u - u_h^{RT^k}\|_{L^2(\Omega)} \le \|u - \Pi_h^k u\|_{L^2(\Omega)} + \beta_*^{-1}\|\sigma - \sigma_h^{RT^k}\|_{L^2(\Omega)^d}, \tag{12.4.3}$$

where β_* *is the constants appearing in Lemma 12.3.2.*

Proof. We first have

$$\begin{aligned}\|\sigma - \sigma_h^{RT^k}\|_{L^2(\Omega)^d}^2 &= a(\sigma - \sigma_h^{RT^k}, \sigma - \sigma_h^{RT^k}) \\ &= a(\sigma - \sigma_h^{RT^k}, \sigma - I_h^{RT^k}\sigma) + a(\sigma - \sigma_h^{RT^k}, I_h^{RT}\sigma - \sigma_h^{RT^k}).\end{aligned}$$

From the definition of $I_h^{RT^k}$ and (12.4.1b), we have

$$b(I_h^{RT^k}\sigma - \sigma_h^{RT^k}, q_h) = 0, \quad \forall q_h \in Q_h.$$

Indeed,

$$\begin{aligned}b(I_h^{RT^k}\sigma, q_h) &= \sum_{T\in\mathbb{T}_h} \int_T \operatorname{div}(I_h^{RT^k}\sigma) q_h dx \\ &= \sum_{T\in\mathbb{T}_h} \sum_{F\subset\partial T} \int_F (I_h^{RT^k}\sigma)\cdot n_F q_h ds - \sum_{T\in\mathbb{T}_h} \int_T (I_h^{RT^k}\sigma\cdot\nabla) q_h dx \\ &= \sum_{T\in\mathbb{T}_h} \sum_{F\subset\partial T} \int_F \sigma\cdot n_F q_h ds - \sum_{T\in\mathbb{T}_h} \int_T (\sigma\cdot\nabla) q_h dx \\ &= \sum_{T\in\mathbb{T}_h} \int_T \operatorname{div}\sigma q_h dx = b(\sigma, q_h).\end{aligned}$$

Because $\operatorname{div}(I_h^{RT^k}\sigma - \sigma_h^{RT^k}) \in \mathcal{P}^k$, we can substitute $\operatorname{div}(I_h^{RT^k}\sigma - \sigma_h^{RT^k})$ for q_h to conclude that

$$\operatorname{div}(I_h^{RT^k}\sigma - \sigma_h^{RT^k}) = 0.$$

Therefore, setting $v_h := I_h^{RT^k}\sigma - \sigma_h^{RT^k}$ in (12.4.1a), we have

$$a(\sigma - \sigma_h^{RT^k}, I_h^{RT^k}\sigma - \sigma_h^{RT^k}) = 0.$$

We thus obtain, using the Hölder's inequality,

$$\|\sigma - \sigma_h^{RT^k}\|^2_{L^2(\Omega)^d} \leq \|\sigma - \sigma_h^{RT^k}\|_{L^2(\Omega)^d} \|\sigma - I_h^{RT^k}\sigma\|_{L^2(\Omega)^d},$$

which concludes (12.4.2).

From (12.4.1a) and the definition of the L^2-projection, we have

$$a(\sigma - \sigma_h^{RT^k}, v_h) + b(v_h, \Pi_h^k u - u_h^{RT^k}) = 0 \quad \forall v_h \in V_h,$$

because $\operatorname{div} v_h \in \mathcal{P}^k$ and

$$b(v_h, u - \Pi_h^k u) = 0.$$

With help of the discrete inf-sup stability (12.3.4) and Hölder's inequality, we obtain

$$\begin{aligned}
\|u - u_h^{RT^k}\|_{L^2(\Omega)} &\leq \|u - \Pi_h^k u\|_{L^2(\Omega)} + \|\Pi_h^k u - u_h^{RT^k}\|_{L^2(\Omega)} \\
&\leq \|u - \Pi_h^k u\|_{L^2(\Omega)} + \beta_*^{-1} \sup_{v_h \in V_h} \frac{b(v_h, \Pi_h^k u - u_h^{RT^k})}{\|v_h\|_{V_h}} \\
&\leq \|u - \Pi_h^k u\|_{L^2(\Omega)} + \beta_*^{-1} \sup_{v_h \in V_h} \frac{a(\sigma_h^{RT^k} - \sigma, v_h)}{\|v_h\|_{V_h}} \\
&\leq \|u - \Pi_h^k u\|_{L^2(\Omega)} + \beta_*^{-1} \|\sigma_h^{RT^k} - \sigma\|_{L^2(\Omega)^d},
\end{aligned}$$

where we used that $\|v_h\|_{L^2(\Omega)^d} \leq \|v_h\|_{V_h}$. □

Theorem 12.4.2. *For $k \in \mathbb{N}_0$, let $\ell \in \mathbb{N}_0$ be such that $0 \leq \ell \leq k$. Let $(\sigma, u) \in H^{\ell+1}(\Omega)^d \times H^{\ell+1}(\Omega)$ and $(\sigma_h^{RT^k}, u_h^{RT^k}) \in V_h \times Q_h$ be the solutions of* (12.2.2) *and* (12.3.1)*, respectively. We impose Condition 4.3.1 with $h \leq 1$. It then holds that*

(I) *if all elements are composed of the type $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is not imposed,*

$$\begin{aligned}
&\|\sigma - \sigma_h^{RT^k}\|_{L^2(\Omega)^d} \\
&\quad \leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon| = \ell+1} h^{\varepsilon} \|\partial_r^{\varepsilon} \sigma\|_{L^2(T)^d} + h_T \sum_{|\beta| = \ell} h^{\beta} \|\partial_r^{\beta} \nabla \cdot \sigma\|_{L^2(T)} \right);
\end{aligned} \tag{12.4.4}$$

and

$$\|u - u_h^{RT^k}\|_{L^2(\Omega)} \le c \sum_{T \in \mathbb{T}_h} \sum_{|\epsilon| = \ell+1} h^{\varepsilon} \|\partial_r^{\epsilon} u\|_{L^2(T)} + \beta_*^{-1} \|\sigma - \sigma_h^{RT^k}\|_{L^2(\Omega)^d}; \tag{12.4.5}$$

(II) *if all elements are composed of the type $T_2 \in \mathfrak{T}_2^{(3)}$ and Condition 3.3.1 is not imposed,*

$$\|\sigma - \sigma_h^{RT^k}\|_{L^2(\Omega)^3} \le c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon| = \ell+1} h^{\varepsilon} \|\partial_r^{\varepsilon} \sigma\|_{L^2(T)^3} + h_T \sum_{k=1}^{3} \sum_{|\varepsilon| = \ell} h^{\varepsilon} \left\| \partial_r^{\varepsilon} \frac{\partial \sigma}{\partial r_k} \right\|_{L^2(T)^3} \right); \tag{12.4.6}$$

and

$$\|u - u_h^{RT^k}\|_{L^2(\Omega)} \le c \sum_{T \in \mathbb{T}_h} \sum_{|\epsilon| = \ell+1} h^{\varepsilon} \|\partial_r^{\epsilon} u\|_{L^2(T)} + \beta_*^{-1} \|\sigma - \sigma_h^{RT^k}\|_{L^2(\Omega)^d}. \tag{12.4.7}$$

Proof. Using Theorem 12.4.1 and the interpolation error estimates of Theorems 6.3.1 and 9.7.3, we thus have the error estimates of the mixed finite element approximation (12.3.1) on anisotropic meshes. □

Chapter 13

Relationship between the Raviart–Thomas and Crouzeix–Raviart Finite Element Approximation for $d = 3$

13.1 Preliminaries for Analysis

Let us consider a tetrahedron $K \subset \mathbb{R}^3$ such as that in Figure 13.1. Let x_i $(i = 1, 2, 3, 4)$ be the vertices and $m_{i,j}$ the midpoints of edges of the tetrahedron; that is, $m_{i,j} := \frac{1}{2}(x_i + x_j)$. Furthermore, for $1 \le i \le 4$, let F_i be the face of the tetrahedron opposite x_i. Then, by simple calculation, we find the equality

$$L := \sum_{i=1}^{4} |x_i - x_K|^2 = |m_{1,4} - m_{2,3}|^2 + |m_{1,3} - m_{2,4}|^2 + |m_{1,2} - m_{3,4}|^2, \tag{13.1.1}$$

holds, where x_K is the barycentre of K such that $x_K := \frac{1}{4}\sum_{i=1}^{4} x_i$.

We present a quadrature scheme over a simplex $K \subset \mathbb{R}^3$ (e.g., [82, p.307]) that is easily conformed.

Lemma 13.1.1. *For any $f \in \mathcal{C}^0(K)$, the quadrature scheme*

$$\int_K f(x)dx \sim -\frac{|K|}{20}\sum_{i=1}^{4} f(x_i) + \frac{|K|}{5}\sum_{1\le i<j\le 4} f(m_{i,j})$$

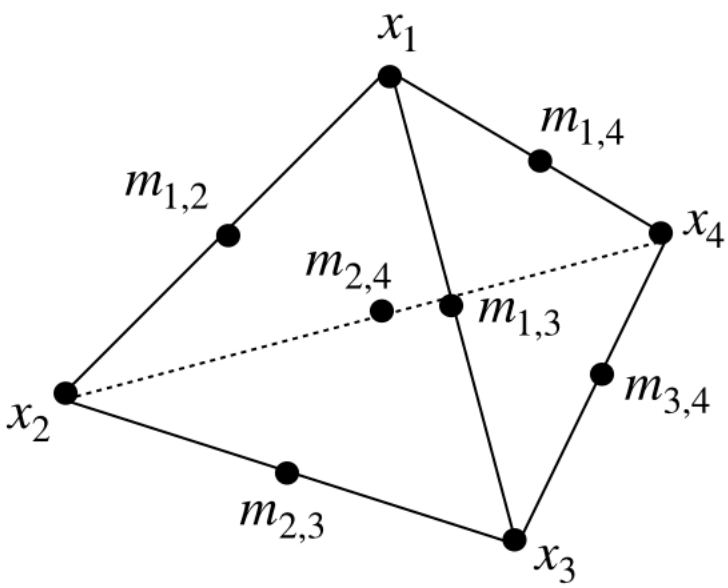

Fig. 13.1: Tetrahedron

is exact for polynomials of degree less than or equal to 2*;*

$$\int_K f(x)dx + \frac{|K|}{20}\sum_{i=1}^{4} f(x_i) - \frac{|K|}{5}\sum_{1\le i<j\le 4} f(m_{i,j}) = 0 \quad \forall f \in \mathcal{P}^2(K). \tag{13.1.2}$$

Define the function φ_T by

$$\varphi_K(x) := L - 12|x - x_K|^2, \quad \text{on } K. \tag{13.1.3}$$

We then have the following lemma.

Lemma 13.1.2. *It holds that*

$$\frac{1}{|F_i|}\int_{F_i} \varphi_K(x)ds = 0, \quad i = 1,2,3,4, \tag{13.1.4}$$

$$\frac{1}{|K|}\int_K \varphi_K(x)dx = \frac{2}{5}L, \tag{13.1.5}$$

$$\frac{1}{|K|}\int_K |\nabla\varphi_K(x)|^2 dx = \frac{144}{5}L. \tag{13.1.6}$$

Proof. From second-order three-point numerical integration over F_1,

$$\int_{F_1} f(x)ds = \frac{|F_1|}{3}\left(f(m_{2,3}) + f(m_{3,4}) + f(m_{2,4})\right) \quad \forall f \in \mathcal{P}^2(T),$$

we have

$$
\begin{aligned}
&\frac{1}{|F_1|}\int_{F_1}\varphi_K(x)ds \\
&= \frac{1}{3}\left(\varphi_K(m_{2,3})+\varphi_K(m_{3,4})+\varphi_K(m_{2,4})\right) \\
&= \frac{1}{3}\left(3L-12\left(|m_{2,3}-x_K|^2+|m_{3,4}-x_K|^2+|m_{2,4}-x_K|^2\right)\right) \\
&= \frac{1}{3}\left(3L-\frac{12}{4}\left(|m_{2,3}-m_{1,4}|^2+|m_{3,4}-m_{1,2}|^2+|m_{2,4}-m_{1,3}|^2\right)\right)=0,
\end{aligned}
$$

which leads to (13.1.4).

Next, using (13.1.2), we have

$$
\begin{aligned}
&\frac{1}{|K|}\int_{T}\varphi_K(x)dx \\
&= -\frac{1}{20}\sum_{i=1}^{4}\varphi_K(x_i)+\frac{1}{5}\sum_{1\le i<j\le 4}\varphi_K(m_{i,j}) \\
&= -\frac{1}{20}\left(4L-12\sum_{i=1}^{4}|x_i-x_K|^2\right)+\frac{1}{5}\left(6L-12\sum_{1\le i<j\le 4}|m_{i,j}-x_K|^2\right) \\
&= \frac{2}{5}L,
\end{aligned}
$$

which leads to (13.1.5). We here used

$$
\begin{aligned}
&\sum_{1\le i<j\le 4}|m_{i,j}-x_K|^2 \\
&= |m_{1,2}-x_K|^2+|m_{1,3}-x_K|^2+|m_{1,4}-x_K|^2 \\
&\quad +|m_{2,3}-x_K|^2+|m_{2,4}-x_K|^2+|m_{3,4}-x_K|^2 \\
&= \frac{1}{4}\left(2|m_{1,2}-m_{3,4}|^2+2|m_{1,3}-m_{2,4}|^2+2|m_{1,4}-m_{2,3}|^2\right)=\frac{L}{2}.
\end{aligned}
$$

We similarly obtain

$$
\begin{aligned}
&\frac{1}{|K|}\int_{T}|\nabla\varphi_K(x)|^2dx \\
&= \frac{24^2}{|K|}\int_{T}|x-x_K|^2dx \\
&= -\frac{24^2}{20}\sum_{i=1}^{4}|x_i-x_K|^2+\frac{24^2}{5}\sum_{1\le i<j\le 4}|m_{i,j}-x_K|^2=\frac{144}{5}L,
\end{aligned}
$$

which leads to (13.1.6). □

13.2 Relationship

This section shows the relationship between the Raviart–Thomas and Crouzeix–Raviart finite element problems. Find $(\bar{\sigma}_h^{RT^0}, \bar{u}_h^{RT^0}) \in V_h^{RT^0} \times M_h^0$ such that

$$a(\bar{\sigma}_h^{RT^0}, v_h) + b(v_h, \bar{u}_h^{RT^0}) = 0 \quad \forall v_h \in V_h^{RT^0}, \tag{13.2.1a}$$

$$b(\bar{\sigma}_h^{RT^0}, q_h) = -(\Pi_h^0 f, q_h) \quad \forall q_h \in M_h^0 \tag{13.2.1b}$$

and find $\bar{u}_h^{CR} \in V_{h0}^{CR}$ such that

$$a_h^{CR}(\bar{u}_h^{CR}, \varphi_h) = (\Pi_h^0 f, \varphi_h) \quad \forall \varphi_h \in V_{h0}^{CR}. \tag{13.2.2}$$

Here, (13.2.2) is the Crouzeix–Raviart finite element approximation of the Poisson equation

$$-\Delta \bar{u} = \Pi_h^0 f \quad \text{in } \Omega, \quad \bar{u} = 0 \quad \text{on } \partial\Omega. \tag{13.2.3}$$

In the case of $d = 2$, it is well known that there exists a relationship between $(\bar{\sigma}_h^{RT^0}, \bar{u}_h^{RT^0})$ and $\bar{u}_h^{CR}$ introduced by Marini; for example, [69]. See also [66, 55, 67]. We here show the relation in the three dimensional case.

For $T \in \mathbb{T}_h$, let L be defined such as (13.1.1) on each T. Define the function φ_T by

$$\varphi_T(x) := \begin{cases} L - 12|x - x_T|^2, & \text{on } T, \\ 0, & \text{otherwise}, \end{cases} \tag{13.2.4}$$

where x_T is the barycentre of T. We set the bubble space B_h by

$$B_h := \{b_h \in L^2(\Omega);\ b_h|_T \in \operatorname{span}\{\varphi_T\},\ \forall T \in \mathbb{T}_h\}. \tag{13.2.5}$$

Then, for any $\psi_h \in V_{h0}^{CR}$ and $b_h \in B_h$, because one writes $b_h|_T = c_b \varphi_T$ for $c_b \in \mathbb{R}$, it holds that

$$\begin{aligned}(\nabla_h \psi_h, \nabla_h b_h) &= \sum_{T \in \mathbb{T}_h} c_b \int_T \nabla \psi_h \cdot \nabla \varphi_T dx \\ &= \sum_{T \in \mathbb{T}_h} c_b \left\{ \sum_{F \subset \partial T} (n_F \cdot \nabla \psi_h) \int_F \varphi_T ds - \int_T \Delta \psi_h \varphi_T dx \right\} = 0.\end{aligned}$$

We here used the facts that (13.1.4), $n_F \cdot \nabla \psi_h$ is constant on F, and $\Delta \psi_h = 0$ on T. That is to say, two finite element spaces V_{h0}^{CR} and B_h are orthogonal to each other. Furthermore, we define the finite element space X_h^{bCR} by

$$X_h^{bCR} := V_{h0}^{CR} + B_h = \{\psi_h + b_h;\ \psi_h \in V_{h0}^{CR},\ b_h \in B_h\}. \tag{13.2.6}$$

We consider the following finite element problem. Find $u_h^{bCR} \in X_h^{bCR}$ such that

$$a_h^{CR}(u_h^{bCR}, \varphi_h) = (\nabla_h u_h^{bCR}, \nabla_h \varphi_h) = (\Pi_h^0 f, \varphi_h) \quad \forall \varphi_h \in X_h^{bCR}. \tag{13.2.7}$$

The solution $u_h^{bCR} \in X_h^{bCR}$ is then decomposed as $u_h^{bCR} = \bar{u}_h^{CR} + b_h$ with $\bar{u}_h^{CR} \in V_{h0}^{CR}$ and $b_h \in B_h$. Note that $\bar{u}_h^{CR}$ and b_h respectively satisfy (13.2.2) and the equation

$$a_h^{CR}(b_h, c_h) = (\nabla_h b_h, \nabla_h c_h) = (\Pi_h^0 f, c_h) \quad \forall c_h \in B_h. \tag{13.2.8}$$

On each element $T \in \mathbb{T}_h$, (13.2.8) has the form

$$\gamma_T \int_T \nabla \varphi_T \cdot \nabla \varphi_T dx = \int_T \Pi_T^0 f \varphi_T dx, \quad \gamma_T \in \mathbb{R},$$

where $b_h := \gamma_0 \varphi_T$ and $c_h := \varphi_T$ on T. From (13.1.3) and (13.1.4), we have

$$\gamma_T = \frac{1}{72} \Pi_T^0 f \quad \forall T \in \mathbb{T}_h. \tag{13.2.9}$$

Theorem 13.2.1. *Let $u_h^{bCR} \in X_h^{bCR}$ be the solution of* (13.2.7) *and* $(\bar{\sigma}_h^{RT^0}, \bar{u}_h^{RT^0}) \in V_h^{RT^0} \times M_h^0$ *the solution of* (13.2.1). *We then have $\nabla_h u_h^{bCR} \in V_h^{RT^0}$ and*

$$\bar{\sigma}_h^{RT^0} = \nabla u_h^{bCR} \quad \forall T \in \mathbb{T}_h, \tag{13.2.10}$$

$$\bar{u}_h^{RT^0} = \Pi_T^0 u_h^{bCR} \quad \forall T \in \mathbb{T}_h. \tag{13.2.11}$$

Proof. The proof can be found in [45]. □

From Theorem 13.2.1, for $d = 3$, the following lemma holds.

Lemma 13.2.2. *Let $\overline{u}_h^{CR} \in V_{h0}^{CR}$ be the solution of* (13.2.2) *and* $(\bar{\sigma}_h^{RT^0}, \bar{u}_h^{RT^0}) \in V_h^{RT^0} \times M_h^0$ *be the solution of* (13.2.1). *We then have the relationships*

$$\bar{\sigma}_h^{RT^0}|_T = \nabla \bar{u}_h^{CR} - \frac{1}{3} \Pi_T^0 f (x - x_T) \quad \forall T \in \mathbb{T}_h, \tag{13.2.12}$$

$$\bar{u}_h^{RT^0}|_T = \Pi_T^0 \bar{u}_h^{CR} + \frac{1}{180} \Pi_T^0 f \sum_{i=1}^{4} |x_i - x_T|^2 \quad \forall T \in \mathbb{T}_h, \tag{13.2.13}$$

where x_i, $i \in \{1 : 4\}$ are the vertices of $T \in \mathbb{T}_h$.

Proof. Recall that we $u_h^{bCR} = \bar{u}_h^{CR} + b_h$ with $\bar{u}_h^{CR} \in V_{h0}^{CR}$ and $b_h = \gamma_0 \varphi_T \in B_h$. From (13.2.10), for any $T \in \mathbb{T}_h$,

$$\begin{aligned} \bar{\sigma}_h^{RT^0} &= \nabla u_h^{bCR} = \nabla \bar{u}_h^{CR} + \gamma_T \nabla \varphi_T \\ &= \nabla \bar{u}_h^{CR} + \left(\frac{1}{72} \Pi_T^0 f \right) \cdot (-24(x - x_T)), \end{aligned}$$

which leads to (13.2.12). From (13.2.11), for any $T \in \mathbb{T}_h$,

$$\begin{aligned}\bar{u}_h^{RT^0} &= \Pi_T^0 u_h^{bCR} = \Pi_T^0 \bar{u}_h^{CR} + \gamma_T \Pi_T^0 \varphi_T \\ &= \Pi_T^0 \bar{u}_h^{CR} + \left(\frac{1}{72} \Pi_T^0 f \right) \cdot \left(\frac{2}{5} \sum_{i=1}^{4} |x_i - x_T|^2 \right),\end{aligned}$$

which leads to (13.2.13). Here, we used that, from (13.1.5),

$$\Pi_T^0 \varphi_T = \frac{1}{|T|} \int_T \varphi_T dx = \frac{2}{5} L.$$

□

Using the relationship between the Raviart–Thomas and Crouzeix–Raviart finite element methods, we have the error estimate of the Crouzeix–Raviart finite element approximation with the bubble function.

Lemma 13.2.3. *We assume that Ω is convex. We impose Condition 4.3.1 with $h \leq 1$. Let $\bar{u} \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of* (13.2.3) *and $u_h^{bCR} \in X_h^{bCR}$ be the solution of the Crouzeix–Raviart finite element problem* (13.2.7). *There then exists a constant $c > 0$ independent of $\bar{u}$, h, such that*

(I) *if all elements are composed of the type $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is not imposed,*

$$|\bar{u} - u_h^{bCR}|_{H^1(\mathbb{T}_h)} \leq c \sum_{T \in \mathbb{T}_h} \sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{\varepsilon} \nabla \bar{u}\|_{L^2(T)^d} + h \|\Pi_h^0 f\|_{L^2(\Omega)};$$

(II) *if all elements are composed of the type $T_2 \in \mathfrak{T}_2^{(3)}$ and Condition 3.3.1 is not imposed,*

$$|\bar{u} - u_h^{bCR}|_{H^1(\mathbb{T}_h)} \leq c \sum_{T \in \mathbb{T}_h} \left(\sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{\varepsilon} \nabla \bar{u}\|_{L^2(T)^3} + h_T \sum_{k=1}^{3} \left\| \frac{\partial \nabla \bar{u}}{\partial r_k} \right\|_{L^2(T)^3} \right).$$

Proof. Let $(\bar{\sigma}_h^{RT^0}, \bar{u}_h^{RT^0}) \in V_h^{RT^0} \times M_h^0$ be the solution of (13.2.1). From Theorem 13.2.1, it holds that $\nabla_h u_h^{bCR} \in V_h^{RT^0}$ and $\bar{\sigma}_h^{RT} = \nabla_h u_h^{bCR}$. Setting $\bar{\sigma} := \nabla \bar{u} \in H^1(\Omega)^d$, we then have

$$|\bar{u} - u_h^{bCR}|_{H^1(\mathbb{T}_h)} = \left(\sum_{T \in \mathbb{T}_h} \|\bar{\sigma} - \bar{\sigma}_h^{RT^0}\|_{L^2(T)^d}^2 \right)^{1/2},$$

which leads to the desired results using Theorem 12.4.2. □

Chapter 14

Stokes Equation

14.1 Continuous Problem

This section treats continuous settings of the Stokes equations, e.g., see [32, Section 53.1], [37], [52], [81], and [84, 85]. In this section, let Ω be a Lipschitz domain in $\mathbb{R}^d$. The (scaled) Stokes problem is to find $(u, p) : \Omega \to \mathbb{R}^d \times \mathbb{R}$ such that

$$-\nu \Delta u + \nabla p = f \quad \text{in } \Omega, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \qquad (14.1.1)$$

where ν is a non-negative parameter and $f : \Omega \to \mathbb{R}^d$ is a given function. We set function spaces as follows.

$$V := H_0^1(\Omega)^d, \quad Q := L_0^2(\Omega) := \left\{ q \in L^2(\Omega);\ \int_\Omega q dx = 0 \right\},$$

with norms:

$$|\cdot|_V := |\cdot|_{H^1(\Omega)^d}, \quad \|\cdot\|_Q := \|\cdot\|_{L^2(\Omega)}.$$

The variational formulation for the Stokes equations (14.1.1) is then as follows. For any $f \in L^2(\Omega)^d$, find $(u, p) \in V \times Q$ such that

$$a(u, \varphi) + b(\varphi, p) = (f, \varphi) \quad \forall \varphi \in V, \qquad (14.1.2a)$$

$$b(u, q) = 0 \quad \forall q \in Q, \qquad (14.1.2b)$$

where $a : H^1(\Omega)^d \times H^1(\Omega)^d \to \mathbb{R}$ and $b : H^1(\Omega)^d \times L^2(\Omega) \to \mathbb{R}$ respectively denote bilinear forms defined by

$$a(v, \psi) := \nu \int_\Omega \nabla v : \nabla \psi dx = \nu \sum_{i=1}^d \int_\Omega \nabla v_i \cdot \nabla \psi_i dx,\ b(\psi, q) := -\int_\Omega \operatorname{div} \psi q dx.$$

Here, the colon denotes the scalar product of tensors.

Lemma 14.1.1 (Inf-sup condition)**.** *It holds that*

$$\inf_{q\in Q}\sup_{v\in V}\frac{|b(v,q)|}{|v|_V\|q\|_Q}\geq\beta>0. \tag{14.1.3}$$

Proof. The proof is found in [52, Theorem 3.46], [32, Lemma 53.9], and [37, Lemma 4.1]. □

Remark 14.1.2 (Inf-sup condition in $W^{1,p}$-$L^{p'}$)**.** Let $p\in(1,\infty)$ and let $p'\in(1,\infty)$ be such that $\frac{1}{p}+\frac{1}{p'}=1$. Then, the operator $\mathrm{div}:W_0^{1,p}(\Omega)^d\to L_0^p(\Omega):=\{q\in L^p(\Omega):\ \int_\Omega q\,dx=0\}$ is surjective, that is, identifying $(L_0^p(\Omega))'$ with $L_0^{p'}(\Omega)$, we have

$$\inf_{q\in L_0^{p'}(\Omega)}\sup_{v\in W_0^{1,p}(\Omega)^d}\frac{|b(v,q)|}{|v|_{W^{1,p}(\Omega)^d}\|q\|_{L^{p'}(\Omega)}}:=\beta_{\Omega,p}>0,$$

see [33, Remark 53.10].

Remark 14.1.3 (Helmholtz decomposition [33, 52, 81])**.** We set

$$H_*^1(\Omega):=H^1(\Omega)\cap L_0^2(\Omega),\quad \mathcal{H}:=\{v\in L^2(\Omega)^d:\ \mathrm{div}\,v=0,\ v|_{\partial\Omega}\cdot n=0\},$$

where $\mathrm{div}\,v=0$ and $v|_{\partial\Omega}\cdot n=0$ mean that $\int_\Omega(v\cdot\nabla)q\,dx=0$ for any $q\in H_*^1(\Omega)$. The following L^2-orthogonal decomposition then holds true:

$$L^2(\Omega)^d=\mathcal{H}\oplus\nabla(H_*^1(\Omega)),$$

see [33, Lemma 74.1]. The L^2-orthogonal projection $P_{\mathcal{H}}:L^2(\Omega)^d\to\mathcal{H}$ resulting from this decomposition is often *Leray projection.*

Theorem 14.1.4 (Well-posedness, Stability)**.** *For any $f\in L^2(\Omega)^d$ or $f\in V'$, the weak formulation* (14.1.2) *of the Stokes problem is well-posed. Furthermore, if $f\in V'$, it holds that*

$$|u|_V\leq\frac{1}{\nu}\|f\|_{V'}, \tag{14.1.4}$$

$$\|p\|_Q\leq\frac{2}{\beta}\|f\|_{V'}. \tag{14.1.5}$$

If $f\in L^2(\Omega)^d$,

$$|u|_V\leq\frac{C_P}{\nu}\|P_{\mathcal{H}}(f)\|_{L^2(\Omega)^d}, \tag{14.1.6}$$

where C_P is the Poincaré constant.

Proof. We apply Theorem 12.1.1. We set $X_0 := \{v \in V;\ b(v,q) = 0\ \forall q \in Q\}$. We observe that

$$a(v,v) = \nu |v|_V^2 \quad \forall v \in X_0,$$

and the bilinear form $b(.,.)$ satisfies the inf-sup condition (14.1.3). Therefore, the problem (14.1.2) is well-posed.

Let $f \in V'$. Setting $v := u$ in (14.1.2a) and $q := p$ in (14.1.2b) yields

$$\nu |u|_V^2 = \int_\Omega f \cdot u dx \le \|f\|_{V'} |u|_V,$$

which leads to (14.1.4). Here, we used the Hölder's inequality. For the estimate of the pressure, using the inf-sup condition (14.1.3), the equation (14.1.2a), the Hölder's inequality, and (14.1.4) yields

$$\begin{aligned}\beta \|p\|_Q &\le \sup_{v \in V} \frac{|b(v,p)|}{|v|_V} = \sup_{v \in V} \frac{|\int_\Omega f \cdot v dx - a(u,v)|}{|v|_V} \\ &\le \sup_{v \in V} \frac{\|f\|_{V'} |v|_V + \nu |u|_V |v|_V}{|v|_V} = \|f\|_{V'} + \nu |u|_V \le 2\|f\|_{V'},\end{aligned}$$

which leads to (14.1.5).

Let $f \in L^2(\Omega)^d$. Because u is divergence-free and vanishes at the boundary, we have $u \in \mathcal{H}$, and

$$\int_\Omega f \cdot u dx = \int_\Omega P_{\mathcal{H}}(f) \cdot u dx.$$

Setting $v := u$ in (14.1.2a) and $q := p$ in (14.1.2b) yields

$$\begin{aligned}\nu |u|_V^2 = \int_\Omega P_{\mathcal{H}}(f) \cdot u dx &\le \|P_{\mathcal{H}}(f)\|_{L^2(\Omega)^d} \|u\|_{L^2(\Omega)^d} \\ &\le C_P \|P_{\mathcal{H}}(f)\|_{L^2(\Omega)^d} |u|_{H^1(\Omega)^d},\end{aligned}$$

which leads to (14.1.6). Here, we used the Hölder's and Poincaré inequalities. □

Remark 14.1.5. If $f \in L^2(\Omega)^d$, the inequality (14.1.6) can be estimated as

$$|u|_V \le \frac{C_P}{\nu} \|f\|_{L^2(\Omega)^d}. \qquad (14.1.7)$$

However, a priori estimate (14.1.6) is shaper than this estimate.

14.2 Crouzeix–Raviart Finite Element Approximation

14.2.1 Finite Element Approximation

Let $p \in (1,\infty)$. We define the Stokes elements (V_h, Q_h) as

$$V_h := (V_{h0}^{CR})^d, \quad Q_h := M_h^0 \cap L_0^{p'}(\Omega),$$

with norms

$$|v_h|_{V_h} := |v_h|_{W^{1,p}(\mathbb{T}_h)^d} = \left(\sum_{i=1}^d |v_{h,i}|^p_{W^{1,p}(\mathbb{T}_h)} \right)^{1/p}, \quad \|q_h\|_{Q_h} := \|q_h\|_{L^{p'}(\Omega)}$$

for any $v_h = (v_{h,1}, \dots, v_{h,d})^T \in V_h$ and $q_h \in Q_h$. Observe that V_h is nonconforming in $W_0^{1,p}(\Omega)^d$. Therefore, we define $a_h : (V+V_h) \times (V+V_h) \to \mathbb{R}$ and $b_h : (V+V_h) \times Q_h \to \mathbb{R}$ which is the discrete counterparts of the bilinear forms a and b as follows.

$$a_h(u_h, v_h) := \nu \sum_{i=1}^d \int_\Omega \nabla_h u_{h,i} \cdot \nabla_h v_{h,i} dx,$$
$$b_h(v_h, q_h) := - \int_\Omega \operatorname{div}_h v_h q_h dx,$$

We consider the Crouzeix–Raviart finite element approximate problem for the Stokes equation (14.1.1) as follows. Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$a_h(u_h, v_h) + b_h(v_h, p_h) = (f, v_h) \quad \forall v_h \in V_h, \tag{14.2.1a}$$
$$b_h(u_h, q_h) = 0 \quad \forall q_h \in Q_h. \tag{14.2.1b}$$

Definition 14.2.1. The vector-valued local interpolation operator

$$\mathcal{I}_T^{CR} : W^{1,1}(T)^d \to \mathcal{P}^1(T)^d \quad \forall T \in \mathbb{T}_h,$$

is defined component-wise, that is,

$$\mathcal{I}_T^{CR} v := (I_T^{CR} v_1, \dots, I_T^{CR} v_d)^T \quad \forall v = (v_1, \dots, v_d)^T \in W^{1,1}(T)^d.$$

We define the global interpolation operator $\mathcal{I}_h^{CR} : W_0^{1,1}(\Omega)^d \to V_h$ by

$$(\mathcal{I}_h^{CR} v)|_T = \mathcal{I}_T^{CR}(v|_T) \quad \forall T \in \mathbb{T}_h, \quad \forall v \in W_0^{1,1}(\Omega)^d.$$

14.2.2 Discrete Inf-sup Condition, Stability

Lemma 14.2.2 (Discrete inf-sup condition). *Let $p, p' \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. We impose Condition 4.3.1 with $h \le 1$. It holds that*

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{|b(v_h, q_h)|}{|v_h|_{W^{1,p}(\mathbb{T}_h)^d} \|q_h\|_{L^{p'}(\Omega)}} \ge \beta_0 > 0. \tag{14.2.2}$$

Proof. We follows [32, Lemma 55.18].

For any $r \in L_0^p(\Omega)$, there exists $v_r \in W_0^{1,p}(\Omega)^d$ such that $\operatorname{div} v_r = r$ and $|v_r|_{W^{1,p}(\Omega)^d} \le c\|r\|_{L^p(\Omega)}$ (see Remark 14.1.2).

Using $\nabla q_h \equiv 0$ on T and the definition of $\mathcal{I}_h^{CR} v_r$ yields

$$\begin{aligned} b_h(v_r, q_h) &= \sum_{T \in \mathbb{T}_h} \int_T \operatorname{div} v q_h dx = \sum_{T \in \mathbb{T}_h} \int_{\partial T} (v_r \cdot n_T) q_h ds \\ &= \sum_{T \in \mathbb{T}_h} \int_{\partial T} (\mathcal{I}_h^{CR} v_r \cdot n_T) q_h ds = \sum_{T \in \mathbb{T}_h} \int_T \operatorname{div}(\mathcal{I}_h^{CR} v_r) q_h dx \\ &= b_h(\mathcal{I}_h^{CR} v_r, q_h). \end{aligned}$$

Because

$$\int_T r q_h dx = b(v_r, q_h) = b_h(v_r, q_h) = b_h(\mathcal{I}_h^{CR} v_r, q_h),$$

using the $W_0^{1,p}(\Omega)^d$-stability (see (7.3.5)) of $\mathcal{I}_h^{CR}$ together with the above bound on v_r yields

$$\begin{aligned} \|q_h\|_{L^{p'}} &\le \sup_{r \in L_0^p(\Omega)} \frac{|\int_\Omega r q_h dx|}{\|r\|_{L^p(\Omega)}} = \sup_{r \in L_0^p(\Omega)} \frac{|b_h(\mathcal{I}_h^{CR} v_r, q_h)|}{\|r\|_{L^p(\Omega)}} \\ &= \sup_{r \in L_0^p(\Omega)} \frac{|b_h(\mathcal{I}_h^{CR} v_r, q_h)|}{|\mathcal{I}_h^{CR} v_r|_{W^{1,p}(\mathbb{T}_h)^d}} \frac{|\mathcal{I}_h^{CR} v_r|_{W^{1,p}(\mathbb{T}_h)^d}}{\|r\|_{L^p(\Omega)}} \\ &\le \sup_{v_h \in V_h} \frac{|b_h(v_h, q_h)|}{|v_h|_{W^{1,p}(\mathbb{T}_h)^d}} \times \sup_{r \in L_0^p(\Omega)} \frac{|\mathcal{I}_h^{CR} v_r|_{W^{1,p}(\mathbb{T}_h)^d}}{\|r\|_{L^p(\Omega)}} \\ &\le c \sup_{v_h \in V_h} \frac{|b_h(v_h, q_h)|}{|v_h|_{W^{1,p}(\mathbb{T}_h)^d}} \times \sup_{r \in L_0^p(\Omega)} \frac{|v_r|_{W^{1,p}(\mathbb{T}_h)^d}}{\|r\|_{L^p(\Omega)}}, \end{aligned}$$

which leads to (14.2.2). □

Remark 14.2.3 (Fortin operator [35]). Let $p = 2$. The Crouzeix–Raviart interpolation operator acts as a nonconforming Fortin operator. To show this, a simpler proof is possible than the proof of Lemma 14.2.2.

Let $v \in V$ and $q_h \in Q_h$. We have, using $\nabla q_h \equiv 0$ on T and the definition of $\mathcal{I}_{,h}^{CR} v$,

$$\begin{aligned}
b_h(v, q_h) &= \sum_{T \in \mathbb{T}_h} \int_T \operatorname{div} v q_h dx = \sum_{T \in \mathbb{T}_h} \int_{\partial T} (v \cdot n_T) q_h ds \\
&= \sum_{T \in \mathbb{T}_h} \int_{\partial T} (\mathcal{I}_h^{CR} v \cdot n_T) q_h ds = \sum_{T \in \mathbb{T}_h} \int_T \operatorname{div}(\mathcal{I}_h^{CR} v) q_h dx \\
&= b_h(\mathcal{I}_h^{CR} v, q_h).
\end{aligned}$$

Because that $\Delta(I_h^{CR} v_i) \equiv 0$ on $T \in \mathbb{T}_h$ and $n_T \cdot \nabla(I_h^{CR} v_i) \in \mathcal{P}^0$ on a face of T, we have, for $i = 1, \ldots, d$,

$$\begin{aligned}
|I_h^{CR} v_i|_{H^1(T)}^2 &= \int_T |\nabla I_h^{CR} v_i|^2 dx \\
&= \int_{\partial T} n_T \cdot \nabla(I_h^{CR} v_i) I_h^{CR} v_i ds - \int_T \Delta(I_h^{CR} v_i) I_h^{CR} v_i dx \\
&= \int_{\partial T} n_T \cdot \nabla(I_h^{CR} v_i) v_i ds \\
&\quad - \int_{\partial T} n_T \cdot \nabla(I_h^{CR} v_i) v_i ds + \int_T \nabla(I_h^{CR} v_i) \cdot \nabla v_i dx \\
&\leq |I_h^{CR} v_i|_{H^1(T)} |v_i|_{H^1(T)}.
\end{aligned}$$

This concludes that

$$|\mathcal{I}_h^{CR} v|_{H^1(\mathbb{T}_h)^d}^2 = \sum_{i=1}^d |I_h^{CR} v_i|_{H^1(T)}^2 \leq \sum_{i=1}^d |v_i|_{H^1(T)}^2 = |v|_{H^1(\mathbb{T}_h)^d}^2.$$

Lemma 14.2.4 (Stability)**.** *For any $f \in L^2(\Omega)^d$, let $(u_h, p_h) \in V_h \times Q_h$ be the solution of* (14.2.1)*. It then holds that*

$$|u_h|_{V_h} \leq \frac{c}{\nu} \|f\|_{L^2(\Omega)^d}, \tag{14.2.3a}$$

$$\|p_h\|_{Q_h} \leq \frac{c}{\beta_0} \|f\|_{L^2(\Omega)^d}. \tag{14.2.3b}$$

Proof. Setting $v_h := u_h$ in (14.2.1a) and $q_h := p_h$ in (14.2.1b) and using the discrete Poincaré inequality (11.2.2) yields

$$\nu |u_h|_{H^1(\mathbb{T}_h)^d}^2 \leq \|f\|_{L^2(\Omega)^d} \|u_h\|_{L^2(\Omega)^d} \leq c \|f\|_{L^2(\Omega)^d} |u_h|_{H^1(\mathbb{T}_h)^d},$$

which leads to (14.2.3a). For the estimate of the pressure, using the inf-sup condition (14.2.2), the equation (14.2.1a), the Hölder's inequality, the discrete Poincaré inequality (11.2.2), and (14.2.3a) yields

$$\begin{aligned}\beta_0\|p_h\|_{Q_h} &\leq \sup_{v_h\in V_h}\frac{|b_h(v_h,p_h)|}{|v_h|_{V_h}} = \sup_{v_h\in V_h}\frac{|\int_\Omega f\cdot v_h dx - a_h(u_h,v_h)|}{|v_h|_{V_h}}\\ &\leq \sup_{v_h\in V_h}\frac{c\|f\|_{L^2(\Omega)^d}|v_h|_{V_h}+\nu|u_h|_{V_h}|v_h|_{V_h}}{|v_h|_{V_h}} = c\|f\|_{L^2(\Omega)^d}+\nu|u_h|_{V_h}\\ &\leq c\|f\|_{L^2(\Omega)^d},\end{aligned}$$

which leads to (14.2.3b). □

Remark 14.2.5. For any $f\in L^2(\Omega)^d$, let $(u_h,p_h)\in V_h\times Q_h$ be the solution of (14.2.1). In general, the following equality does not hold.

$$\int_\Omega f\cdot u_h dx = \int_\Omega P_{\mathcal{H}}(f)\cdot u_h dx. \tag{14.2.4}$$

Indeed, because $f\in L^2(\Omega)^d$, by the Helmholtz decomposition,

$$f = P_{\mathcal{H}}(f)+\nabla q, \quad q\in H^1_*(\Omega).$$

We then have

$$\int_\Omega f\cdot u_h dx = \int_\Omega P_{\mathcal{H}}(f)\cdot u_h dx + \int_\Omega (u_h\cdot\nabla) q dx. \tag{14.2.5}$$

Setting $q_h := \operatorname{div}_h u_h$ in (14.2.1b) yields

$$\operatorname{div}_h u_h = 0 \quad \text{in } L^2(\Omega).$$

The second term of the right hand side in (14.2.5) does not vanish because

$$\begin{aligned}\int_\Omega (u_h\cdot\nabla) q dx &= \sum_{T\in\mathbb{T}_h}\int_T (u_h\cdot\nabla) q dx\\ &= \sum_{F\in\mathcal{F}_h}\int_F [\![(u_h\cdot n_F)q]\!] ds - \sum_{T\in\mathbb{T}_h}\int_T \operatorname{div} u_h q dx\\ &= \sum_{F\in\mathcal{F}_h}\int_F [\![(u_h\cdot n_F)q]\!] ds,\end{aligned}$$

and V_h is nonconforming in $H(\operatorname{div};\Omega)$. Remark that the normal component $[\![u_h\cdot n_F]\!]$ in V_h can jump across the mesh interfaces. The elements of vector space V_h are generally not divergence free in Ω. This means that the discretisation is not well-balanced ([32, Remark 53.22]). For this difficulty, in [63], a well-balanced scheme is is proposed, c.f., see [5] on anisotropic meshes. The scheme is constructed by using a lifting operator mapping the velocity test functions to the lowest-order Raviart–Thomas space in order to recover the property of (14.2.4).

Remark 14.2.6. We set

$$V_{h0}^{RT^k} := \{v_h \in V_h^{RT^k} :\ v_h \cdot n = 0\ \forall F \in \mathcal{F}_h^\partial\}. \tag{14.2.6}$$

If $u_h \in V_{h0}^{RT^0}$, it holds that

$$\begin{aligned}\int_\Omega (u_h \cdot \nabla) q dx &= \sum_{F \in \mathcal{F}_h} \int_F [\![(u_h \cdot n_F) q]\!] ds \\ &= \sum_{F \in \mathcal{F}_h^i} \int_F [\![u_h \cdot n_F]\!] q ds = 0,\end{aligned}$$

which leads to (14.2.4).

14.3 Second Strang Lemma

Lemma 14.3.1. *Let* $(u,p) \in V \times Q$ *be the solution of* (14.1.2) *and* $(u_h, p_h) \in V_h \times Q_h$ *be the solution of* (14.2.1)*. It then holds that*

$$\begin{aligned}&|u - u_h|_{V_h} + \|p - p_h\|_{Q_h} \\ &\quad \le C(\beta_0) \left\{ \inf_{v_h \in V_h} |u - v_h|_{V_h} + \frac{1}{\nu} \inf_{q_h \in Q_h} \|p - q_h\|_{Q_h} + \frac{1}{\nu} E_h(u,p) \right\},\end{aligned} \tag{14.3.1}$$

where

$$E_h(u,p) := \sup_{w_h \in V_h} \frac{|a_h(u, w_h) - (f, w_h) + b_h(w_h, p)|}{|w_h|_{V_h}}. \tag{14.3.2}$$

Proof. Let $v_h \in V_h$. Setting $\varphi_h := u_h - v_h \in V_h$, we get the decomposition

$$u - u_h = u - v_h - \varphi_h.$$

With an arbitrary $q_h \in Q_h$, we have, using (14.1.2b) and (14.2.1b),

$$\begin{aligned}b_h(\varphi_h, p - p_h) &= b_h(u_h - u, p - p_h) + b_h(u - v_h, p - p_h) \\ &= b_h(u_h - u, p - q_h) + b_h(u - v_h, p - p_h),\end{aligned}$$

because $Q_h \subset Q$. Using (14.2.1) and he Hölder's inequality yields

$$\begin{aligned}\nu |\varphi_h|_{V_h}^2 &= a_h(u_h - v_h, \varphi_h) \\ &= a_h(u - v_h, \varphi_h) + a_h(u_h, \varphi_h) - a_h(u, \varphi_h) \\ &= a_h(u - v_h, \varphi_h) + (f, \varphi_h) - b_h(\varphi_h, p_h) - a_h(u, \varphi_h) \\ &= a_h(u - v_h, \varphi_h) + (f, \varphi_h) - b_h(\varphi_h, p) + b_h(\varphi_h, p - p_h) - a_h(u, \varphi_h) \\ &\le \nu |u - v_h|_{V_h} |\varphi_h|_{V_h} + E_h(u,p) |\varphi_h|_{V_h} \\ &\quad + M_1 |u_h - u|_{V_h} \|p - q_h\|_{Q_h} + M_1 |u - v_h|_{V_h} \|p - p_h\|_{Q_h}.\end{aligned}$$

Using the above inequality, we have

$$
\begin{aligned}
\nu|u-u_h|_{V_h}^2 &\le 2\nu|u-v_h|_{V_h}^2 + 2\nu|\varphi_h|_{V_h}^2 \\
&\le 2\nu|u-v_h|_{V_h}^2 + 2\nu|u-v_h|_{V_h}|\varphi_h|_{V_h} + 2E_h(u,p)|\varphi_h|_{V_h} \\
&\quad + 2M_1|u_h-u|_{V_h}\|p-q_h\|_{Q_h} + 2M_1|u-v_h|_{V_h}\|p-p_h\|_{Q_h} \\
&\le 2\nu|u-v_h|_{V_h}^2 + 2\nu|u-v_h|_{V_h}|u_h-u|_{V_h} + 2\nu|u-v_h|_{V_h}^2 \\
&\quad + 2E_h(u,p)|u_h-u|_{V_h} + 2E_h(u,p)|u-v_h|_{V_h} \\
&\quad + 2M_1|u_h-u|_{V_h}\|p-q_h\|_{Q_h} + 2M_1|u-v_h|_{V_h}\|p-p_h\|_{Q_h}.
\end{aligned}
$$

Using Young's inequality $ab \le \frac{\varepsilon}{4}a^2 + \frac{1}{\varepsilon}b^2$ with $a,b\in\mathbb{R}_+$ and arbitrary $\varepsilon>0$, we have, for any $\varepsilon_1,\varepsilon_2>0$,

$$
\begin{aligned}
2\nu|u-v_h|_{V_h}|u_h-u|_{V_h} &\le \frac{\nu\varepsilon_1}{4}|u_h-u|_{V_h}^2 + \frac{4\nu}{\varepsilon_1}|u-v_h|_{V_h}^2, \\
2E_h(u,p)|u_h-u|_{V_h} &\le \frac{\nu\varepsilon_1}{4}|u_h-u|_{V_h}^2 + \frac{4}{\nu\varepsilon_1}E_h(u,p)^2, \\
M_1|u_h-u|_{V_h}\|p-q_h\|_{Q_h} &\le \frac{\nu\varepsilon_1}{4}|u_h-u|_{V_h}^2 + \frac{M_1^2}{\nu\varepsilon_1}\|p-q_h\|_{Q_h}^2, \\
2M_1|u-v_h|_{V_h}\|p-p_h\|_{Q_h} &\le \frac{\varepsilon_2}{4\nu}\|p-p_h\|_{Q_h}^2 + \frac{4M_1^2\nu}{\varepsilon_2}|u-v_h|_{V_h}^2, \\
2E_h(u,p)|u-v_h|_{V_h} &\le \frac{\nu}{4}|u-v_h|_{V_h}^2 + \frac{4}{\nu}E_h(u,p)^2.
\end{aligned}
$$

Setting $\varepsilon_1 := 1$, we obtain

$$
\begin{aligned}
|u-u_h|_{V_h}^2 \le \left(33 + \frac{16M_1^2}{\varepsilon_2}\right)|u-v_h|_{V_h}^2 + \frac{4M_1^2}{\nu^2}\|p-q_h\|_{Q_h}^2 + \frac{32}{\nu^2}E_h(u,p)^2 \\
+ \frac{\varepsilon_2}{\nu^2}\|p-p_h\|_{Q_h}^2.
\end{aligned}
$$

The intermediate result will be used later on.

For estimating the pressure error $\|p-p_h\|_{Q_h}$, we use the inf–sup stability relation (14.2.2) with $p=2$. With an arbitrary $q_h\in Q_h$, it follows that

$$
\begin{aligned}
\|p-p_h\|_{Q_h} &\le \|p-q_h\|_{Q_h} + \|q_h-p_h\|_{Q_h} \\
&\le \|p-q_h\|_{Q_h} + \frac{1}{\beta_0}\sup_{v_h\in V_h}\frac{b_h(v_h,q_h-p_h)}{|v_h|_{V_h}} \\
&\le \|p-q_h\|_{Q_h} \\
&\quad + \frac{1}{\beta_0}\sup_{v_h\in V_h}\frac{b_h(v_h,q_h-p)}{|v_h|_{V_h}} + \frac{1}{\beta_0}\sup_{v_h\in V_h}\frac{b_h(v_h,p-p_h)}{|v_h|_{V_h}} \\
&\le \left(1+\frac{1}{\beta_0}\right)\|p-q_h\|_{Q_h} + \frac{1}{\beta_0}\sup_{v_h\in V_h}\frac{b_h(v_h,p-p_h)}{|v_h|_{V_h}}.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\sup_{v_h \in V_h} \frac{b_h(v_h, p - p_h)}{|v_h|_{V_h}} \\
&\quad = \sup_{v_h \in V_h} \frac{a_h(u, v_h) + b_h(v_h, p) - (f, v_h) + a_h(u_h - u, v_h)}{|v_h|_{V_h}} \\
&\quad \le E_h(u,p) + \nu |u - u_h|_{V_h}.
\end{aligned}
$$

Therefore, it holds that

$$
\|p - p_h\|_{Q_h}^2 \le 9\left(1 + \frac{1}{\beta_0}\right)^2 \|p - q_h\|_{Q_h}^2 + \frac{9}{\beta_0^2} E_h(u,p)^2 + \frac{9\nu^2}{\beta_0^2}|u - u_h|_{V_h}^2.
$$

Combination of the intermediate results and choosing $\varepsilon_2 := \frac{1}{18}\beta_0^2$ yields

$$
\begin{aligned}
&|u - u_h|_{V_h}^2 \\
&\quad \le \left(33 + \frac{16 \times 18 M_1^2}{\beta_0^2}\right)|u - v_h|_{V_h}^2 + \frac{4M_1^2}{\nu^2}\|p - q_h\|_{Q_h}^2 + \frac{32}{\nu^2}E_h(u,p)^2 \\
&\qquad + \frac{\beta_0^2}{2\nu^2}\left(1 + \frac{1}{\beta_0}\right)^2 \|p - q_h\|_{Q_h}^2 + \frac{1}{2\nu^2}E_h(u,p)^2 + \frac{1}{2}|u - u_h|_{V_h}^2,
\end{aligned}
$$

which leads to

$$
|u - u_h|_{V_h} \le C_1(\beta_0)\left\{|u - v_h|_{V_h} + \frac{1}{\nu}\|p - q_h\|_{Q_h} + \frac{1}{\nu}E_h(u,p)\right\},
$$

for some a positive constant $C_1(\beta_0)$. We thus have

$$
\|p - p_h\|_{Q_h} \le C_2(\beta_0)\left\{|u - v_h|_{V_h} + \frac{1}{\nu}\|p - q_h\|_{Q_h} + \frac{1}{\nu}E_h(u,p)\right\},
$$

for some a positive constant $C_2(\beta_0)$. These estimates implies the desired result. □

Lemma 14.3.2. *Let $(u,p) \in V \times Q$ be the solution of* (14.1.2) *and $(u_h, p_h) \in V_h \times Q_h$ be the solution of* (14.2.1)*. It then holds that*

$$
|u - u_h|_{V_h} \le 2 \inf_{v_h \in X_{h0}} |u - v_h|_{V_h} + \frac{1}{\nu}E_{h0}(u,p), \tag{14.3.3a}
$$

$$
\|p - p_h\|_{Q_h} \le \left(1 + \frac{1}{\beta_0}\right)\|p - q_h\|_{Q_h} + \frac{1}{\beta_0}\left(E_h(u,p) + \nu|u - u_h|_{V_h}\right), \tag{14.3.3b}
$$

where

$$
E_{h0}(u,p) := \sup_{w_h \in X_{h0}} \frac{|a_h(u, w_h) - (f, w_h)|}{|w_h|_{V_h}}. \tag{14.3.4}
$$

Proof. Let $v_h \in X_{h0}$. Setting $\varphi_h := u_h - v_h \in X_{h0}$, we get the decomposition

$$u - u_h = u - v_h - \varphi_h.$$

Using (14.2.1) and he Hölder's inequality yields

$$\begin{aligned}\nu|\varphi_h|_{V_h}^2 &= a_h(u_h - v_h, \varphi_h) \\ &= a_h(u - v_h, \varphi_h) + a_h(u_h, \varphi_h) - a_h(u, \varphi_h) \\ &= a_h(u - v_h, \varphi_h) + (f, \varphi_h) - a_h(u, \varphi_h) \\ &\le \nu|u - v_h|_{V_h}|\varphi_h|_{V_h} + E_{h0}(u,p)|\varphi_h|_{V_h}.\end{aligned}$$

We hence have

$$\begin{aligned}\nu|u - u_h|_{V_h} &\le \nu|u - v_h|_{V_h} + \nu|\varphi_h|_{V_h} \\ &\le 2\nu|u - v_h|_{V_h} + E_{h0}(u,p),\end{aligned}$$

which leads to (14.3.3a).

By analogous argument in Lemma 14.3.1, we have (14.3.3b). □

14.4 Consistency Error Analysis on Anisotropic Meshes

The essential parts for error estimates are the consistency error terms (14.3.2) or (14.3.4).

Lemma 14.4.1. *We impose Condition 4.3.1. Let $(u,p) \in (V \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$ be the solution of the homogeneous Dirichlet Stokes problem (14.1.1) with data $f \in L^2(\Omega)^d$. If all elements are composed of the type $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ and Condition 3.3.1 is not imposed, it then holds that*

$$\begin{aligned}&E_h(u,p) = \sup_{w_h \in V_h} \frac{|a_h(u, w_h) - (f, w_h) + b_h(w_h, p)|}{|w_h|_{V_h}} \\ &\le c\nu \sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} h_j \left\| \frac{\partial}{\partial r_j} \nabla u_i \right\|_{L^2(T)^d} + c\nu h \|\Delta u\|_{L^2(\Omega)^d} + ch|p|_{H^1(\Omega)} + ch\|f\|_{L^2(\Omega)^d},\end{aligned} \tag{14.4.1}$$

Proof. For $i = 1, \dots, d$, we have

$$\operatorname{div}(I_h^{RT^0}(\nu\nabla u_i - pe_i)) = \Pi_h^0 \operatorname{div}(\nu\nabla u_i - pe_i) = \Pi_h^0 \left(\nu\Delta u_i - \frac{\partial p}{\partial x_i}\right) = -\Pi_h^0 f_i,$$

where $(e_1, \dots, e_d)$ denotes the Cartesian basis of $\mathbb{R}^d$.

Setting $v_h := I_h^{RT^0}(\nu \nabla u_i - p e_i)$ and using (11.6.1), we have, for any $w_h \in V_h$ and $i = 1, \dots, d$,

$$
\begin{aligned}
&\nu \int_\Omega \nabla_h u_i \cdot \nabla_h w_{h,i} dx - \int_\Omega f_i w_{h,i} dx - \int_\Omega (p e_i) \cdot \nabla_h w_{h,i} dx \\
&\quad = \int_\Omega (\nu \nabla u_i - v_h) \cdot \nabla_h w_{h,i} dx - \int_\Omega (f_i + \operatorname{div} v_h) w_{h,i} dx - \int_\Omega (p e_i) \cdot \nabla_h w_{h,i} dx \\
&\quad = \nu \int_\Omega (\nabla u_i - I_h^{RT^0} \nabla u_i) \cdot \nabla_h w_{h,i} dx - \int_\Omega (f_i - \Pi_h^0 f_i)(w_{h,i} - \Pi_h^0 w_{h,i}) dx \\
&\qquad + \int_\Omega (I_h^{RT^0}(p e_i) - p e_i) \cdot \nabla_h w_{h,i} dx,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
a_h(u, w_h) - (f, w_h) &= \nu \sum_{i=1}^d \int_\Omega (\nabla u_i - I_h^{RT^0} \nabla u_i) \cdot \nabla_h w_{h,i} dx \\
&\quad - \sum_{i=1}^d \int_\Omega (f_i - \Pi_h^0 f_i)(w_{h,i} - \Pi_h^0 w_{h,i}) dx \\
&\quad + \sum_{i=1}^d \int_\Omega (I_h^{RT^0}(p e_i) - p e_i) \cdot \nabla_h w_{h,i} dx \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

Using the Hölder's inequality and (9.7.3), the term I_1 is estimated as

$$
\begin{aligned}
|I_1| &\le \nu \sum_{i=1}^d \sum_{T \in \mathbb{T}_h} \| \nabla u_i - I_h^{RT^0} \nabla u_i \|_{L^2(T)^d} |w_{h.i}|_{H^1(T)} \\
&\le \nu \left(\sum_{i=1}^d \sum_{T \in \mathbb{T}_h} \| \nabla u_i - I_h^{RT^0} \nabla u_i \|^2_{L^2(T)^d} \right)^{1/2} \left(\sum_{i=1}^d \sum_{T \in \mathbb{T}_h} |w_{h.i}|^2_{H^1(T)} \right)^{1/2} \\
&\le c\nu \left(\sum_{i=1}^d \sum_{T \in \mathbb{T}_h} \sum_{|\varepsilon|=1} h^{2\varepsilon} \| \partial_r^\varepsilon \nabla u_i \|^2_{L^2(T)^d} + \sum_{i=1}^d \sum_{T \in \mathbb{T}_h} h_T^2 \| \nabla \cdot \nabla u_i \|^2_{L^2(T)} \right)^{1/2} |w_h|_{H^1(\mathbb{T}_h)^d} \\
&\le c\nu \left(\sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^d h_j \left\| \frac{\partial}{\partial r_j} \nabla u_i \right\|_{L^2(T)^d} + h \left(\sum_{i=1}^d \sum_{T \in \mathbb{T}_h} \| \Delta u_i \|^2_{L^2(T)} \right)^{1/2} \right) |w_h|_{H^1(\mathbb{T}_h)^d}.
\end{aligned}
$$

The term I_2 is estimated as, using the Hölder's inequality, (6.2.1) and

(6.3.2),

$$|I_2| \leq \sum_{i=1}^{d} \|f_i - \Pi_h^0 f_i\|_{L^2(\Omega)} \|w_{h,i} - \Pi_h^0 w_{h,i}\|_{L^2(\Omega)}$$
$$\leq ch \|f\|_{L^2(\Omega)^d} |w_h|_{H^1(\mathbb{T}_h)^d}.$$

We estimate I_3 as follows. Using the Hölder's inequality and (9.7.3), we have

$$\begin{aligned}
|I_3| &\leq \sum_{i=1}^{d} \|I_h^{RT^0}(pe_i) - pe_i\|_{L^2(\Omega)} |w_{h,i}|_{H^1(\mathbb{T}_h)} \\
&\leq c \left(\sum_{i=1}^{d} \sum_{T \in \mathbb{T}_h} \|I_h^{RT^0}(pe_i) - pe_i\|_{L^2(T)}^2 \right)^{1/2} |w_h|_{H^1(\mathbb{T}_h)^d} \\
&\leq c \left(\sum_{i=1}^{d} \sum_{T \in \mathbb{T}_h} \sum_{|\varepsilon|=1} h^{\varepsilon} \|\partial_r^{2\varepsilon}(pe_i)\|_{L^2(T)^d}^2 + \sum_{i=1}^{d} \sum_{T \in \mathbb{T}_h} h_T^2 \|\nabla \cdot (pe_i)\|_{L^2(T)}^2 \right)^{1/2} |w_h|_{H^1(\mathbb{T}_h)^d} \\
&\leq ch \left(\sum_{i=1}^{d} \sum_{T \in \mathbb{T}_h} \left\| \frac{\partial p}{\partial x_i} \right\|_{L^2(T)}^2 \right)^{1/2} |w_h|_{H^1(\mathbb{T}_h)^d}.
\end{aligned}$$

Gathering the above inequalities and using the Cauchy–Schwarz inequality, we conclude that

$$\begin{aligned}
&\frac{|a_h(u, w_h) - (f, w_h) + b_h(w_h, p)|}{|w_h|_{V_h}} \\
&\quad \leq c\nu \sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} h_j \left\| \frac{\partial}{\partial r_j} \nabla u_i \right\|_{L^2(T)^d} + c\nu h \|\Delta u\|_{L^2(\Omega)^d} + ch|p|_{H^1(\Omega)} + ch\|f\|_{L^2(\Omega)^d},
\end{aligned}$$

which is the desired result. □

Remark 14.4.2. For $i = 1, \ldots, d$, we have

$$\mathrm{div}(I_h^{RT^0}(\nu \nabla u_i)) = \Pi_h^0 \,\mathrm{div}(\nu \nabla u_i) = \Pi_h^0 \left(\nu \Delta u_i\right).$$

Setting $v_h := I_h^{RT^0}(\nu \nabla u_i)$ and using (11.6.1), we have, for any $w_h \in X_{h0}$

and $i = 1, \ldots, d$,

$$
\begin{aligned}
&\nu \int_\Omega \nabla_h u_i \cdot \nabla_h w_{h,i} dx - \int_\Omega f_i w_{h,i} dx \\
&\quad = \int_\Omega (\nu \nabla u_i - v_h) \cdot \nabla_h w_{h,i} dx - \int_\Omega (f_i + \operatorname{div} v_h) w_{h,i} dx \\
&\quad = \nu \int_\Omega (\nabla u_i - I_h^{RT^0} \nabla u_i) \cdot \nabla_h w_{h,i} dx \\
&\qquad - \int_\Omega (-\nu \Delta u_i + \Pi_h^0 \nu \Delta u_i) w_{h,i} dx - \int_\Omega \frac{\partial p}{\partial x_i} w_{h,i} dx,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
a_h(u, w_h) - (f, w_h) &= \nu \sum_{i=1}^d \int_\Omega (\nabla u_i - I_h^{RT^0} \nabla u_i) \cdot \nabla_h w_{h,i} dx \\
&\quad - \nu \sum_{i=1}^d \int_\Omega (-\Delta u_i + \Pi_h^0 \Delta u_i) w_{h,i} dx - \int_\Omega (w_h \cdot \nabla) p dx \\
&=: J_1 + J_2 + J_3.
\end{aligned}
$$

Because $p \in H_*^1(\Omega)$, as described in Remark 14.2.6, if $w_h \in V_{h0}^{RT^0}$, it holds that

$$
J_3 = \int_\Omega (w_h \cdot \nabla) p dx = 0. \tag{14.4.2}
$$

Using the Hölder's, Cauchy–Schwarz inequalites and (9.7.3), the term J_1 can be estimated as I_1 in Lemma 14.4.1. The term J_2 can be estimated as I_2 in Lemma 14.4.1 by using the Hölder's inequality, (6.2.1) and (6.3.2). Therefore if $J_3 = 0$, it holds that, for any $w_h \in X_{h0}$,

$$
\frac{|a_h(u, w_h) - (f, w_h)|}{|w_h|_{V_h}} \leq c\nu \sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^d h_j \left\| \frac{\partial}{\partial r_j} \nabla u_i \right\|_{L^2(T)^d} + c\nu h \|\Delta u\|_{L^2(\Omega)^d},
$$

which leads to

$$
\frac{1}{\nu} E_{h0}(u, p) \leq c \sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^d h_j \left\| \frac{\partial}{\partial r_j} \nabla u_i \right\|_{L^2(T)^d} + ch \|\Delta u\|_{L^2(\Omega)^d}.
$$

Then, the error estimate (14.3.3a) is described as

$$
\begin{aligned}
&|u - u_h|_{V_h} \\
&\quad \le 2 \inf_{v_h \in X_{h0}} |u - v_h|_{V_h} + \frac{1}{\nu} E_{0h}(u,p) \\
&\quad \le 2 \inf_{v_h \in X_{h0}} |u - v_h|_{V_h} + c \sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} h_j \left\| \frac{\partial}{\partial r_j} \nabla u_i \right\|_{L^2(T)^d} + ch \|\Delta u\|_{L^2(\Omega)^d}.
\end{aligned}
\tag{14.4.3}
$$

A notable feature is that ν does not appear on the right-hand side of (14.4.3). To achieve this, schemes should be constructed so that $J_3 = 0$.

14.5 Well-balanced Scheme

In this section, we introduce a well-balanced scheme proposed in [63] and in [5] under the maximum-angle condition. The schemes have features described in Remark 14.4.2.

We define the global lowest-order Raviart–Thomas interpolation $I_{h0}^{RT^0} : H_0^1(\Omega)^d \oplus V_h \to V_{h0}^{RT^0}$ as

$$
(I_h^{RT^0} v)|_T = I_T^{RT^0}(v|_T) \quad \forall T \in \mathbb{T}_h, \quad \forall v \in H^1(\Omega)^d \oplus V_h.
$$

We then consider the well-balanced type Crouzeix–Raviart finite element approximate problem for the Stokes equation (14.1.1) as follows. Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$
a_h(u_h, v_h) + b_h(v_h, p_h) = (f, I_{h0}^{RT^0} v_h) \quad \forall v_h \in V_h, \tag{14.5.1a}
$$

$$
b_h(u_h, q_h) = 0 \quad \forall q_h \in Q_h. \tag{14.5.1b}
$$

Lemma 14.5.1 (Stability). *Assume that Ω is convex. We impose Condition 4.3.1 with $h \le 1$. For any $f \in L^2(\Omega)^d$, let $(u_h, p_h) \in V_h \times Q_h$ be the solution of* (14.5.1). *It then holds that*

$$
|u_h|_{V_h} \le \frac{c}{\nu} \|f\|_{L^2(\Omega)^d}, \tag{14.5.2}
$$

Proof. Let $v_h \in V_h$. Using the discrete Poincaré inequality (11.2.2), (9.4.8), Lemmata 9.5.1, and 9.5.2,

$$
\|I_{h0}^{RT^0} v_h\|_{L^2(\Omega)^d}^2 = \sum_{T \in \mathbb{T}_h} \|I_T^{RT^0} v_h\|_{L^2(T)^d}^2 \le c \sum_{T \in \mathbb{T}_h} \|v_h\|_{H^1(T)^d}^d \le c |v_h|_{V_h}^2,
$$

which leads to

$$\|I_{h0}^{RT^0} v_h\|_{L^2(\Omega)^d} \le c|v_h|_{H^1(\mathbb{T}_h)^d}. \tag{14.5.3}$$

Setting $v_h := u_h$ in (14.2.1a) and $q_h := p_h$ in (14.2.1b) yields

$$\nu|u_h|^2_{H^1(\mathbb{T}_h)^d} \le \|f\|_{L^2(\Omega)^d}\|I_{h0}^{RT^0} u_h\|_{L^2(\Omega)^d} \le c\|f\|_{L^2(\Omega)^d}|u_h|_{H^1(\mathbb{T}_h)^d},$$

which leads to (14.5.2). □

14.6 Consistency Error Analysis of the Well-balanced Scheme

Lemma 14.6.1. *We impose Condition 4.3.1. Let* $(u,p) \in (V \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$ *be the solution of the homogeneous Dirichlet Stokes problem* (14.1.1) *with data* $f \in L^2(\Omega)^d$. *If all elements are composed of the type* $T_1 \in \mathfrak{T}^{(2)}, \mathfrak{T}_1^{(3)}$ *and Condition 3.3.1 is not imposed, it then holds that*

$$\begin{aligned} \frac{1}{\nu} E_{h0}^w(u,p) &:= \frac{1}{\nu} \sup_{w_h \in X_{h0}} \frac{|a_h(u,w_h) - (f, I_{h0}^{RT^0} w_h)|}{|w_h|_{V_h}} \\ &\le c \sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^d h_j \left\| \frac{\partial}{\partial r_j} \nabla u_i \right\|_{L^2(T)^d} + ch\|\Delta u\|_{L^2(\Omega)^d}. \end{aligned} \tag{14.6.1}$$

Proof. Setting $v_h := I_h^{RT^0}(\nu \nabla u_i)$ and using (11.6.1), we have, for any $w_h \in X_{h0}$ and $i = 1, \dots, d$,

$$\begin{aligned} &\nu \int_\Omega \nabla_h u_i \cdot \nabla_h w_{h,i} dx - \int_\Omega f_i (I_{h0}^{RT^0} w_h)_i dx \\ &= \int_\Omega (\nu \nabla u_i - v_h) \cdot \nabla_h w_{h,i} dx - \int_\Omega f_i (I_{h0}^{RT^0} w_h)_i dx - \int_\Omega \operatorname{div} v_h w_{h,i} dx \\ &= \nu \int_\Omega (\nabla u_i - I_h^{RT^0} \nabla u_i) \cdot \nabla_h w_{h,i} dx - \int_\Omega \left(-\nu \Delta u_i + \frac{\partial p}{\partial x_i} \right) (I_{h0}^{RT^0} w_h)_i dx \\ &\quad - \nu \int_\Omega (\Pi_h^0 \Delta u_i)(w_{h,i} - (I_{h0}^{RT^0} w_h)_i) dx - \nu \int_\Omega (\Pi_h^0 \Delta u_i)(I_{h0}^{RT^0} w_h)_i dx \\ &= \nu \int_\Omega (\nabla u_i - I_h^{RT^0} \nabla u_i) \cdot \nabla_h w_{h,i} dx + \nu \int_\Omega (\Delta u_i - \Pi_h^0 \Delta u_i)(I_{h0}^{RT^0} w_h)_i dx \\ &\quad - \nu \int_\Omega (\Pi_h^0 \Delta u_i)(w_{h,i} - (I_{h0}^{RT^0} w_h)_i) dx - \int_\Omega \frac{\partial p}{\partial x_i} (I_{h0}^{RT^0} w_h)_i dx, \end{aligned}$$

which leads to

$$
\begin{aligned}
a_h(u, w_h) - (f, w_h) &= \nu \sum_{i=1}^{d} \int_\Omega (\nabla u_i - I_h^{RT^0} \nabla u_i) \cdot \nabla_h w_{h,i} dx \\
&\quad + \nu \sum_{i=1}^{d} \int_\Omega (\Delta u_i - \Pi_h^0 \Delta u_i)(I_{h0}^{RT^0} w_h)_i dx \\
&\quad + \nu \sum_{i=1}^{d} \int_\Omega (\Pi_h^0 \Delta u_i)(w_{h,i} - (I_{h0}^{RT^0} w_h)_i) dx \\
&\quad - \int_\Omega ((I_{h0}^{RT^0} w_h) \cdot \nabla) p dx \\
&=: K_1 + K_2 + K_3 + K_4.
\end{aligned}
$$

Using the Hölder's, Cauchy–Schwarz inequalities and (9.7.3), the term K_1 can be estimated as I_1 in Lemma 14.4.1, that is,

$$
|K_1| \le c\nu \left(\sum_{T \in \mathbb{T}_h} \sum_{i,j=1}^{d} h_j \left\| \frac{\partial}{\partial r_j} \nabla u_i \right\|_{L^2(T)^d} + h \left(\sum_{T \in \mathbb{T}_h} \sum_{i=1}^{d} \|\Delta u_i\|_{L^2(T)}^2 \right)^{1/2} \right) |w_h|_{H^1(\mathbb{T}_h)^d}.
$$

Using the Hölder's, Cauchy–Schwarz inequalities, (9.4.8), (6.2.3), (6.2.12) and (9.6.2), the term K_2 can be estimated as

$$
\begin{aligned}
|K_2| &\le \nu \sum_{i=1}^{d} \int_\Omega |\Delta u_i - \Pi_h^0 \Delta u_i| |(I_{h0}^{RT^0} w_h)_i - w_{h,i}| dx \\
&\quad + \nu \sum_{i=1}^{d} \int_\Omega |\Delta u_i - \Pi_h^0 \Delta u_i| |w_{h,i} - \Pi_h^0 w_{h,i}| dx \\
&\le c\nu h \|\Delta u\|_{L^2(\Omega)^d} |w_h|_{H^1(\mathbb{T}_h)^d},
\end{aligned}
$$

where we used the fact that

$$
\int_\Omega (\Delta u_i - \Pi_h^0 \Delta u_i) \Pi_h^0 w_{h,i} dx = 0.
$$

Using the Hölder's, Cauchy–Schwarz inequalities, (6.2.12) and (9.6.2), the term K_3 can be estimated as

$$
|K_3| \le c\nu h \|\Delta u\|_{L^2(\Omega)^d} |w_h|_{H^1(\mathbb{T}_h)^d}.
$$

Because $p \in H_*^1(\Omega)$ and $I_{h0}^{RT^0} w_h \in V_{h0}^{RT^0}$,

$$
K_4 = - \int_\Omega ((I_{h0}^{RT^0} w_h) \cdot \nabla) p dx = 0.
$$

Gathering the above inequalities yields the desired result. □

Remark 14.6.2 (L^2 error estimate)**.** The L^2 error estimate of the well-balanced scheme on shape-regular meshes is proven in [65]. However, the L^2 error analysis on anisotropic meshes is still open. For the standard Crouzeix–Raviart approximate problem, it is possible to obtain the L^2 error estimate.

14.7 Further Topics

14.7.1 The k-th order Crouzeix–Raviart Finite Element Methods

We consider the case of $d = 2$ and $k \geq 1$. The k-th order Crouzeix–Raviart finite element method is constructed in [71] and is analysed on shape-regular meshes.

For any $T \in \mathbb{T}_h$, the enriched space $\mathcal{V}^k(T)$ is defined by

$$\mathcal{V}^k(T) := \mathcal{P}^k(T) + \Sigma^{k+1}(T), \quad k \in \mathbb{N},$$

where $\Sigma^2(T) = \emptyset$ for $k = 1$. Examples of the subspace $\Sigma^{k+1}(T) \subset \mathcal{P}^{k+1}(T)$ is introduced later. We define discontinuous finite element spaces by

$$\begin{aligned} P^k_{dc,h} &:= \left\{ q_h \in L^2(\Omega);\ q_h|_T \in \mathcal{V}^k(T)\ \forall T \in \mathbb{T}_h \right\}, \\ Q^{k-1}_{dc,h} &:= \left\{ q_h \in L^2(\Omega);\ q_h|_T \in \mathcal{P}^{k-1}(T)\ \forall T \in \mathbb{T}_h \right\}. \end{aligned}$$

Furthermore, we define the (weakly continuous) nonconforming finite element spaces by

$$\begin{aligned} P^k_{wc,h} &:= \left\{ v_h \in P^k_{dc,h} :\ \int_F p_{k-1} [\![v_h]\!]_F ds = 0\ \forall F \in \mathcal{F}_h,\ \forall p_{k-1} \in \mathcal{P}^{k-1}(F) \right\}, \\ V^k_{wc,h} &:= (P^k_{wc,h})^d, \\ Q^{k-1}_h &:= Q^{k-1}_{dc,h} \cap Q, \end{aligned}$$

with norms

$$|v_h|_{V^k_{wc,h}} := \left(\sum_{i=1}^d |v_{h,i}|^2_{H^1(\mathbb{T}_h)} \right)^{1/2}, \quad \|q_h\|_{Q^{k-1}_h} := \|q_h\|_{L^2(\Omega)}$$

for any $v_h = (v_{h,1}, \ldots, v_{h,d})^T \in V^k_{wc,h}$ and $q_h \in Q^{k-1}_h$. When $k = 1$, $P^1_{dc,h}$ is just the standard Crouzeix–Raviart finite element space.

Let $k \geq 2$. Let $\lambda_{T,i} : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, \ldots, 3$ be the barycentric coordinates of the element $T \in \mathbb{T}_h$ such that the edge $E_{T,i}$ of T corresponds to $\lambda_{T,i} = 0$, $i = 1, 2, 3$. We define the subspace $\Sigma^{k+1}(T) \subset \mathcal{P}^{k+1}(T)$ by

$$\Sigma^{k+1}(T) := \mathrm{span}\{ b_T \lambda_{T,1}^{k-2-i} \lambda_{T,2}^i;\ i = 0, \ldots, k-2 \},$$

where

$$b_T := (\lambda_{T,1} - \lambda_{T,2})(\lambda_{T,2} - \lambda_{T,3})(\lambda_{T,3} - \lambda_{T,1}).$$

We give the local nodal functionals on $H^1(\mathbb{T}_h)$ by

$$N^j_{E_{T,i}}(v) := \frac{1}{2}\int_{-1}^{1} v(x_{E_{T,i}}(s), y_{E_{T,i}}(s))|_T L^j(s)ds, \quad j \geq 0, \quad i = 1, 2, 3,$$
$$N^j_T(v) := \frac{1}{|T|}\int_T v(x, y)M^j_T(x, y)dx, \quad j = 1, \ldots, \frac{k(k-1)}{2},$$

where $\{M^j_T\}$ is an arbitrary but fixed basis of $\mathcal{P}^{k-2}(T)$ and L^j is the j-th Legendre polynomials. The edge $E_{T,i}$ with end points $A = (x_A, y_A)^T$ and $B = (x_B, y_B)^T$ is parametrised by $s \in (-1, 1)$ such that

$$x_{E_{T,i}}(s) := x_A + \frac{s+1}{2}(x_B - x_A), \quad y_{E_{T,i}}(s) := y_A + \frac{s+1}{2}(y_B - y_A).$$

For each element $T \in \mathbb{T}_h$ and any integer $k \geq 2$, we define the set of nodal functionals by

$$\mathcal{N}^k_T := \{N^j_{E_{T,i}} : \ i = 1, 2, 3, \ j = 0, \ldots, k-1\} \cup \{N^j_T : \ j = 1, \ldots, k(k-1)/2\}.$$

The set $\mathcal{N}^k_T$ of nodal functionals is $\mathcal{V}^k(T)$-unisolvent ([71, Theorem 1]). The triple $(T, \mathcal{V}^k(T), \mathcal{N}^k_T)$ is then a finite element. We define the local interpolation operator

$$I^k_{wc,T} : H^1(T) \to V^k(T)$$

as

$$N_i(I^k_{wc,T}\varphi) = N_i(\varphi) \quad \forall \varphi \in H^1(T), \quad \forall N_i \in \mathcal{N}^k_T.$$

14.7.2 Inf-sup Conditions

It is an interesting theme to consider the Stokes element (V_h, Q_h) on anisotropic meshes satisfying the inf-sup condition (12.1.8b).

14.7.2.1 The k-th order Crouzeix–Raviart Finite Element Pair $(V_h, Q_h) := (V^k_{wc,h}, Q^{k-1}_h)$

When $k = 1$ and $d \in \{2, 3\}$, also see Remark 14.2.3.

Lemma 14.7.1. *Assume that there exists a local interpolation operator*

$$I^k_{wc,T} : H^1(T) \to \mathcal{V}^k(T) \quad \forall T \in \mathbb{T}_h$$

satisfying the following properties: for any $\varphi \in H^1(T)$,

$$\int_F p_{k-1}(I^k_{wc,T}\varphi - \varphi)ds = 0 \quad \forall p_{k-1} \in \mathcal{P}^{k-1}(F),\ F \subset \partial T \tag{14.7.1a}$$

$$\int_T q_{k-2}(I^k_{wc,T}\varphi - \varphi)dx = 0 \quad \forall q_{k-2} \in \mathcal{P}^{k-2}(T), \tag{14.7.1b}$$

Let $\mathcal{I}^k_{wc,h} : V = H^1_0(\Omega)^d \to V^k_{wc,h}$ *be the global interpolation operator defined as*

$$(\mathcal{I}^k_{wc,h}v)|_T = \mathcal{I}^k_{wc,T}(v|_T) \quad \forall T \in \mathbb{T}_h, \quad \forall v \in V.$$

Then, the operator $\mathcal{I}^k_{wc,h}$ *has the following properties:*

$$b_h(\mathcal{I}^k_{wc,h}v, q_h) = b_h(v, q_h) \quad \forall v \in V, \quad \forall q_h \in Q^{k-1}_h, \tag{14.7.2}$$

$$|\mathcal{I}^k_{wc,h}v|_{H^1(\mathbb{T}_h)^d} \le |v|_{H^1(\Omega)^d} \quad \forall v \in V, \tag{14.7.3}$$

that is, the operator $\mathcal{I}^k_{wc,h}$ *acts as the Fortin operator.*

Proof. Let $v \in V$ and $q_h \in Q^{k-1}_h$. Using $\nabla q_h \equiv 0$ on T if $k = 1$, $\nabla q_h \in \mathcal{P}^{k-2}(T)^d$ if $k \ge 2$ and the definition of $\mathcal{I}^k_{wc,h}v$ yields

$$\begin{aligned}
b_h(v, q_h) &= \sum_{T\in\mathbb{T}_h} \int_T \operatorname{div} v q_h dx \\
&= \sum_{T\in\mathbb{T}_h} \left\{ \int_{\partial T} (v \cdot n_T) q_h ds - \int_T (v \cdot \nabla) q_h dx \right\} \\
&= \sum_{T\in\mathbb{T}_h} \left\{ \int_{\partial T} (\mathcal{I}^k_{wc,h}v \cdot n_T) q_h ds - \int_T (\mathcal{I}^k_{wc,h}v \cdot \nabla) q_h dx \right\} \\
&= \sum_{T\in\mathbb{T}_h} \int_T \operatorname{div}(\mathcal{I}^k_{wc,h}v) q_h dx = b_h(\mathcal{I}^k_{wc,h}v, q_h).
\end{aligned}$$

The H^1-stability is shown as follows. Remake that $\Delta(I^k_{wc,h}v_i) \equiv 0$ if $k = 1$, $\Delta(I^k_{wc,h}v_i) \in \mathcal{P}^{k-2}(T)$ if $k \ge 2$ and $n \cdot \nabla(I^k_{wc,h}v_i) \in \mathcal{P}^{k-1}(F)$, where F is a face of $T \in \mathbb{T}_h$. We then have, for $i = 1, \dots, d$,

$$\begin{aligned}
|I^k_{wc,h} v_i|^2_{H^1(T)} &= \int_T |\nabla I^k_{wc,h} v_i|^2 dx \\
&= \int_{\partial T} n_T \cdot \nabla (I^k_{wc,h} v_i) I^k_{wc,h} v_i ds - \int_T \Delta (I^k_{wc,h} v_i) I^k_{wc,h} v_i dx \\
&= \int_{\partial T} n_T \cdot \nabla (I^k_{wc,h} v_i) v_i ds - \int_T \Delta (I^k_{wc,h} v_i) v_i dx \\
&= \int_{\partial T} n_T \cdot \nabla (I^k_{wc,h} v_i) v_i ds - \int_{\partial T} n_T \cdot \nabla (I^k_{wc,h} v_i) v_i ds \\
&\quad + \int_T \nabla (I^k_{wc,h} v_i) \cdot \nabla v_i dx \\
&\le |I^k_{wc,h} v_i|_{H^1(T)} |v_i|_{H^1(T)}.
\end{aligned}$$

This conclude that

$$|\mathcal{I}^k_{wc,h} v|^2_{H^1(\mathbb{T}_h)^d} = \sum_{i=1}^d |I^k_{wc,h} v_i|^2_{H^1(T)} \le \sum_{i=1}^d |v_i|^2_{H^1(T)} = |v|^2_{H^1(\mathbb{T}_h)^d}.$$

□

14.7.2.2 Taylor–Hood Element

We define the H^1-conforming finite element space by

$$\begin{aligned}
P^k_{c,h} &:= \left\{ v_h \in P^k_{dc,h} : \ [\![v_h]\!]_F = 0 \ \forall F \in \mathcal{F}^i_h \right\}, \\
P^k_{c,h0} &:= \{ v_h \in P^k_{c,h} : \ v_h|_F = 0 \ \forall F \in \mathcal{F}^\partial_h \}, \\
V^k_{c,h0} &:= (P^k_{c,h0})^d, \\
Q^{k-1}_h &:= P^{k-1}_{c,h} \cap Q.
\end{aligned}$$

It is known that the Stokes element $(V_h, Q_h) := (V^k_{c,h0}, Q^{k-1}_h)$ which is the lowest order Taylor–Hood element satisfies the discrete inf-sup condition on anisotropic triangulation meshes ([16]).

14.8 Numerical Tests

Let $\Omega := (0,1)^2$. We set $\varphi(x_1, x_2) := x_1^2 (x_1 - 1)^2 x_2^2 (x_2 - 1)^2$. The function f of the Stokes equation

$$-\Delta u + \nabla p = f \quad \text{in } \Omega, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

is given so that the exact solution is

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \operatorname{rot} \varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial x_2} \\ -\frac{\partial \varphi}{\partial x_1} \end{pmatrix}, \quad p = x_1^2 - x_2^2.$$

Let M be the division number of each side of the bottom edge and N the division number of the height of Ω with $N \sim M^\gamma$. We set $h := \frac{1}{M}$.

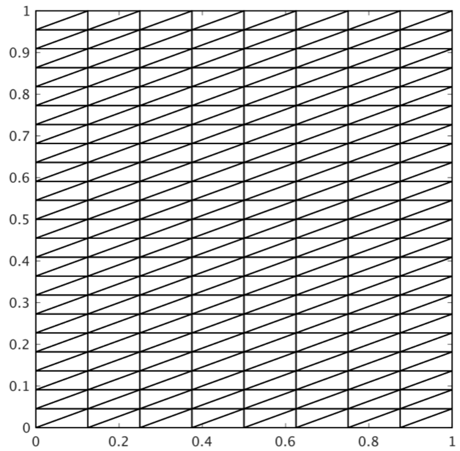

Fig. 14.1: Anisotropic Meshes $\gamma = 1.5$ (M=8, N=22)

If an exact solution u is known, the error $e_h := u - u_h$ and $e_{h/2} := u - u_{h/2}$ are computed numerically for two mesh sizes h and $h/2$. The convergence indicator r is defined by

$$r = \frac{1}{\log(2)} \log \left(\frac{\|e_h\|_X}{\|e_{h/2}\|_X} \right).$$

We compute the convergence order concerning norms defined by

$$Err_h^{(1)}(V_h; Q_h) := \frac{|u - u_h|_{V_h} + \|p - p_h\|_{Q_h}}{|u|_{V_h} + \|p\|_{Q_h}},$$
$$Err_h^{(2)}(L^2) := \frac{\|u - u_h\|_{L^2(\Omega)^2}}{\|u\|_{L^2(\Omega)^2}},$$

for a case: $\gamma = 1.5$.

FreeFem++

For numerical computation, we use *FreeFem++* which is an open source PDE Solver using finite element methods [43], also see [72]. We consider the following approximate problem with a penalty. For a sufficient small $\varepsilon > 0$, find $(u_h^\varepsilon, p_h^\varepsilon) \in V_h \times \widetilde{Q}_h$ such that

$$\begin{aligned} a_h(u_h^\varepsilon, v_h) + b_h(v_h, p_h^\varepsilon) &= (f, v_h) \quad \forall v_h \in V_h, \\ b_h(u_h^\varepsilon, q_h) - \varepsilon(p_h^\varepsilon, q_h) &= 0 \quad \forall q_h \in \widetilde{Q}_h, \end{aligned}$$

where V_h and $\widetilde{Q}_h$ are finite element spaces. Remark that $\widetilde{Q}_h \subset L^2(\Omega)$. However, in FreeFem++, one cannot calculate norms such as

$$|u - u_h|_{V_h}, \quad \|p - p_h\|_{Q_h}, \quad \|u - u_h\|_{L^2(\Omega)^2}.$$

Therefore, some ingenuity is required. Let Π_{fine} be the interpolation into a finite element space on the finest mesh. Using this operator, one calculates the following norms:

$$|\Pi_{fine} u - \Pi_{fine} u_h|_{V_h}, \quad \|\Pi_{fine} p - \Pi_{fine} p_h\|_{Q_h}, \quad \|\Pi_{fine} u - \Pi_{fine} u_h\|_{L^2(\Omega)^2}.$$

Listing 14.1: FreeFem++ Code

```
mesh Th0 = square(256,4096);
mesh Th = square(4,8);

real alpha=1.0;

fespace Vh0(Th0, P1nc);
fespace Qh0(Th0, P0);

fespace Vh1(Th, P1nc);
Vh1 u1, u2;
Vh1 v1, v2;
fespace Qh1(Th, P0);
Qh1 p, q;

Vh0 ue1= alpha*x*x*(x-1)*(x-1)*(2*y*(y-1)*(y-1)+2*y*y*(y
    -1));
Vh0 ue2= -alpha*(2*x*(x-1)*(x-1)+2*x*x*(x-1))*y*y*(y-1)*(y
    -1);
Qh0 pe=x*x-y*y;

func f1 = -alpha*(12*x*x-12*x+2)*(2*y*(y-1)*(y-1)+2*y*y*(y
    -1))-alpha*12*x*x*(x-1)*(x-1)*(2*y-1)+2*x;
func f2= alpha*12*(2*x-1)*y*y*(y-1)*(y-1)+alpha*(2*x*(x-1)
    *(x-1)+2*x*x*(x-1))*(12*y*y-12*y+2)-2*y;
```

```
func gd1 = 0;
func gd2 = 0;

solve stokes([u1,u2,p],[v1,v2,q]) = int2d(Th)( dx(u1)*dx(v1) +dy(u1)*dy(v1)+dx(u2)*dx(v2)+dy(u2)*dy(v2) ) - int2d(Th) (p*(dx(v1)+dy(v2))) - int2d(Th)(q*(dx(u1)+dy(u2))) + int2d(Th)(p*q*0.000001) - int2d(Th) ( f1*v1+f2*v2) + on (1,2,3,4,u1=gd1,u2=gd2);

Vh0 uu1=u1;
Vh0 uu2=u2;
Qh0 pp=p;

real erruH01 = sqrt(int2d(Th0)((dx(uu1)-dx(ue1))^2 + (dy(uu1)-dy(ue1))^2 + (dx(uu2)-dx(ue2))^2 + (dy(uu2)-dy(ue2))^2));
real exuH01 = sqrt(int2d(Th0)((dx(ue1))^2+(dy(ue1))^2+(dx(ue2))^2 + (dy(ue2))^2));

real erruL2 = sqrt(int2d(Th0)((uu1-ue1)^2 + (uu2-ue2)^2));
real exuL2 = sqrt(int2d(Th0)((ue1)^2+(ue2)^2));

real errpL2 = sqrt(int2d(Th0)((pp-pe)^2));
real expL2 = sqrt(int2d(Th0)((pe)^2));

cout << "(erruH01+errpL2)/(exuH01+expL2) = "<<(erruH01+errpL2)/(exuH01+expL2) <<endl;
cout << "erruL2/exuL2 u = "<<erruL2/exuL2 <<endl;
```

Note 14.8.1. The finite element spaces available in FreeFem++ can be found in [42].

14.8.1 Standard Crouzeix–Raviart Finite Element Approximation

The first case gives numerical results for the pair $(V_h, Q_h) := ((V_{h0}^{CR})^d, M_h^0 \cap L_0^2(\Omega))$ in Section 14.2. As mentioned above, we use the penalty method with the space $\widetilde{Q}_h$ instead of Q_h. The theoretical results are as follows.

$$|u - u_h|_{V_h} + \|p - p_h\|_{Q_h} = \mathcal{O}(h),$$
$$\|u - u_h\|_{L^2(\Omega)^d} = \mathcal{O}(h^2).$$

Table 14.1: Error of $(P1nc, P0)$ $(\gamma = 1.5)$

M	N	h	$Err_h^{(1)}$	r	$Err_h^{(2)}$	r
4	8	2.50e-01	5.5680		6.0771e-01	
8	22	1.25e-01	2.6855	1.05	1.7162e-01	1.82
16	64	6.25e-02	1.2336	1.12	4.6020e-02	1.90
32	182	3.13e-02	5.4162e-01	1.19	1.2049e-02	1.93
64	512	1.56e-02	2.2997e-01	1.24	3.0752e-03	1.97
256 (Finest mesh)	4,096	3.91e-03				

14.8.2 Taylor–Hood Element

For $k \in \mathbb{N}$, we consider the Taylor–Hood element introduced in Section 14.7.2.2. We set $(V_h, Q_h) := (V_{c,h0}^k, Q_h^{k-1})$. For any k, the theoretical results are anticipated as follows.

$$|u - u_h|_{V_h} + \|p - p_h\|_{Q_h} = \mathcal{O}(h^k), \quad \|u - u_h\|_{L^2(\Omega)^d} = \mathcal{O}(h^{k+1}).$$

We change the finite element space parts of FreeFem++ Code as

Listing 14.2: Taylor-Hood Element: P2-P1

```
fespace Vh1(Th, P2);
fespace Qh1(Th, P1);
```

Table 14.2: Error of $(P2, P1)$ $(\gamma = 1.5)$

M	N	h	$Err_h^{(1)}$	r	$Err_h^{(2)}$	r
4	8	2.50e-01	2.2935e-02		2.2127e-02	
8	22	1.25e-01	5.3493e-03	2.10	2.3790e-03	3.22
16	64	6.25e-02	1.2819e-03	2.06	2.6868e-04	3.15
32	182	3.13e-02	3.1180e-04	2.03	3.6669e-05	2.87
64	512	1.56e-02	7.7957e-05	2.00	1.8446e-05	0.99
256 (Finest mesh)	4,096	3.91e-03				

We change the finite element space parts of FreeFem++ Code as

Listing 14.3: Taylor-Hood Element: P3-P2

```
load "Element_P3"
fespace Vh1(Th, P3);
fespace Qh1(Th, P2);
```

Table 14.3: Error of $(P3, P2)$ $(\gamma = 1.5)$

M	N	h	$Err_h^{(1)}$	r	$Err_h^{(2)}$	r
4	8	2.50e-01	2.5136e-03		1.8342e-03	
8	22	1.25e-01	2.1973e-04	3.52	7.9844e-05	4.52
16	64	6.25e-02	2.1433e-05	3.36	1.8368e-05	2.12
32	182	3.13e-02	5.0865e-06	2.08	1.8031e-05	0.27
64	512	1.56e-02	4.1195e-06	0.30	1.8030e-05	7.20e-05
256 (Finest mesh)	4,096	3.91e-03				

14.8.3 Mini Element

We consider the MINI element which is $(V_h, Q_h) := (V_h^b, P_{c,h}^1 \cap L_0^2(\Omega))$. On shape regular meshes, the theoretical results are as follows.

$$|u - u_h|_{V_h} + \|p - p_h\|_{Q_h} = \mathcal{O}(h), \quad \|u - u_h\|_{L^2(\Omega)^d} = \mathcal{O}(h^2).$$

However, when using anisotropic meshes, it is anticipated that a convergence rate is not optimal, e.g., see Example 5.4.6. We change the finite element space parts of FreeFem++ Code as

Listing 14.4: Mini Element: P1b-P1

```
fespace Vh1(Th, P1b);
fespace Qh1(Th, P1);
```

Table 14.4: Error of $(P1b, P1)$ $(\gamma = 1.5)$

M	N	h	$Err_h^{(1)}$	r	$Err_h^{(2)}$	r
4	8	2.50e-01	1.5424e-01		3.1793e-01	
8	22	1.25e-01	7.4447e-02	1.05	7.9766e-02	1.99
16	64	6.25e-02	4.2053e-02	0.82	1.8909e-02	2.08
32	182	3.13e-02	2.8596e-02	0.56	4.5285e-03	2.06
64	512	1.56e-02	2.1982e-02	0.38	1.0866e-03	2.06
256 (Finest mesh)	4,096	3.91e-03				

Table 14.5: Error of $(P1b, P1)$ $(\gamma = 2.0)$

M	N	h	$Err_h^{(1)}$	r	$Err_h^{(2)}$	r
4	16	2.50e-01	3.3410e-01		3.2072e-01	
8	64	1.25e-01	2.2491e-01	5.70e-01	7.9418e-02	2.01
16	256	6.25e-02	2.0708e-01	1.19e-01	1.9515e-02	2.02
32	1,024	3.13e-02	2.0382e-01	2.29e-02	4.8150e-03	2.02
64	4,096	1.56e-02	2.0306e-01	5.43e-03	1.16340e-03	2.05
128 (Finest mesh)	16,384	7.81e-03				

From numerical results, we observe thta

$$|u - u_h|_{V_h} + \|p - p_h\|_{Q_h} = \mathcal{O}(h^{2-\gamma}), \quad \|u - u_h\|_{L^2(\Omega)^d} = \mathcal{O}(h^2).$$

14.8.4 Discontinuous Pressure Element: $(\mathcal{P}^2, \mathcal{P}^0_{dc})$

We consider the pair $(V_h, Q_h) := (V^2_{c,h0}, M^0_h \cap L^2_0(\Omega))$. On shape regular meshes, the theoretical results are as follows.

$$|u - u_h|_{V_h} + \|p - p_h\|_{Q_h} = \mathcal{O}(h), \quad \|u - u_h\|_{L^2(\Omega)^d} = \mathcal{O}(h^2).$$

We change the finite element space parts of FreeFem++ Code as

Listing 14.5: Discontinuous Pressure Element: P2-P0

```
fespace Vh1(Th, P2);
fespace Qh1(Th, P0);
```

Table 14.6: Error of $(P2, P0)$ $(\gamma = 1.5)$

M	N	h	$Err_h^{(1)}$	r	$Err_h^{(2)}$	r
4	8	2.50e-01	2.5407e-01		4.2074e-01	
8	22	1.25e-01	1.2983e-01	9.69e-01	1.2097e-01	1.80
16	64	6.25e-02	6.6121e-02	9.73e-01	3.2709e-02	1.89
32	182	3.13e-02	3.3488e-02	9.81e-01	8.5259e-03	1.94
256 (Finest mesh)	4,096	3.91e-03				

Table 14.7: Error of $(P2, P0)$ $(\gamma = 2.0)$

M	N	h	$Err_h^{(1)}$	r	$Err_h^{(2)}$	r
4	16	2.50e-01	2.5854e-01		4.6615e-01	
8	64	1.25e-01	1.3534e-01	9.34e-01	1.3269e-01	1.81
16	256	6.25e-02	6.9370e-02	9.64e-01	3.5091e-02	1.92
128 (Finest mesh)	16,384	7.81e-03				

Table 14.8: Error of $(P2, P0)$ $(\gamma = 2.5)$

M	N	h	$Err_h^{(1)}$	r	$Err_h^{(2)}$	r
4	32	2.50e-01	2.6683e-01		4.9044e-01	
8	181	1.25e-01	1.3869e-01	9.44e-01	1.3683e-01	1.84
64 (Finest mesh)	32,768	1.56e-02				

From numerical results, we observe that

$$|u - u_h|_{V_h} + \|p - p_h\|_{Q_h} = \mathcal{O}(h), \quad \|u - u_h\|_{L^2(\Omega)^d} = \mathcal{O}(h^2),$$

where when $\gamma \geq 3$, numerical verification may be necessary.

Part V

Appendices

Appendix A

Proof of Theorem 4.3.2 for $d = 3$

In this chapter, we show the proof of Theorem 4.3.2 for $d = 3$. For simplicity, we prove that there exists $\gamma_9 > 0$ such that

$$\frac{H_{T^s}}{h_{T^s}} \leq \gamma_9. \tag{4.3.2}$$

if and only if there exists a constant $0 < \gamma_{11} < \pi$ such that

$$\theta_{T^s,\max} \leq \gamma_{11}, \quad \psi_{T^s,\max} \leq \gamma_{11}. \tag{4.3.4}$$

We prove for each standard positions (Type i) and (Type ii) defined in Section 3.2.

In this chapter, we locally use the following notation.

A.1 Notation

Let $\{\mathbb{T}_h\}$ be a family of conformal meshes. Let T^s be the standard element in $\mathbb{R}^3$ with vertices, P_1, P_2, P_3 and P_4. Let F_i be the face of a simplex T^s opposite to the vertex P_i. We denote by $\psi^{i,j}$ the angle between the face F_i and the face F_j, see Figure A.1. Note that $\psi^{i,j} = \psi^{j,i}$. Furthermore, we denote by θ^i_j the internal angle at the vertex P_j on the face F_i and by ϕ^i_j the angle between the face F_i and the segment $\overline{P_j P_i}$.

Table A.1: $\psi^{i,j}$

	F_1	F_2	F_3	F_4
F_1	-	$\psi^{1,2}$	$\psi^{1,3}$	$\psi^{1,4}$
F_2	$\psi^{2,1}$	-	$\psi^{2,3}$	$\psi^{2,4}$
F_3	$\psi^{3,1}$	$\psi^{3,2}$	-	$\psi^{3,4}$
F_4	$\psi^{4,1}$	$\psi^{4,2}$	$\psi^{4,3}$	-

Table A.2: θ_j^i

	F_1	F_2	F_3	F_4
P_1	-	θ_1^2	θ_1^3	θ_1^4
P_2	θ_2^1	-	θ_2^3	θ_2^4
P_3	θ_3^1	θ_3^2	-	θ_3^4
P_4	θ_4^1	θ_4^2	θ_4^3	-

Table A.3: ϕ_j^i

	F_1	F_2	F_3	F_4
P_1	-	ϕ_1^2	ϕ_1^3	ϕ_1^4
P_2	ϕ_2^1	-	ϕ_2^3	ϕ_2^4
P_3	ϕ_3^1	ϕ_3^2	-	ϕ_3^4
P_4	ϕ_4^1	ϕ_4^2	ϕ_4^3	-

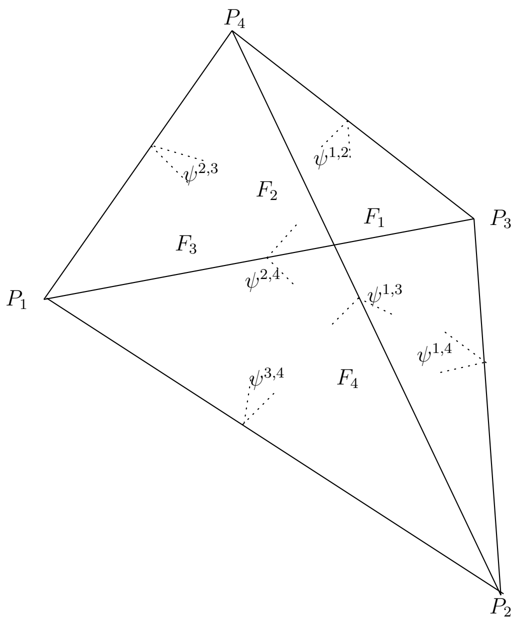

Fig. A.1: Tetrahedra

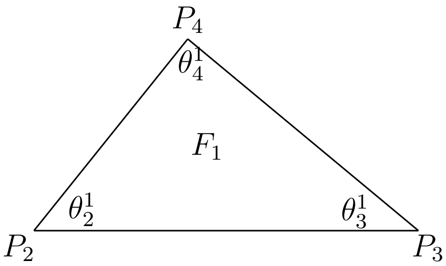

Fig. A.2: Face 1

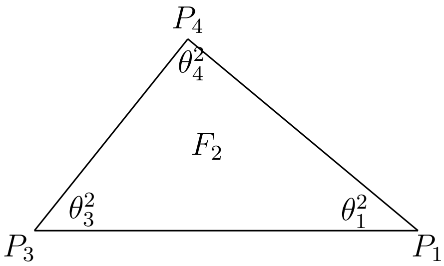

Fig. A.3: Face 2

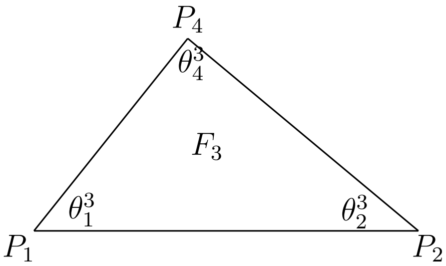

Fig. A.4: Face 3

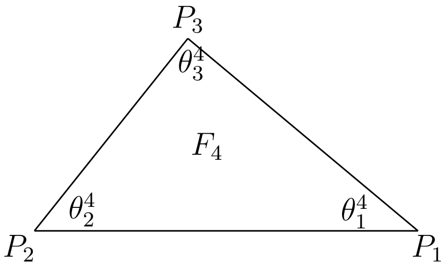

Fig. A.5: Face 4

A.2 Preliminaries: Part 1

Lemma A.2.1. *Let $K \subset \mathbb{R}^2$ be a simplex and let θ_1, θ_2 and θ_3 be internal angles of K with $\theta_1 \le \theta_2 \le \theta_3$. If there exists $0 < \theta_0 < \pi$, $\theta_0 \in \mathbb{R}$, such that $\theta_3 \le \theta_0$, we then have*

$$\sin\theta_2,\ \sin\theta_3 \ge \min\left\{ \sin\frac{\pi - \theta_0}{2}, \sin\theta_0 \right\}.$$

Proof. Because $\theta_1 + \theta_2 + \theta_3 = \pi$ and $\theta_1 \le \theta_2 \le \theta_3$, we have

$$\theta_0 \ge \theta_3 \ge \theta_2 \ge \frac{\theta_1 + \theta_2}{2} \ge \frac{\pi - \theta_3}{2} \ge \frac{\pi - \theta_0}{2},$$

which leads to the target inequality. □

Lemma A.2.2. *Let $K \subset \mathbb{R}^2$ be a simplex with internal angles θ_1, θ_2 and θ_3. For any fixed $\gamma \in \mathbb{R}$ with $0 < \gamma < \pi$, we assume that $\pi - \gamma \le \theta_i$, $i \in \{1, 2, 3\}$. We then have θ_{i+1}, $\theta_{i+2} \le \gamma$, where the indices i, $i+1$ and $i+2$ have to be understood " mod 3".*

Proof. Because $\theta_1 + \theta_2 + \theta_3 = \pi$, we have

$$\theta_{i+1} = \pi - \theta_i - \theta_{i+2} < \pi - \theta_i \le \pi - (\pi - \gamma) = \gamma.$$

□

Lemma A.2.3. *Let $\gamma \in \mathbb{R}$ with $\frac{\pi}{3} \le \gamma < \pi$. It then holds that*

$$0 < \frac{\cos\gamma + 1}{\sin\frac{\gamma}{2} + 1} \le 1.$$

Proof. Because $\cos\gamma = 1 - 2\sin^2\frac{\gamma}{2}$, we have

$$\frac{\cos\gamma + 1}{\sin\frac{\gamma}{2} + 1} = \frac{2 - 2\sin^2\frac{\gamma}{2}}{\sin\frac{\gamma}{2} + 1} = 2\left(1 - \sin\frac{\gamma}{2}\right).$$

Therefore, for $\frac{\pi}{3} \le \gamma < \pi$, the target inequality holds. □

A.3 Preliminaries: Part 2

Let $\{\mathbb{T}_h\}$ be a family of conformal meshes. Let T^s be the standard element in $\mathbb{R}^3$.

Lemma A.3.1 (Cosine rules for the sides and for the angles)**.** *It holds that*

$$\cos\theta_j^{j+3} = \cos\theta_j^{j+1}\cos\theta_j^{j+2} + \sin\theta_j^{j+1}\sin\theta_j^{j+2}\cos\psi^{j+1,j+2}, \tag{A.3.1a}$$
$$\cos\theta_j^{j+1} = \cos\theta_j^{j+2}\cos\theta_j^{j+3} + \sin\theta_j^{j+2}\sin\theta_j^{j+3}\cos\psi^{j+2,j+3}, \tag{A.3.1b}$$
$$\cos\theta_j^{j+2} = \cos\theta_j^{j+3}\cos\theta_j^{j+1} + \sin\theta_j^{j+3}\sin\theta_j^{j+1}\cos\psi^{j+3,j+1}, \tag{A.3.1c}$$
$$\cos\psi^{j+1,j+2} = \sin\psi^{j+2,j+3}\sin\psi^{j+3,j+1}\cos\theta_j^{j+3} - \cos\psi^{j+2,j+3}\cos\psi^{j+3,j+1}, \tag{A.3.1d}$$
$$\cos\psi^{j+2,j+3} = \sin\psi^{j+3,j+1}\sin\psi^{j+1,j+2}\cos\theta_j^{j+1} - \cos\psi^{j+3,j+1}\cos\psi^{j+1,j+2}, \tag{A.3.1e}$$
$$\cos\psi^{j+3,j+1} = \sin\psi^{j+1,j+2}\sin\psi^{j+2,j+3}\cos\theta_j^{j+2} - \cos\psi^{j+1,j+2}\cos\psi^{j+2,j+3}, \tag{A.3.1f}$$

where the indices j, $j+1$, $j+2$ and $j+3$ have to be understood " mod 4".

Proof. The proof is found in in [36, 86]. □

Lemma A.3.2. *Let $\gamma_{\max} \in \mathbb{R}$ with $\frac{\pi}{3} \le \gamma_{\max} < \pi$ satisfy the maximum-angle conditions* (4.3.4) *for the maximum solid $\theta_{T^s,\max}$ and the maximum dihedral $\psi_{T^s,\max}$ of T^s. Assume that for each $j = 1, 2$, θ_j^4 is not the minimum angle of $\triangle P_1P_2P_3$ and $\theta_j^4 < \frac{\pi}{2}$. Then, setting $\delta := \delta(\gamma_{\max})$, $0 < \delta \le \frac{\pi}{2}$ such that*

$$\sin\delta = \left(\frac{\cos\gamma_{\max} + 1}{\sin\frac{\gamma_{\max}}{2} + 1}\right)^{1/2},$$

it holds that

$$\psi^{j+1,4} \ge \delta, \quad \textit{or} \quad \psi^{3,4} \ge \delta, \tag{A.3.2}$$

where the indices j and $j+1$ have to be understood " mod 2".

Proof. From Lemma A.2.3, we have

$$0 < \frac{\cos\gamma_{\max} + 1}{\sin\frac{\gamma_{\max}}{2} + 1} \le 1,$$

because $\frac{\pi}{3} \le \gamma_{\max} < \pi$. Therefore, δ is well-defined.

We use proof by contradiction. Suppose that

$$0 < \psi^{j+1,4} < \delta, \quad 0 < \psi^{3,4} < \delta,$$

that is,

$$0 < \sin\psi^{j+1,4}\sin\psi^{3,4} < \sin^2\delta, \text{ and } 1 > \cos\psi^{j+1,4}\cos\psi^{3,4} > \cos^2\delta \ge 0.$$

From Lemma A.2.1 and assumption, we have

$$\frac{\pi - \gamma_{\max}}{2} \le \theta_j^4 < \frac{\pi}{2},$$

which implies

$$0 < \cos\theta_j^4 \le \cos\left(\frac{\pi - \gamma_{\max}}{2}\right) = \sin\frac{\gamma_{\max}}{2}.$$

We thus obtain

$$\sin\psi^{j+1,4}\sin\psi^{3,4}\cos\theta_j^4 < \sin^2\delta\sin\frac{\gamma_{\max}}{2}.$$

Using the cosine rule (A.3.1d) with $j = 1$ and the above inequalities yield

$$\begin{aligned}\cos\psi_{2,3} &= \sin\psi^{3,4}\sin\psi^{4,2}\cos\theta_1^4 - \cos\psi^{3,4}\cos\psi^{4,2}\\ &< \sin^2\delta\sin\frac{\gamma_{\max}}{2} - (1 - \sin^2\delta)\\ &= \frac{\cos\gamma_{\max} + 1}{\sin\frac{\gamma_{\max}}{2} + 1}\left(\sin\frac{\gamma_{\max}}{2} + 1\right) - 1 = \cos\gamma_{\max}.\end{aligned}$$

This is contradiction for the maximum-angle condition $0 < \psi^{2,3} \le \gamma_{\max} < \pi$, that is, $\cos\psi^{2,3} \ge \cos\gamma_{\max}$.

Analogously, using the cosine rule (A.3.1f) with $j = 2$ and the above inequalities yield

$$\begin{aligned}\cos\psi^{1,3} &= \sin\psi^{3,4}\sin\psi^{4,1}\cos\theta_2^4 - \cos\psi^{3,4}\cos\psi^{4,1}\\ &< \cos\gamma_{\max}.\end{aligned}$$

This is contradiction for the maximum-angle condition $0 < \psi^{1,3} \le \gamma_{\max} < \pi$, that is, $\cos\psi^{1,3} \ge \cos\gamma_{\max}$. □

Corollary A.3.3. *For each $j = 1, 2$, under assumptions in Lemma A.3.2, it holds that setting $C_0 := \min\{\delta, \gamma_{\max}\}$,*

$$\sin\psi^{j+1,4} \ge C_0, \quad \text{or} \quad \sin\psi^{3,4} \ge C_0$$

where the indices j and $j+1$ have to be understood " mod 2".

Lemma A.3.4. *For any $i, j \in \{1, 2, 3, 4\}$, $i \ne j$ and $k \in \{1, 2, 3, 4\}$, $k \ne i, j$, it holds that*

$$\sin\phi_j^i = \sin\theta_j^k \sin\psi^{k,i}.$$

Proof. We only show the case $i = 4$, $j = 1$ and $k = 2$. We then have

$$\sin\phi_1^4 = |\overline{P_1P_4}| \sin\theta_1^2 \times \frac{1}{|\overline{P_1P_4}|} \sin\psi^{2,4} = \sin\theta_1^2 \sin\psi^{2,4}.$$

□

Lemma A.3.5. *Assume that there exists a positive constant M_j ($j = 1, 2$) with $0 < M_j < 1$ such that*

$$\sin\theta_j^4 \sin\phi_1^4 > M_j, \quad j = 1, 2.$$

Setting $\gamma(M_j) := \pi - \sin^{-1} M_j$ ($j = 1, 2$), we have $\frac{\pi}{2} < \gamma(M_j) < \pi$ and it holds that for each $j = 1, 2$,

$$\begin{aligned} &\theta_1^4,\ \theta_2^4,\ \theta_3^4 < \gamma(M_j), \\ &\theta_2^3,\ \theta_4^3,\ \theta_3^2,\ \theta_4^2,\ \theta_1^2,\ \theta_1^3,\ \psi^{2,4},\ \psi^{3,4} < \gamma(M_j). \end{aligned}$$

Proof. From assumption, we have, for each $j = 1, 2$,

$$\begin{aligned} &\sin\theta_j^4 \geq \sin\theta_j^4 \sin\phi_1^4 > M_j, \\ &\sin\phi_1^4 > M_j. \end{aligned}$$

The definition of $\gamma(M_j)$ and Lemma A.2.2 yield, for each $j = 1, 2$,

$$\begin{aligned} &\pi - \gamma < \theta_j^4 < \gamma(M_j), \quad \theta_{j+1}^4 < \gamma(M_j), \quad \theta_{j+2}^4 < \gamma(M_j), \\ &\pi - \gamma < \phi_1^4 < \gamma(M_j), \end{aligned}$$

where the indices j, $j + 1$ and $j + 2$ have to be understood "mod 3".

We obtain, from Lemma A.3.4,

$$\sin\phi_1^4 = \sin\theta_1^2 \sin\psi^{2,4} = \sin\theta_1^3 \sin\psi^{3,4} > M_j, \quad j = 1, 2.$$

We then have, for each $j = 1, 2$,

$$\sin\theta_1^2,\ \sin\psi^{2,4},\ \sin\theta_1^3,\ \sin\psi^{3,4} > M_j,$$

that is,

$$\pi - \gamma(M_j) < \theta_1^2,\ \theta_1^3,\ \psi^{2,4},\ \psi^{3,4} < \gamma(M_j).$$

On $\triangle P_1P_2P_4$ and $\triangle P_1P_3P_4$, using Lemma A.2.2 yields

$$\theta_2^3,\ \theta_4^3,\ \theta_3^2,\ \theta_4^2 < \gamma(M_j), \quad j = 1, 2.$$

□

By analogous argument with Lemma A.3.5, we get the following two lemmata.

Lemma A.3.6. *Assume that there exists* M_3 *with* $0 < M_3 < 1$ *such that*

$$\sin\theta_3^1 \sin\phi_3^1 > M_3.$$

Setting $\gamma(M_3) := \pi - \sin^{-1} M_3$*, we have* $\frac{\pi}{2} < \gamma(M_3) < \pi$ *and it holds that*

$$\theta_3^2,\ \theta_3^4,\ \theta_2^1,\ \theta_4^1,\ \theta_3^1,\ \psi^{2,1},\ \psi^{4,1} < \gamma(M_3).$$

Proof. From assumption, we have

$$\sin\theta_3^1 \geq \sin\theta_3^1 \sin\phi_3^1 > M_3, \quad \sin\phi_3^1 > M_3.$$

Using the definition of $\gamma(M_3)$ yields

$$\pi - \gamma < \theta_1^3 < \gamma(M_3), \quad \pi - \gamma < \phi_1^3 < \gamma(M_3).$$

We obtain, from Lemma A.3.4,

$$\sin\phi_3^1 = \sin\theta_3^2 \sin\psi^{2,1} = \sin\theta_3^4 \sin\psi^{4,1} > M_3.$$

We then have

$$\sin\theta_3^2,\ \sin\psi^{2,1},\ \sin\theta_3^4,\ \sin\psi^{4,1} > M_3,$$

that is,

$$\pi - \gamma(M_3) < \theta_3^2,\ \theta_3^4,\ \psi^{2,1},\ \psi^{4,1} < \gamma(M_3).$$

Meanwhile, on $\triangle P_2P_3P_4$, using Lemma A.2.2, we have

$$\theta_2^1,\ \theta_4^1 < \gamma(M_3).$$

□

Lemma A.3.7. *Assume that there exists* M_4 *with* $0 < M_4 < 1$ *such that*

$$\sin\theta_2^1 \sin\phi_4^1 > M_4.$$

Setting $\gamma(M_4) := \pi - \sin^{-1} M_4$*, we have* $\frac{\pi}{2} < \gamma(M_4) < \pi$ *and it holds that*

$$\theta_4^2,\ \theta_4^3,\ \theta_2^1,\ \theta_3^1,\ \theta_4^1,\ \psi^{1,2},\ \psi^{1,3} < \gamma(M_4).$$

Proof. The proof is obtained by using an analogous argument with Lemma A.3.6. □

A.4 Proof of Theorem 4.3.2 in (Type i)

A.4.1 (4.3.4) ⇒ (4.3.2)

We set $t_1 := \sin \theta_1^4$ and $t_2 := \sin \phi_1^4$. We then have

$$\frac{H_{T^s}}{h_{T^s}} = \frac{h_1 h_2 h_3}{|T^s|} = \frac{6}{\sin \theta_1^4 \sin \phi_1^4}.$$

We here used the fact that $|T^s| = \frac{1}{6} h_1 h_2 h_3 \sin \theta_1^4 \sin \phi_1^4$. By construct of the standard element (Type i), the angle θ_3^4 and θ_2^4 are respectively the maximum angle and the minimum angle of the base $\triangle P_1 P_2 P_3$ of T^s. We hence have $\theta_1^4 < \frac{\pi}{2}$. From Lemma A.2.1, we have

$$\frac{\pi - \gamma_{11}}{2} \le \theta_1^4 \le \gamma_{11}, \quad \sin \theta_1^4 \ge \min \left\{ \sin \frac{\pi - \gamma_{11}}{2}, \sin \gamma_{11} \right\} =: C_1.$$

Due to Lemma A.3.2, setting $\delta := \delta(\gamma_{11})$, $0 < \delta \le \frac{\pi}{2}$ such that

$$\sin \delta = \left(\frac{\cos \gamma_{11} + 1}{\sin \frac{\gamma_{11}}{2} + 1} \right)^{1/2},$$

it holds that

$$\psi^{2,4} \ge \delta, \quad \text{or} \quad \psi^{3,4} \ge \delta.$$

Suppose that $\psi^{2,4} \ge \delta$. By Corollary A.3.3 and Lemma A.3.4, we have

$$\sin \phi_1^4 = \sin \theta_1^2 \sin \psi^{2,4} \ge C_0 \sin \theta_1^2.$$

By construct of the standard element (Type i), the angle θ_1^2 is not the minimum angle of $\triangle P_1 P_3 P_4$. From Lemma A.2.1, we have

$$\frac{\pi - \gamma_{11}}{2} \le \theta_1^2 \le \gamma_{11}, \quad \sin \theta_1^2 \ge C_1.$$

We thus obtain

$$\sin \phi_1^4 \ge C_0 C_1.$$

Suppose that $\psi^{3,4} \ge \delta$. By Corollary A.3.3 and Lemma A.3.4, we have

$$\sin \phi_1^4 = \sin \theta_1^3 \sin \psi^{3,4} \ge C_0 \sin \theta_1^3.$$

By construct of the standard element (Type i), the angle θ_1^3 is not the minimum angle of $\triangle P_1P_2P_4$. From Lemma A.2.1, we have

$$\frac{\pi - \gamma_{11}}{2} \le \theta_1^3 \le \gamma_{11}, \quad \sin\theta_1^3 \ge C_1.$$

We thus obtain

$$\sin\phi_1^4 \ge C_0C_1.$$

In both cases

$$\psi^{2,4} \ge \delta, \quad \text{or} \quad \psi^{3,4} \ge \delta,$$

gathering the above results yield

$$\frac{H_{T^s}}{h_{T^s}} = \frac{6}{\sin\theta_1^4 \sin\phi_1^4} \ge \frac{6}{C_0C_1^2} =: D_1 > 0,$$

that is, (4.3.2) holds. □

A.4.2 (4.3.2) ⇒ (4.3.4)

From assumption, it holds that

$$\frac{H_{T^s}}{h_{T^s}} = \frac{h_1h_2h_3}{|T^s|} = \frac{6}{\sin\theta_1^4 \sin\phi_1^4} \le \gamma_9.$$

Remark that $\frac{6}{\gamma_9} < 1$ because $\theta_1^4 < \frac{\pi}{2}$ and $\sin\theta_1^4 \sin\phi_1^4 < 1$. Therefore, we have

$$\sin\theta_1^4 \sin\phi_1^4 \ge \frac{6}{\gamma_9} =: C_2.$$

From Lemma A.3.5 with $j = 1$, setting $\gamma(C_2) := \pi - \sin^{-1} C_2$, we have $\frac{\pi}{2} < \gamma(C_2) < \pi$ and it holds that

$$\begin{aligned}
&\theta_1^4,\ \theta_2^4,\ \theta_3^4 < \gamma(C_2), \\
&\theta_2^3,\ \theta_4^3,\ \theta_1^3,\ \theta_3^2,\ \theta_4^2,\ \theta_1^2,\ \psi^{2,4},\ \psi^{3,4} < \gamma(C_2).
\end{aligned}$$

Furthermore, we write

$$\begin{aligned}
|T^s| &= \frac{1}{3} \times \frac{1}{2}|\overline{P_2P_3}||\overline{P_3P_4}| \sin\theta_3^1 \times h_2 \sin\phi_3^1 = \frac{1}{6}h_2|\overline{P_2P_3}||\overline{P_3P_4}| \sin\theta_3^1 \sin\phi_3^1 \\
&< \frac{1}{3}h_1h_2h_3 \sin\theta_3^1 \sin\phi_3^1,
\end{aligned}$$

where we used the fact that $|\overline{P_3P_4}| < |\overline{P_1P_4}| + |\overline{P_1P_3}| \leq 2h_3$ on $\triangle P_1P_3P_4$ and $|\overline{P_2P_3}| \leq h_1$. We thus have

$$\gamma_9 \geq \frac{H_{T^s}}{h_{T^s}} > \frac{3}{\sin\theta_3^1 \sin\phi_3^1},$$

that is,

$$\sin\theta_3^1 \sin\phi_3^1 > \frac{3}{\gamma_9} =: C_3.$$

From Lemma A.3.6, setting $\gamma(C_3) := \pi - \sin^{-1} C_3$, we have $\frac{\pi}{2} < \gamma(C_3) < \pi$ and it holds that

$$\theta_2^1,\ \theta_4^1,\ \theta_3^1,\ \psi^{2,1},\ \psi^{4,1} < \gamma(C_3).$$

Due to the cosine rule (A.3.1f) with $j = 2$, we get

$$\cos\psi^{1,3} = \sin\psi^{3,4} \sin\psi^{4,1} \cos\theta_2^4 - \cos\psi^{3,4} \cos\psi^{4,1}.$$

By construct of the standard element (Type i), the angle θ_2^4 is the minimum angle of $\triangle P_1P_2P_3$. Therefore, we have

$$\begin{aligned} &\cos\theta_2^4 \geq \frac{1}{2} \quad \text{because } \theta_2^4 \leq \frac{\pi}{3}, \\ &\sin\psi^{3,4} \sin\psi^{4,1} \cos\theta_2^4 > 0, \quad \text{because } \sin\psi^{3,4} \sin\psi^{4,1} > 0, \end{aligned}$$

and thus

$$\cos\psi^{1,3} > -\cos\psi^{3,4} \cos\psi^{4,1}.$$

Using $\sin\psi^{3,4} > C_2$ and $\sin\psi^{4,1} > C_3$ yields

$$\begin{aligned} \cos\psi^{1,3} &> -\cos\psi^{3,4} \cos\psi^{4,1} \\ &\geq -|\cos\psi^{3,4}||\cos\psi^{4,1}| = -\sqrt{1-\sin^2\psi^{3,4}}\sqrt{1-\sin^2\psi^{4,1}} \\ &> -\sqrt{1-C_2^2}\sqrt{1-C_3^2} =: C_4 > -1. \end{aligned}$$

Setting $\gamma(C_4) := \cos^{-1} C_4$, it holds that

$$\psi^{1,3} < \gamma(C_4) < \pi.$$

Due to the cosine rule (A.3.1d) with $j = 1$, we get

$$\cos\psi^{2,3} = \sin\psi^{3,4} \sin\psi^{4,2} \cos\theta_1^4 - \cos\psi^{3,4} \cos\psi^{4,2}.$$

By construct of the standard element (Type i), the angle θ_3^4 and θ_2^4 are respectively the maximum angle and the minimum angle of the base $\triangle P_1P_2P_3$ of T^s. We hence have $\theta_1^4 < \frac{\pi}{2}$. Therefore, we have

$$\begin{aligned}
&\cos\theta_1^4 > 0 \quad \text{because } \theta_1^4 \leq \frac{\pi}{2}, \\
&\sin\psi^{3,4}\sin\psi^{4,2}\cos\theta_1^4 > 0, \quad \text{because } \sin\psi^{3,4}\sin\psi^{4,2} > 0,
\end{aligned}$$

and thus

$$\cos\psi^{2,3} > -\cos\psi^{3,4}\cos\psi^{4,2}.$$

Using $\sin\psi^{3,4} > C_2$ and $\sin\psi^{4,2} > C_2$ yield

$$\begin{aligned}
\cos\psi^{2,3} &> -\cos\psi^{3,4}\cos\psi^{4,2} \\
&\geq -|\cos\psi^{3,4}||\cos\psi^{4,2}| = -\sqrt{1-\sin^2\psi^{3,4}}\sqrt{1-\sin^2\psi^{4,2}} \\
&> -(1-C_2^2) =: C_5 > -1.
\end{aligned}$$

Setting $\gamma(C_5) := \cos^{-1} C_5$, it holds that

$$\psi^{2,3} < \gamma(C_5) < \pi.$$

We set $\gamma_{\max} := \max\{\gamma(C_3), \gamma(C_4), \gamma(C_5)\}$. We then have $0 < \gamma_{\max} < \pi$, that is, (4.3.4) holds. □

A.5 Proof of Theorem 4.3.2 in (Type ii)

A.5.1 (4.3.4) ⇒ (4.3.2)

We set $t_1 := \sin\theta_2^4$ and $t_2 := \sin\phi_1^4$. We then have

$$\frac{H_{T^s}}{h_{T^s}} = \frac{h_1h_2h_3}{|T^s|} = \frac{6}{\sin\theta_2^4\sin\phi_1^4}.$$

We here used the fact that $|T^s| = \frac{1}{6}h_1h_2h_3\sin\theta_2^4\sin\phi_1^4$. By construct of the standard element (Type ii), the angle θ_3^4 and θ_1^4 are respectively the maximum angle and the minimum angle of the base $\triangle P_1P_2P_3$ of T^s. We hence have $\theta_2^4 < \frac{\pi}{2}$. From Lemma A.2.1, we have

$$\frac{\pi-\gamma_{11}}{2} \leq \theta_2^4 \leq \gamma_{11}, \quad \sin\theta_2^4 \geq C_1.$$

Due to Lemma A.3.2, it holds that

$$\psi^{1,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta.$$

Suppose that $\psi^{1,4} \geq \delta$. By Corollary A.3.3 and Lemma A.3.4, we have

$$\sin \phi_2^4 = \sin \theta_2^1 \sin \psi^{1,4} \geq C_0 \sin \theta_2^1.$$

Furthermore, it holds that

$$\sin \phi_1^4 = \frac{|\overline{P_2P_4}| \sin \phi_2^4}{h_3}.$$

By construct of the standard element (Type ii), the angle θ_2^1 is not the minimum angle of $\triangle P_2P_3P_4$. From Lemma A.2.1, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_2^1 \leq \gamma_{11}, \quad \sin \theta_2^1 \geq C_1.$$

Because $h_3 = |\overline{P_1P_4}| < |\overline{P_2P_4}|$ on $\triangle P_1P_2P_4$, we thus obtain

$$\sin \phi_1^4 = \frac{|\overline{P_2P_4}|}{h_3} \sin \phi_2^4 > C_0C_1.$$

Suppose that $\psi^{3,4} \geq \delta$. By Corollary A.3.3 and Lemma A.3.4, we have

$$\sin \phi_1^4 = \sin \theta_1^3 \sin \psi^{3,4} \geq C_0 \sin \theta_1^3.$$

By construct of the standard element (Type ii), the angle θ_1^3 is not the minimum angle of $\triangle P_1P_2P_4$. From Lemma A.2.1, we have

$$\frac{\pi - \gamma_{11}}{2} \leq \theta_1^3 \leq \gamma_{11}, \quad \sin \theta_1^3 \geq C_1.$$

We thus obtain

$$\sin \phi_1^4 > C_0C_1.$$

In both cases

$$\psi^{1,4} \geq \delta, \quad \text{or} \quad \psi^{3,4} \geq \delta,$$

gathering the above results yields

$$\frac{H_{T^s}}{h_{T^s}} = \frac{6}{\sin \theta_2^4 \sin \phi_1^4} \geq \frac{6}{C_0C_1^2} = D_1 > 0,$$

that is, (4.3.2) holds. □

A.5.2 (4.3.2) $\Rightarrow$ (4.3.4)

From assumption, it holds that

$$\frac{H_{T^s}}{h_{T^s}} = \frac{h_1 h_2 h_3}{|T^s|} = \frac{6}{\sin\theta_2^4 \sin\phi_1^4} \leq \gamma_9.$$

Remark that $\frac{6}{\gamma_9} < 1$ because $\theta_2^4 < \frac{\pi}{2}$ and $\sin\theta_2^4 \sin\phi_1^4 < 1$. Therefore, we have

$$\sin\theta_2^4 \sin\phi_1^4 \geq \frac{6}{\gamma_9} = C_2.$$

From Lemma A.3.5 with $j = 2$, it holds that

$$\begin{aligned}
&\theta_1^4,\ \theta_2^4,\ \theta_3^4 < \gamma(C_2),\\
&\theta_2^3,\ \theta_4^3,\ \theta_1^3,\ \theta_3^2,\ \theta_4^2,\ \theta_1^2,\ \psi^{2,4},\ \psi^{3,4} < \gamma(C_2).
\end{aligned}$$

Furthermore, we write

$$\begin{aligned}
|T^s| &= \frac{1}{3} \times \frac{1}{2}|\overline{P_2P_4}||\overline{P_2P_3}| \sin\theta_2^1 \times h_3 \sin\phi_4^1\\
&< \frac{1}{6} h_1 h_2 h_3 \sin\theta_2^1 \sin\phi_4^1,
\end{aligned}$$

where we used the fact that $|\overline{P_3P_2}| = h_2$ and $|\overline{P_2P_4}| \leq h_1$. We thus have

$$\gamma_9 \geq \frac{H_{T^s}}{h_{T^s}} > \frac{6}{\sin\theta_2^1 \sin\phi_4^1},$$

that is,

$$\sin\theta_2^1 \sin\phi_4^1 > \frac{6}{\gamma_9} = C_2.$$

From Lemma A.3.7, it holds that

$$\theta_2^1,\ \theta_4^1,\ \theta_3^1,\ \psi^{1,2},\ \psi^{1,3} < \gamma(C_2).$$

Due to the cosine rule (A.3.1e) with $j = 2$, we get

$$\cos\psi^{4,1} = \sin\psi^{1,3} \sin\psi^{3,4} \cos\theta_2^3 - \cos\psi^{1,3} \cos\psi^{3,4}.$$

By construct of the standard element (Type ii), the angle θ_2^3 is the minimum angle of $\triangle P_1P_2P_4$. Therefore, we have

$$\begin{aligned}
&\cos\theta_2^3 \geq \frac{1}{2} \quad \text{because } \theta_2^3 \leq \frac{\pi}{3},\\
&\sin\psi^{1,3} \sin\psi^{3,4} \cos\theta_2^3 > 0, \quad \text{because } \sin\psi^{1,3} \sin\psi^{3,4} > 0,
\end{aligned}$$

and thus

$$\cos\psi^{4,1} > -\cos\psi^{1,3}\cos\psi^{3,4}.$$

Using $\sin\psi^{1,3} > C_2$ and $\sin\psi^{3,4} > C_2$ yield

$$\begin{aligned}\cos\psi^{4,1} &> -\cos\psi^{1,3}\cos\psi^{3,4}\\ &\geq -\sqrt{1-\sin^2\psi^{1,3}}\sqrt{1-\sin^2\psi^{3,4}}\\ &> -(1-C_2^2) = C_5 > -1.\end{aligned}$$

It then holds that

$$\psi^{4,1} < \gamma(C_5) < \pi.$$

Due to the cosine rule (A.3.1d) with $j = 1$, we get

$$\cos\psi^{2,3} = \sin\psi^{3,4}\sin\psi^{4,2}\cos\theta_1^4 - \cos\psi^{3,4}\cos\psi^{4,2}.$$

By construct of the standard element (Type ii), the angle θ_1^4 is the minimum angle of $\triangle P_1P_2P_3$. We hence have $\theta_1^4 < \frac{\pi}{3}$. Therefore, we have

$$\begin{aligned}&\cos\theta_1^4 \geq \frac{1}{2} \quad \text{because } \theta_1^4 \leq \frac{\pi}{3},\\ &\sin\psi^{3,4}\sin\psi^{4,2}\cos\theta_1^4 > 0, \quad \text{because } \sin\psi^{3,4}\sin\psi^{4,2} > 0,\end{aligned}$$

and thus

$$\cos\psi^{2,3} > -\cos\psi^{3,4}\cos\psi^{4,2}.$$

Using $\sin\psi^{3,4} > C_2$ and $\sin\psi^{4,2} > C_2$ yield

$$\begin{aligned}\cos\psi^{2,3} &> -\cos\psi^{3,4}\cos\psi^{4,2}\\ &\geq -\sqrt{1-\sin^2\psi^{3,4}}\sqrt{1-\sin^2\psi^{4,2}}\\ &> -(1-C_2^2) = C_5 > -1.\end{aligned}$$

It then holds that

$$\psi^{2,3} < \gamma(C_5) < \pi.$$

We set $\gamma_{\max} := \max\{\gamma(C_2), \gamma(C_5)\}$. We then have $0 < \gamma_{\max} < \pi$, that is, (4.3.4) holds. □

Appendix B

$H(\mathrm{div}; D)$ Finite Elements

In this chapter, we refer to [17, 18, 30, 31, 37, 53, 79].

B.1 Normal Trace

In this section, we follow [31, p. 43].

Let D be a Lipschitz domain of $\mathbb{R}^d$. Let $p \in (1, \infty)$. We consider the following Banach space:

$$Z^{d,p}(D) := \left\{ v \in L^p(D)^d;\ \mathrm{div}\, v \in L^p(D) \right\}. \tag{B.1.1}$$

When $p = 2$, we denote the function space by

$$H(\mathrm{div}; D) := Z^{d,2}(D),$$

which is a Hilbert space with the inner product and norm:

$$\begin{aligned} (u, v)_{H(\mathrm{div};D)} &:= (u, v) + (\mathrm{div}\, u, \mathrm{div}\, v), \\ \|v\|_{H(\mathrm{div};D)} &:= (v, v)^{1/2}_{H(\mathrm{div};D)} = \left(\|v\|^2_{L^2(D)^d} + \|\,\mathrm{div}\, v\|^2_{L^2(D)} \right)^{1/2}. \end{aligned}$$

The trace operator $\gamma^g : W^{1,p'}(D) \to W^{\frac{1}{p},p'}(\partial D)$ begin surjective, we infer that there exists C^{γ^c} such that, for all $\ell \in W^{\frac{1}{p},p'}(\partial D)^d$, there exists $w(\ell) \in W^{1,p'}(D)^d$ such that $\gamma^g(w(\ell)) = \ell$ and $\|w(\ell)\|_{W^{1,p'}(D)} \le C^{\gamma^c} \|\ell\|_{W^{\frac{1}{p},p'}(\partial D)}$.

We define the linear operator $\gamma^d : Z^{d,p}(D) \to W^{-\frac{1}{p},p}(\partial D)$ as

$$\langle \gamma^d(v), \ell \rangle_{\partial D} := \int_D (v \cdot \nabla) q(\ell) dx + \int_D \mathrm{div}\, v q(\ell) dx, \tag{B.1.2}$$

for all $v \in Z^{d,p}(D)$ and all $\ell \in W^{\frac{1}{p},p}(\partial D)$, where $q(\ell) \in W^{1,p'}(D)$ is such that $\gamma^g(q(\ell)) = \ell$, and $\langle \cdot, \cdot \rangle_{\partial D}$ denotes the duality pair between $W^{-\frac{1}{p},p}(\partial D)$ and $W^{\frac{1}{p},p'}(\partial D)$. One can verify that $\gamma^d(v) = v|_{\partial D} \cdot n$ when v is smooth, γ^d is bounded, and the definition of γ^d is independent of the choice of $q(\ell)$.

Theorem B.1.1. *The following holds true:*

(I) $\gamma^d(v) = v|_{\partial D} \cdot n$ *whenever* v *is smooth.*

(II) γ^d *is surjective.*

(III) Density: setting $Z_0^{d,p}(D) := \overline{\mathcal{C}_0^\infty(D)}^{Z^{d,p}(D)}$, *we have* $Z_0^{d,p}(D) = \ker(\gamma^d)$.

Proof. The proof is found in [31, Theorem 4.15]. □

B.2 The Function Space $H(\mathrm{div}; D)$

Let D be a Lipschitz domain of $\mathbb{R}^d$.

Theorem B.2.1. *The space* $\mathcal{C}^\infty(\overline{D})^d$ *is dense in* $H(\mathrm{div}; D)$.

Proof. The proof is found in [37, Theorem 2.4]. The condition of "boundedness" is entered the assumptions because we use the space $\mathcal{C}^\infty(\overline{D})^d$. □

Theorem B.2.2. *The trace operator* $\gamma^d : \mathcal{C}^\infty(\overline{D})^d \to \mathcal{C}^\infty(\overline{\partial D})$ *which maps* $\varphi \mapsto \varphi \cdot n|_{\partial D}$ *can be extended to a continuous, linear mapping*

$$\gamma^d : H(\mathrm{div}; D) \to H^{-\frac{1}{2}}(\partial D),$$

where $H^{-\frac{1}{2}}(\partial D)$ *is the dual space* $H^{\frac{1}{2}}(\partial D)$.

Proof. The proof is found in [37, Theorem 2.5]. □

Theorem B.2.3. *The trace theorem is optimal in the sense that*

$$\gamma^d : H(\mathrm{div}; D) \to H^{-\frac{1}{2}}(\partial D),$$

is surjective.

Proof. Let $\mu \in H^{-\frac{1}{2}}(\partial D)$. To show is that there exists $v \in H(\mathrm{div}; D)$ such that

$$\begin{aligned} v \cdot n &= \mu \quad \text{on } \partial D, \\ \|v\|_{H(\mathrm{div};D)} &\leq \|v \cdot n\|_{H^{-\frac{1}{2}}(\partial D)}. \end{aligned}$$

We know that the problem

$$-\Delta\varphi + \varphi = 0 \quad \text{in } D, \quad \frac{\partial \varphi}{\partial n} = \mu \quad \text{on } \partial D$$

has a unique solution $\varphi \in H^1(D)$ satisfying

$$\|\varphi\|^2_{H^1(D)} = \langle \mu, \varphi \rangle_{\partial D} \le \|\mu\|_{H^{-\frac{1}{2}}(\partial D)} \|\varphi\|_{H^1(D)}, \tag{B.2.1}$$

see [37, Section 1.4 and (1.16)]. Setting $v = \nabla\varphi$, we have $v \in H(\text{div}; D)$, $v \cdot n = \mu$, and

$$\begin{aligned} \|v\|_{H(\text{div};D)} &= \left(\|v\|^2_{L^2(D)^d} + \|\operatorname{div} v\|^2_{L^2(D)} \right)^{1/2} = \|\varphi\|_{H^1(D)} \\ &\le \|\mu\|_{H^{-\frac{1}{2}}(\partial D)} = \|\mu\|_{H^{-\frac{1}{2}}(\partial D)} = \|v \cdot n\|_{H^{-\frac{1}{2}}(\partial D)}. \end{aligned}$$

□

Theorem B.2.4. *It holds that*

$$H_0(\text{div}; D) := \ker(\gamma^d) = \{v \in H(\text{div}; D) : \ v \cdot n|_{\partial D} = 0\}.$$

Proof. The proof is found in [37, Theorem 2.6]. □

Theorem B.2.5. *Let*

$$H_\sigma := \{v \in H_0(\text{div}; D) : \operatorname{div} v = 0\}.$$

It then holds that

$$L^2(D)^d = H_\sigma \oplus H^\perp,$$

where $H^\perp$ denotes the orthogonal of H_σ in $L^2(D)^d$ for the scalar product, that is,

$$H^\perp := \{v = \nabla q : \ q \in H^1(D)\}.$$

Proof. The proof is found in [37, Theorem 2.7]. Remark that D is open, bounded, connected, and a Lipschitz set, because D is a Lipschitz domain of $\mathbb{R}^d$. □

B.3 Conforming Subspaces of $H(\mathrm{div}; D)$

Let $\mathbb{T}_h$ be a subdivision of the domain D. Let $k \in \mathbb{N}_0$. Recall that the Raviart–Thomas polynomial space of order k as $RT^k := (\mathcal{P}^k)^d + x\mathcal{P}^k$ for any $x \in \mathbb{R}^d$. We set

$$\begin{aligned} W_h &:= \{v_h \in L^1(D)^d : \ v_h|_T \in RT^k(T) \ \forall T \in \mathbb{T}_h\}, \\ V_h &:= \{v_h \in W_h : \ [\![v_h \cdot n]\!]_F = 0 \ \forall F \in \mathcal{F}_h^i\}. \end{aligned}$$

Lemma B.3.1. *It holds that*

$$V_h \subset H(\mathrm{div}; D).$$

Proof. Let $v_h \in V_h$. Because its restriction to every $T \in \mathbb{T}_h$ is a polynomial, it is differentiable in the classical sense. Let us consider the function $w_h \in L^2(D)$ defined on T by $w_h|_T = \mathrm{div}(v_h)|_T$. Let $\varphi \in C_0^\infty(D)$. Then, using the Green formula yields

$$\begin{aligned} \int_D w_h \varphi dx &= \sum_{T \in \mathbb{T}_h} \int_T w_h \varphi dx \\ &= -\sum_{T \in \mathbb{T}_h} \int_T (v_h)|_T \cdot \nabla \varphi dx + \sum_{F \in \mathcal{F}_h^i} \int_F [\![v_h \cdot n]\!]_F \varphi ds. \end{aligned}$$

Owing to $[\![v_h \cdot n]\!]_F = 0$,

$$\int_D w_h \varphi dx = -\int_D (v_h \cdot \nabla) \varphi dx.$$

Therefore, the distributional divergence of v_h is w_h. Because $w_h \in L^2(D)$, $\mathrm{div}\, v_h \in L^2(D)$. □

Recall that for $v \in RT^k(T)$, the local degrees of freedom are given as

$$\int_{F_i} v \cdot n_{F_i} p_k ds, \quad \forall p_k \in \mathcal{P}^k(F_i), \quad F_i \subset \partial T, \tag{B.3.1}$$

$$\int_T v \cdot q_{k-1} dx, \quad \forall q_{k-1} \in \mathcal{P}^{k-1}(T)^d. \tag{B.3.2}$$

Here, n_{F_i} denotes the outer unit normal vector of T on the face F_i. Note that for $k = 0$, local degrees of freedom of type (B.3.2) are violated.

Lemma B.3.2. *Let $T \in \mathbb{T}_h$. For any $v \in RT^k(T)$ and any $F \subset \partial T$, it holds that*

$$\operatorname{div} v \in \mathcal{P}^k(T), \tag{B.3.3}$$

$$v \cdot n|_F \in \mathcal{P}^k(F). \tag{B.3.4}$$

The divergence operator is surjective from RT^k to $\mathcal{P}^k$, hence,

$$\operatorname{div} RT^k = \mathcal{P}^k. \tag{B.3.5}$$

For the divergence free functions,

$$RT^k_\sigma := \{v \in RT^k : \operatorname{div} v = 0\} \subset (\mathcal{P}^k)^d. \tag{B.3.6}$$

Proof. The proof is found in [53, Lemma 4.2.11]. □

Lemma B.3.3. *For the simplicial Raviart–Thomas element in $\mathbb{R}^d$, it holds that*

$$\begin{aligned}\dim RT^k &= (d+1)\dim \mathcal{P}^k - \dim \mathcal{P}^{k-1} \\ &= (d^2 + kd + d)\frac{(k+d-1)!}{d!k!}.\end{aligned} \tag{B.3.7}$$

Proof. The proof is found in [53, Lemma 4.2.12]. □

Lemma B.3.4. *The Raviart–Thomas element with the nodal values in* (B.3.1) *and* (B.3.2) *is unisolvent.*

Proof. The proof is found in [53, Lemma 4.2.13]. □

B.4 Remarks on the Definition of the Raviart–Thomas Interpolation

We cite [31, Section 17.1], [18, Section 2.5.1] and [53, Example 4.2.23].

Let $T \in \mathbb{T}_h$ and F a face of T. Let v be a vector field defined on T. The goal is to search smoothness requirements on the field v to give a meaning to the quantity $\int_F (v \cdot n_T)\phi ds$, where ϕ is a given smooth function on F (e.g., a polynomial function) and n_T is the outward unit normal vector on ∂T.

Let $v \in H(\operatorname{div}; T)$. As described in (B.1.2), the normal trace $\gamma^d(v) \in H^{-\frac{1}{2}}(\partial T)$ is defined as

$$\langle \gamma^d(v), \psi \rangle_{\partial T} := \int_T ((v \cdot \nabla) w(\psi) + (\operatorname{div} v) w(\psi))\, dx,$$

for any $\psi \in H^{\frac{1}{2}}(\partial T)$, where $w(\psi) \in H^1(T)$ is a lifting of ψ, that is, $\gamma^g(w(\psi)) = \psi$. Here, $\gamma^g : H^1(T) \to H^{\frac{1}{2}}(\partial T)$ is the trace map. However, this situation is too weak for our purpose, because we need to localise the normal trace to functions ψ only defined on a face F, that is, ψ may not be defined on the whole boundary ∂T. As an example, we introduce the following remark ([18, Remark 2.5.1]).

Remark B.4.1. Given a function $\chi \in H^{-\frac{1}{2}}(\partial T)$, even if we are allowed to take

$$\int_{\partial T} \chi ds := \langle \chi, \psi \rangle \quad \text{with } \psi \equiv 1,$$

we cannot take the integral over an edge F of ∂T. The typical answer is: "Because the function identically equal to 1 on the whole boundary ∂T belongs to $H^{\frac{1}{2}}(\partial T)$, while the function that is equal to 1 on the edge F and 0 on the rest of ∂T does not belong to $H^{\frac{1}{2}}(\partial T)$."

As a example to think about this situation, see [18, Remark 2.5.1] and [53, Example 4.2.23].

A key observation is to extend ψ by zero from F to ∂T. Because the extended function is not in $H^{\frac{1}{2}}(\partial T)$, one need to change the functional setting. Therefore, we use the fact the zero-extension of a smooth function defined on a face F of ∂T is in $W^{1-\frac{1}{t},t}(\partial T)$ with $t < 2$.

B.5 Face-to-cell Lifting Operator

In this section, we discuss about a face-to-cell lifting operator, see [31, Section 17.1].

Let p, q be two real numbers such that

$$p > 2, \quad q > \frac{2d}{2+d}.$$

Let v be a vector field on T such that

$$v \in L^p(T), \quad \operatorname{div} v \in L^q(T).$$

Let $\tilde{p} \in (2, p]$ be such that $q \geq \frac{\tilde{p}d}{\tilde{p}+d}$. Let $\tilde{p}'$ be conjugate number such that $\frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1$. Let $L_F^T : W^{\frac{1}{\tilde{p}},\tilde{p}'}(F) \to W^{1,\tilde{p}'}(T)$ be a lifting operator such that, for any $\phi \in W^{\frac{1}{\tilde{p}},\tilde{p}'}(F)$, $L_F^T(\phi)$ is a lifting of the zero-extension of ϕ to ∂T, that is,

$$\gamma^g(L_F^T(\phi))|_{\partial T \setminus F} = 0, \quad \gamma^g(L_F^T(\phi))|_F = \phi. \tag{B.5.1}$$

Note that the domain of L_F^T is $W^{1-\frac{1}{t},t}(F)$ with $t := \tilde{p}' < 2$. We observe that

$$L_F^T(\phi) \in W^{1,p'}(T) \cap L^{q'}(T),$$

with conjugate numbers p' and q' such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Indeed, $L_F^T(\phi) \in W^{1,p'}(T)$ follows from $p' \leq \tilde{p}'$ (that is, $\tilde{p} \leq p$). Meanwhile, $L_F^T(\phi) \in L^{q'}(T)$ follows from $W^{1,\tilde{p}'}(T) \hookrightarrow L^{q'}(T)$ owing to the Sobolev embedding theorem (since $q' \leq \frac{\tilde{p}'d}{d-\tilde{p}'}$ can be verified from $d \geq 2 > \tilde{p}'$ and $\frac{1}{\tilde{p}'} - \frac{1}{d} = 1 - (\frac{1}{\tilde{p}} + \frac{1}{d}) \leq 1 - \frac{1}{q} = \frac{1}{q'}$ because $q \geq \frac{\tilde{p}d}{\tilde{p}+d}$).

With the lifting operator L_F^T and fixing $\phi \in W^{\frac{1}{\tilde{p}},\tilde{p}'}(F)$, we define the linear form χ_ϕ on $V(T) := \{v \in L^p(T)^d : \operatorname{div} v \in L^q(T)\}$ as

$$\chi_\phi(v) := \int_T \big((v \cdot \nabla)L_F^T(\phi) + (\operatorname{div} v)L_F^T(\phi)\big)\, dx. \tag{B.5.2}$$

The right-hand side of (B.5.2) is well defined owing to Hölder's inequality, and whenever the field v is smooth, we have, from (B.5.1),

$$\chi_\phi(v) = \int_{\partial T} (v \cdot n_T)\gamma^g(L_F^T(\phi))ds = \int_{\partial T} (v \cdot n_T)\phi ds.$$

Thus, the linear form $v \mapsto \chi_\phi(v)$ is an extension of the linear form $v \mapsto \int_{\partial T}(v \cdot n_T)\phi ds$, which is meaningful for smooth fields $v \in \mathcal{C}^0(T)^d$. This extension is bounded for all $v \in V(T)$.

Remark B.5.1. This thesis does not address this situation on anisotropic meshes. We leave them for future work.

Conclusions

In this thesis, we have developed the anisotropic interpolation error theory. We found that the new parameter, geometric condition equivalent to the maximum-angle condition, and another geometric condition deduces that the interpolation error estimates are optimal. Therefore, the above two geometric conditions are straightforward and may be useful.

In much literature, as a mesh condition, the shape-regularity condition is widely used and well known to obtain optimal interpolation error estimates. However, in some cases, it is not necessary for the shape-regularity condition. As usual, to do the interpolation error analysis, we need to set an affine mapping. Our idea is to divide it into three mappings. The Euclidean condition number of the Jacobian matrix of the affine transformations is bounded by a geometric quantity. We can naturally consider the new geometric condition as being sufficient to obtain optimal order estimates. The error estimations may be applied to arbitrary meshes, including very " flat " or anisotropic simplices. Furthermore, it is proven that the geometric condition is equivalent to the maximum-angle condition. Therefore, We expect the new mesh condition to become an alternative to the maximum-angle condition.

In anisotropic interpolation error estimates, the heart of our analysis is to do a delicate scaling argument. From this argument, one realises that it is possible to obtain more precise results by adding a specific condition to the maximum-angle condition. As usual, in error analysis, we use the Bramble–Hilbert-type lemma on the reference elements; however, there are some points to take care of in anisotropic analysis. We need to apply the lemma component-wise. However, the proof is generally not valid because the order exchange between differentiation and interpolation does not hold. To overcome this difficulty, we need to construct a set of functionals. The optimal interpolation error estimates can be obtained under slightly stricter assumptions than the normal error ones through these preparations.

We analyse the interpolation properties of the Raviart–Thomas finite element space. The analogous argument above makes it possible to obtain the anisotropic Raviart–Thomas interpolation error estimates. The heart of our analysis is also to do a delicate scaling argument. However, we need to use the Piola transformation for the analysis of a vector field. Our idea is to divide it into three transformations. This argument makes it possible to obtain more precise results by adding a specific condition to the maximum-angle condition. As usual, to deduce the interpolation estimates, we use the Bramble–Hilbert type lemma on the reference elements. However, in general, it is known that the component-wise stability of the Raviart–Thomas does not hold. The answer to this difficulty is to realise that one can try to "kill" degrees of freedom. Through these preparations, the optimal Raviart–Thomas interpolation error estimates can be obtained under slightly stricter assumptions than the normal error ones.

We finally introduced application examples. In this thesis, we teated the Crouzeix–Raviart finite element approximation for the Poisson and Stokes problems and the dual mixed formulation of the Poisson problem. Due to time constraints, we could not give more applications. There is still more work to be addressed. These are left as work for the future.

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